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Star-Kernels and Star-Differentials in Quasidifferential Analysis*

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This paper is devoted to the study of quasidifferential structure. Three concepts, kernelled quasidifferential, star-kernel and star-differential, are proposed. Kernelled quasidifferential is used to describe a special class of quasidifferentiable functions, which covers convex and concave functions. A sufficiency theorem and a sufficiency and necessity theorem for a quasi-kernel being a kernelled quasidifferential are proven. The notion of star-kernel is employed if the quasi-kernel is not a kernelled quasidifferential. The existence theorem for a star-kernel of a quasidifferentiable function is established, which shows that the star-kernel is a pair of star-shaped sets and the sub-/super-derivative is expressed by the gauge of a star-shaped set. The notion of star-differential is used to describe the differential of the class of directionally differentiable functions which contains the class of quasidifferentiable functions. A star-differential is also a pair of star-shaped sets and its operational properties are favourable. A representative of the star-differential can be easily obtained by decomposing the directional derivative into the difference of its positive and negative part.

Keywords: Quasidifferentiable function, directional derivative, quasidifferential, kernelled quasidifferential, star-kernel, star-differential, star-shaped set

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1. Introduction

It is well known that for a quasidifferentiable function f defined on an open set $\mathcal{O} \subset \mathbf{R}^n$ in the sense of Demyanov & Rubinov, the class of quasidifferentials of f at a point in $\mathcal{O}$, say $x \in \mathcal{O}$, is very large so that the whole space $\mathbf{R}^n$ could be covered by the union of sub- or super-differentials, i.e.,

$$\mathbf{R}^n = \bigcup_{Df(x)\in\mathcal{D}f(x)} \underline{\partial} f(x) = \bigcup_{Df(x)\in\mathcal{D}f(x)} \overline{\partial} f(x)$$

where $\mathcal{D}f(x)$ denotes the class of quasidifferentials of f at x, $Df(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ denotes a quasidifferential of f at x consisting of two nonempty compact convex set $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ as its components, called subdifferential and superdifferential, respectively, such that

$$\begin{aligned} f'(x;\cdot) &= \max < \cdot, \underline{\partial} f(x) > + \min < \cdot, \overline{\partial} f(x) > \\ &= \delta^*(\cdot \,|\, \underline{\partial} f(x)) - \delta^*(\cdot \,|\, -\overline{\partial} f(x)) \end{aligned} \tag{1}$$

Here we denote by $< \cdot, A >$ the set $\{< \cdot, a > \,|\, a \in A\}$ and A is a set of $\mathbf{R}^n$. Also it is well known that every $\mathcal{D}f(x)$ and a quasilinear function of the vector space of quasilinear functions are dual to each other in the sense of Minkowski and $\mathcal{D}f(x)$ is an element of a quotient space defined by the equivalence relationship

$$[A_1, B_1] \sim [A_2, B_2] \quad \text{iff} \quad A_1 - B_2 = A_2 - B_1 \tag{2}$$

see for instance, [2], [3] and [4]. Some algebraic properties of $\mathcal{D}f(x)$ as a set are just the consequences of results obtained in the last several years, see for instance, [9], [10], [11], [12], [1] and [4]. The notion of minimal pairs, under an ordering defined by the inclusion relationship, of convex compact subsets of a topological Hausdorff vector space was introduced by Pallaschke, Scholtes and Unbański (1991), and the existence of equivalent minimal pairs of nonempty compact convex sets was also proved. For one dimensional space equivalent minimal pairs are uniquely determined up to translations, due to [9]. For the two dimensional case equivalent minimal pairs are also uniquely determined up to translations, see [16] and [1]. For the case more than the 3-dimensional case Grzybowski (1994) gave an example, in 3-dimensional case, of finitely many equivalent minimal pairs which are not connected by translations, and furthermore, Pallaschke and Unbański (1996) [12] indicated that a continuum of equivalent pairs are not connected by translation for different indices. Some sufficient conditions and both sufficient and necessary conditions for the minimality of pairs of compact convex sets were given and some reduction techniques for the reduction of pairs of compact convex sets via cutting hyperplanes or excision of compact convex subsets was proposed due to Pallaschke and Urbański (1993, 1994). Since a finite sublinear function and a unique nonempty compact convex set are dual in the sense of Minkowski. Hence, there exists an one-to-one correspondence between $\mathcal{D}f(x)$ and the set, associated with a quasilinear function that is the directional derivative $f'(x;\cdot)$ or $f'_x(\cdot)$, of equivalent summation structures with sublinear and superlinear functions, defined by $f'_x(\cdot) = l_{\text{sub}}(\cdot) + l_{\text{sup}}(\cdot)$, or of equivalent subtraction structures with two sublinear functions defined by $f'_x(\cdot) = l^1_{\text{sub}}(\cdot) - l^2_{\text{sub}}(\cdot)$, in other words it is defined by (1). $\mathcal{D}f(x)$ and $\mathcal{L}^+ = \{[l^1_{\text{sub}}(\cdot), l^2_{\text{sub}}(\cdot)] \,|\, f'_x(\cdot) = l^1_{\text{sub}}(\cdot) - l^2_{\text{sub}}(\cdot)\}$ or $\mathcal{L}^- = \{[l_{\text{sub}}(\cdot), l_{\text{sup}}(\cdot)] \,|\, f'_x(\cdot) = l_{\text{sub}}(\cdot) + l_{\text{sup}}(\cdot)\}$ are isomorphism, with respect to addition and multiplication by scalars in the real number field, with order-preserving. Taking $[A, B]$ and $[C, D] \in \mathcal{D}f(x)$ and define

$$[A, B] \preceq [C, D] \quad \text{iff} \quad A \subset C,\, B \subset D \tag{3}$$

one can define a preordering for $\mathcal{L}^+$

$$[l^A_{\text{sub}}(\cdot), l^B_{\text{sub}}(\cdot)] \preceq [l^C_{\text{sub}}(\cdot), l^D_{\text{sub}}(\cdot)] \quad \text{iff} \quad [A, B] \preceq [C, D] \tag{4}$$

Specifically,

$$l^A_{\text{sub}}(\cdot), \leq l^C_{\text{sub}}(\cdot), \quad l^B_{\text{sub}}(\cdot) \leq l^D_{\text{sub}}(\cdot) \tag{5}$$

where " $\leq$ " is defined by normal meaning, or one can define a preordering for $\mathcal{L}^-$

$$[l^A_{\text{sub}}(\cdot), l^B_{\text{sup}}(\cdot)] \preceq [l^C_{\text{sub}}(\cdot), l^D_{\text{sup}}(\cdot)] \quad \text{iff} \quad [A, B] \preceq [C, D] \tag{6}$$

Specifically,

$$l^A_{\text{sub}}(\cdot), \leq l^C_{\text{sub}}(\cdot), \quad l^B_{\text{sup}}(\cdot) \geq l^D_{\text{sup}}(\cdot) \tag{7}$$

It follows from above that (3)-(7) might be used to characterize some properties of minimal elements in $\mathcal{D}f(x)$, mainly in algebraic, in other words, these properties related to minimality in $\mathcal{D}f(x)$ might be determined by decomposition structures of directional derivative $f'(x;\cdot)$ or $f'_x(\cdot)$, indicated with (1). Indeed, the form (1) is suitable to describe $\mathcal{D}f(x)$ in algebraic as has been done by Pallaschke et al.
A decomposition structure, subtraction structure, of $f'(x;\cdot)$ defined by

$$f'(x;\cdot) = \underline{f}'(x;\cdot) - \overline{f}'(x;\cdot) \tag{8}$$

where $\underline{f}'(x;\cdot)$ and $\overline{f}'(x;\cdot)$ are defined by

$$\underline{f}'(x;\cdot) = \inf_{Df(x)\in\mathcal{D}f(x)} \delta^*(\cdot \,|\, (\underline{\partial} + \overline{\partial})f(x)), \quad \overline{f}'(x;\cdot) = \inf_{Df(x)\in\mathcal{D}f(x)} \delta^*(\cdot \,|\, (\overline{\partial} - \overline{\partial})f(x)) \tag{9}$$

respectively. Generally, $\underline{f}'$ and $\overline{f}'$ are positively homogeneous, but not sublinear. It was proved that

$$\underline{S} = \bigcap_{Df(x)\in\mathcal{D}f(x)} (\underline{\partial} + \overline{\partial})f(x), \overline{S} = \bigcap_{Df(x)\in\mathcal{D}f(x)} (\overline{\partial} - \overline{\partial})f(x) \tag{10}$$

are nonempty, due to Deng and Gao (1991). It is easy to be seen that

$$\delta^*(\cdot \,|\, \underline{S}) \leq \underline{f}'(x;\cdot), \quad \delta^*(\cdot \,|\, \overline{S}) \leq \overline{f}'(x;\cdot) \tag{11}$$

The above statements, (8)-(11), lead to explore geometric properties of $\mathcal{D}f(x)$. In one dimensional case, it was proved that

$$\delta^*(\cdot \,|\, \underline{S}) = \underline{f}'(x;\cdot), \quad \delta^*(\cdot \,|\, \overline{S}) = \overline{f}'(x;\cdot) \tag{12}$$

in other words, $[\underline{S}, \overline{S}] \in \mathcal{D}f(x)$, denoted by $[\partial_* f(x), \partial^* f(x)]$ and the structure of $[\underline{S}, \overline{S}]$ was given by

$$\partial_* f(x) = [\alpha_*, \beta_*], \quad \partial^* f(x) = [\alpha^*, \beta^*] \tag{13}$$

where

$$\begin{array}{l} \alpha_* = \min\{f'(x;1), -f'(x;-1)\} \\ \beta_* = \max\{f'(x;1), -f'(x;-1)\} \\ \alpha^* = \min\{0, f'(x;1) + f'(x;-1)\} \\ \beta^* = \max\{0, -f'(x;1) - f'(x;-1)\} \end{array} \tag{14}$$

see Gao (1988), Xia and Gao (1993). Recently, Gao proved, that both in one and two dimensional cases the following equalities are true

$$\underline{S} = \partial_m f(x) + \partial^m f(x), \quad \overline{S} = \partial^m f(x) - \partial^m f(x) \tag{15}$$

and $[\underline{S}, \overline{S}] \in \mathcal{D}f(x)$, where $[\partial_m f(x), \partial^m f(x)] \in \mathcal{D}f(x)$ is a minimal pair of $\mathcal{D}f(x)$ in the sense of Pallaschke, Scholtes and Urbański, see [9], [16], [7] and [8]. However, in more than 3-dimensional case (15) is not true for all minimal quasidifferentials of $\mathcal{D}f(x)$. In this case it might be necessary to explore some geometric properties that is the main purpose of the paper and the work due to Rubinov & Yagubov (1986) would be one of the useful and potential tools that will be quoted frequently as basic references for this purpose.

$\underline{S}$ and $\overline{S}$ (defined by (10)) are called sub- and super-kernel, respectively, and $[\underline{S}, \overline{S}]$ is called a quasi-kernel of $\mathcal{D}f(x)$. Of course $\underline{S}$ and $\overline{S}$ are compact convex.

In Sec. 2, a sufficient condition for a function to be a kernelled quasidifferentiable one and some operations of kernelled quasidifferentiable functions are given. In Sec. 3, definitions of star shaped quasidifferentials associated with subkernels will be introduced. In Sec. 4, we investigate a subset $\mathcal{D}^0 f(x)$ of $\mathcal{D}f(x)$ containing zero and a sufficient condition for the pair of sets

$$\bigcap_{Df(x)\in\mathcal{D}f(x)} \underline{\partial} f(x), \qquad \bigcap_{Df(x)\in\mathcal{D}f(x)} \overline{\partial} f(x)$$

to be a quasidifferential. In Sec. 5, the notion of star shaped differentials is introduced for directionally differentiable functions based upon the results given by Rubinov and Yagubov (1986) and a nonnegative decomposition of directional derivatives. In Sec. 6, some relationships between Penot differentials and sub- and super directional derivatives are investigated briefly.

2. Kernelled Quasidifferentiable Functions

From Deng and Gao (1991), we know that $\underline{S} \neq \emptyset$. It is easy to be seen that, for any $u \in \underline{S}$ there exists at least one sequence $\{u_i \,|\, u_i \in \underline{\partial}_i f(x) + \overline{\partial}_i f(x)\}$ convergent to u, where $[\underline{\partial}_i f(x), \overline{\partial}_i f(x)] \in \mathcal{D}f(x)$. Especially, if $u \in \mathrm{bd}\underline{S}$, then there exists a sequence $\{u_i \,|\, u_i \in \mathrm{bd}(\underline{\partial}_i f(x) + \overline{\partial}_i f(x))\}$ convergent to u. Let us denote by $\underline{S}(d)$ the max-face of $\underline{S}$ with respect to d, i.e.,

$$\underline{S}(d) = \mathrm{Arg}\max_{u'\in\underline{S}} < u', d >$$

and by $N(u, \underline{S})$ the normal cone to $\underline{S}$ at $u \in \underline{S}$, i.e.,

$$N(u, \underline{S}) = \{v \in \mathbf{R}^n \mid \, < v, u' - u > \leq 0, \forall\, u' \in \underline{S}\}$$

then $u \in \underline{S}(d)$ if and only if $d \in N(u, \underline{S})$. Suppose $\{u_i \,|\, u_i \in \underline{\partial}_i f(x) + \overline{\partial}_i f(x)\}_1^\infty \longrightarrow u \in \underline{S}$ and assume furthermore that for each i, $d_i \in N(u_i, \underline{\partial}_i f(x) + \overline{\partial}_i f(x)) \cap B_1(0)$, then the set of clusters of $\{d_i\}_1^\infty$ is included in $N(u, \underline{S})$. The above lines enable us to give a sufficient condition for $d \in \mathbf{R}^n$ satisfying $\delta^*(d \,|\, \underline{S}) = \underline{f}'(x; d)$ as follows: $u \in \underline{S}$ and $d \in \mathbf{R}^n$ such that there exist sequences $\{u_i \in \underline{\partial}_i f(x) + \overline{\partial}_i f(x)\}_1^\infty \longrightarrow u$ and $\{d_i \,|\, d_i \in$ and assume furthermore that for each i, $d_i \in N(u_i, \underline{\partial}_i f(x) + \overline{\partial}_i f(x))\}_1^\infty \longrightarrow d$.

In the sequel, we will discuss the class of quasidifferentiable functions satisfying $[\underline{S}, \overline{S}] \in \mathcal{D}f(x)$ for every $x \in \mathbf{R}^n$. For this purpose, the following definition is introduced.

Definition 2.1. Let f be a quasidifferentiable function defined on $\mathbf{R}^n$. The quasi-kernel is said to be a kernelled quasidifferential of f at x iff

$$\delta^*(\cdot \,|\, \underline{S}) = \underline{f}'(x; \cdot), \quad \delta^*(\cdot \,|\, \overline{S}) = \overline{f}'(x; \cdot) \tag{16}$$

If f has a kernelled quasidifferential at $x \in \mathbf{R}^n$, then f is said to be a kernelled quasidifferentiable function at x. The kernel $[\underline{S}, \overline{S}]$ is a quasidifferential, denoted by $[\partial_* f(x), \partial^* f(x)]$ or $[\partial_*, \partial^*] f(x)$ or $[\partial_* f, \partial^* f](x)$.

Let $\mathcal{F}(\underline{S}, \overline{S})$ be a shape of $(\underline{S}, \overline{S})$ that is defined by a similar way due to [9], such that

$$\overline{\text{co}} \bigcup_{d \in \mathcal{F}(\underline{S}, \overline{S})} \underline{S}(d) = \underline{S}, \quad \overline{\text{co}} \bigcup_{d \in \mathcal{F}(\underline{S}, \overline{S})} \overline{S}(d) = \overline{S} \tag{17}$$

where $\overline{\text{co}}M$ denotes the closed convex hull of a set M. If $[\underline{S}, \overline{S}] \in \mathcal{D}f(x)$ and satisfies conditions stated in [Th.2.1 or Th.2.2, 10], then $[\underline{S}, \overline{S}]$ is a minimal quasidifferential pair. Generally speaking, $[\underline{S}, \overline{S}]$ is not a quasidifferential, in other words, there are no kernelled quasidifferentials under the definition (16). However, the following theorem provides a sufficient condition for $[\underline{S}, \overline{S}]$ to be a kernelled quasidifferential.

Lemma 2.2. *If $d_1, d_2 \in N(u, \overline{C})$ then*

$$\delta^*(d_1 + d_2 \,|\, \overline{C}) = \delta^*(d_1 \,|\, \overline{C}) + \delta^*(d_2 \,|\, \overline{C})$$

where $\overline{C}$ is a closed convex set and $u \in \overline{C}$.

Proof. The conclusion comes from the inequality $\delta^*(d_1 + d_2 \,|\, \overline{C}) \leq \delta^*(d_1 \,|\, \overline{C}) + \delta^*(d_2 \,|\, \overline{C}) =< d_1 + d_2, u >$ and the condition $u \in \overline{C}$. □

Theorem 2.3. *Let f be a quasidifferntiable function on $\mathbf{R}^n$. Suppose that $\underline{f}'(x; \cdot)$ and $\overline{f}'(x; \cdot)$ are continuous with respect to direction, and furthermore there exists a shape $\mathcal{F}(\underline{S}, \overline{S})$ of $(\underline{S}, \overline{S})$ such that for any $u \in \underline{S}$ and $\overline{S}$ one has that*

$$N(u, \underline{S}) = \overline{cone}\{N(u, \underline{S}) \cap \mathcal{F}(\underline{S}, \overline{S})\} \tag{18}$$

$$N(v, \overline{S}) = \overline{cone}\{N(v, \overline{S}) \cap \mathcal{F}(\underline{S}, \overline{S})\} \tag{19}$$

If for any $d \in \mathcal{F}(\underline{S}, \overline{S})\}, u \in \underline{S}(d)$ and $v \in \overline{S}(d)$ there exist sequences

$$\{u_i \,|\, u_i \in (\underline{\partial}_i + \overline{\partial}_i) f(x)\}_1^\infty \longrightarrow u \tag{20}$$

$$\{v_i \,|\, v_i \in (\overline{\partial}_i - \overline{\partial}_i) f(x)\}_1^\infty \longrightarrow v \tag{21}$$

$$\{d_i \,|\, d_i \in N(u_i, (\underline{\partial}_i + \overline{\partial}_i) f(x)) \cap N(v_i, (\overline{\partial}_i - \overline{\partial}_i) f(x))\} \tag{22}$$

such that d is one of clusters of $\{d_i\}_1^\infty$, then $[\underline{S}, \overline{S}] \in \mathcal{D}f(x)$, in other words, one has that

$$\underline{f}'(x; \cdot) = \delta^*(\cdot \,|\, \underline{S}), \quad \overline{f}'(x; \cdot) = \delta^*(\cdot \,|\, \overline{S}) \tag{23}$$

Proof. Let $d \in \mathbf{R}^n$ be an arbitrary nonzero vector. There exist $u \in \underline{S}$ and $v \in \overline{S}$ such that $d \in N(u, \underline{S}) \cap N(v, \overline{S})$. According to (18) and (19) there exists a sequence

$$d_i \in \mathrm{cone}\{N(u, \underline{S}) \cap \mathcal{F}(\underline{S}, \overline{S})\} \cap \mathrm{cone}\{N(v, \overline{S}) \cap \mathcal{F}(\underline{S}, \overline{S})\}$$

$i = 1, 2, \ldots$, convergent to d. For each i there are two index sets $\underline{J}_i$ and $\overline{J}_i$, with finite indices such that

$$\underline{d}_{ij} \in N(u_i, \underline{S}) \cap \mathcal{F}(\underline{S}, \overline{S}), \quad j \in \underline{J}_i \tag{24}$$

$$\overline{d}_{ij} \in N(v_i, \overline{S}) \cap \mathcal{F}(\underline{S}, \overline{S}), \quad j \in \overline{J}_i \tag{25}$$

$$d_i \in \mathrm{co}\{\underline{d}_{ij} \,|\, j \in \underline{J}_i\} \cap \mathrm{co}\{\overline{d}_{ij} \,|\, j \in \overline{J}_i\} \tag{26}$$

It follows from (20)-(22) and (24)-(26) that for each ij there exist $\{\underline{d}_{ij_k}\}_1^\infty$, $\{\overline{d}_{ij_k}\}_1^\infty$, $\{u_{ij_k}\}_1^\infty$ and $\{v_{ij_k}\}_1^\infty$ such that

$$\{u_{ij_k} \in (\underline{\partial}_{ij_k} + \overline{\partial}_{ij_k}) f(x)\}_1^\infty \longrightarrow u \tag{27}$$

$$\{v_{ij_k} \in (\overline{\partial}_{ij_k} - \overline{\partial}_{ij_k}) f(x)\}_1^\infty \longrightarrow v \tag{28}$$

$$\{\underline{d}_{ij_k} \in N(u_{ij_k}, (\underline{\partial}_{ij_k} + \overline{\partial}_{ij_k}) f(x)\}_1^\infty \longrightarrow \underline{d}_{ij}, \quad j \in \underline{J}_i,\, i = 1, 2, \ldots \tag{29}$$

$$\{\overline{d}_{ij_k} \in N(v_{ij_k}, (\overline{\partial}_{ij_k} - \overline{\partial}_{ij_k}) f(x)\}_1^\infty \longrightarrow \overline{d}_{ij}, \quad j \in \overline{J}_i,\, i = 1, 2, \ldots \tag{30}$$

Since each d_i is a convex combination of $\underline{d}_{ij}, j \in \underline{J}_i$, or of $\overline{d}_{ij}, j \in \overline{J}_i$, one has that there are $\underline{\lambda}_{ij} \geq 0$ and $\overline{\lambda}_{ij} \geq 0$ such that

$$\sum_{j \in \underline{J}_i} \underline{\lambda}_{ij} = 1, \quad \text{and} \quad \sum_{j \in \overline{J}_i} \overline{\lambda}_{ij} = 1$$

satisfying

$$d_i = \sum_{j \in \underline{J}_i} \underline{\lambda}_{ij} \underline{d}_{ij} = \sum_{j \in \overline{J}_i} \overline{\lambda}_{ij} \overline{d}_{ij} \tag{31}$$

$$\begin{aligned} \delta^*(d_i \,|\, \underline{S}) &= \textstyle\sum_{j \in \underline{J}_i} \underline{\lambda}_{ij} < \underline{d}_{ij}, u_i > \\ &= \textstyle\sum_{j \in \underline{J}_i} \underline{\lambda}_{ij} \lim_{k \to \infty} < \underline{d}_{ij_k}, u_{ij_k} > \end{aligned} \tag{32}$$

from (20) and (21), where $\underline{d}_{ij_k} \in N(u_{ij_k}, (\underline{\partial}_{ij_k} + \overline{\partial}_{ij_k}) f(x))$. Since $\{u_{ij_k} \in (\underline{\partial}_{ij_k} + \overline{\partial}_{ij_k}) f(x)\}_{k=1}^\infty \longrightarrow u_i$, $\{\underline{d}_{ij_k} \in N(u_{ij_k}, (\underline{\partial}_{ij_k} + \overline{\partial}_{ij_k}) f(x)\}_{k=1}^\infty \longrightarrow d_{ij}$, it follows, from the sufficient condition for $\delta^*(\xi \,|\, \underline{S}) = \underline{f}'(x; xi)$ given at the beginning of this section, that

$$\begin{aligned} \delta^*(\underline{d}_{ij} \,|\, \underline{S}) &= \underline{f}'(x; \underline{d}_{ij}) \\ &= \lim_{k \to \infty} < \underline{d}_{ij_k}, u_{ij_k} > \\ &=< \underline{d}_{ij}, u_i > \end{aligned} \tag{33}$$

Thus, we obtain that from (33) that

$$\delta^*(d_i \,|\, \underline{S}) =< \sum_{j \in \underline{J}_i} \underline{\lambda}_{ij} \underline{d}_{ij}, u_i >= \underline{f}'(x; d_i) \tag{34}$$

Without loss of generality assume $\{d_i\}_1^\infty \longrightarrow d$. Taking the limit to (34), one has that

$$\delta^*(d \,|\, \underline{S}) =< d, u >= \lim_{i\to\infty} \underline{f}'(x; d_i) \tag{35}$$

According to the continuity of $\underline{f}'(x;\cdot)$, (35) becomes

$$\delta^*(d \,|\, \underline{S}) = \underline{f}'(x; d) \tag{36}$$

Likewise, the second assertion in (23) can be proved by the similar way used in proving the first assertion of (23), i.e.,

$$\delta^*(d \,|\, \overline{S}) = \overline{f}'(x; d) \tag{37}$$

The demonstration is completed. □

If f has a kernelled quasidifferential at a point, say x, under the definition (16), then some basic operations on kernel at x can be established. We only list some of these properties without demonstration as follows by Rules 1-5.

Rule 1. Suppose f_i, $i = 1, \dots, m$, are quasidifferentiable at x and have kernelled quasidifferentials $[\partial_* f(x), \partial^* f(x)]$. Then for the sum function $f(x) = \sum_{i=1}^m f_i(x)$ one has

$$\partial_* f(x) = \sum_{i=1}^m \partial_* f_i(x), \quad \partial^* f(x) = \sum_{i=1}^m \partial^* f_i(x)$$

where $\mathcal{D}f(x)$ is understood as $\mathcal{D}f(x) = \sum_{i=1}^m \mathcal{D}f_i(x)$.

Rule 2. Suppose $f_i, i \in I$, are kernelled quasidifferentiable at x and $\lambda_i, i \in I$, are scalars, where I is a finite index set. Then we have

$$D_K(\sum_{i\in I} \lambda_i f_i)(x) = \sum_{i\in I} D_K((\mathrm{sign}\lambda_i) f_i)(x)$$

where $D_K\phi(x)$ is defined by $D_K\phi(x) = [\partial_*\phi(x), \partial^*\phi(x)]$.

Rule 3. Suppose f_1 and f_2 are kernelled quasidifferentiable at x. Then one has

$$D_K(f_1 f_2)(x) = |f_1(x)| D_K((\mathrm{sign} f_1(x)) f_2)(x) + |f_2(x)| D_K((\mathrm{sign} f_2(x)) f_1)(x)$$

where $\mathcal{D}(f_1 f_2)(x)$ is understood as $f_1(x)\mathcal{D}f_2(x) + f_2(x)\mathcal{D}f_1(x)$.

Rule 4. Suppose f_1 and f_2 are kernelled quasidifferentiable at x and $f_2(x) \neq 0$. Then one has

$$D_K(f_1/f_2)(x) = (|f_2(x)| D_K((\mathrm{sign} f_2(x)) f_1)(x) + |f_1(x)| D_K((\mathrm{sign}(-f_1(x)) f_2)(x))/f_2(x)^2$$

Rule 5. For $f = \max_{i\in I} f_i$, where $f_i, i \in I$, are kernelled quasidifferentiable at x, I is a set of finite indices, one has

$$\partial_* f(x) = \mathrm{co} \bigcup_{k\in R(x)} (\partial_* f_k(x) + \sum_{i\in R(x)\setminus\{k\}} \partial^* f_i(x)), \quad \partial^* f(x) = \sum_{k\in R(x)} \partial^* f_k(x)$$

where $R(x) = \{i \in I \,|\, f_i(x) = f(x)\}$.

3. Star Kernels

It is already known that if f has a kernelled quasidifferential in the sense of the definition (16) at x, then for any nonempty convex compact set A one has that $[\partial_* f(x)-A, \partial^* f(x)+A] \in \mathcal{D}f(x)$. This simple fact can be used to explore the case in which $[\underline{S}, \overline{S}] \notin \mathcal{D}f(x)$, i.e., f has no kernelled quasidifferential at x. In this case special structure for sub- and super-derivatives might be studied. To this end define

$$S(x) = \mathrm{co}\{\{0\}, \bigcap_{Df(x)\in\mathcal{D}f(x)} (\underline{\partial}+\overline{\partial})f(x)\} \tag{38}$$

For simplicity S is used instead of $S(x)$ in the rest of this subparagraph. If $0 \not{i}n\underline{S}$, then S is a umbra of $\underline{S}$ with respect to the origin. Obviously, one has

$$(0,0) \in [(\underline{\partial}+\overline{\partial})f(x) - S, (\overline{\partial}-\overline{\partial})f(x) + S] \tag{39}$$

Note that $S \neq \emptyset$ since the second term in the convex hull of (38) is nonempty, see [5]. For any positive M we define a type of structure on sub- and super-derivatives as follows.

Definition 3.1. Let $M > 0$,

$$\underline{f}'_M(x;\cdot) = \inf_{Df(x)\in\mathcal{D}_M f(x)} \delta^*(\cdot\,|\,(\underline{\partial}+\overline{\partial})f(x)) + \delta^*(\cdot\,|\,-S) \tag{40}$$

$$\overline{f}'_M(x;\cdot) = \inf_{Df(x)\in\mathcal{D}_M f(x)} \delta^*(\cdot\,|\,(\overline{\partial}-\overline{\partial})f(x)) + \delta^*(\cdot\,|\,S) \tag{41}$$

where

$$\mathcal{D}_M f(x) = \{Df(x)\in\mathcal{D}f(x)\,|\,\underline{\partial}f(x)\cup\overline{\partial}f(x)\subseteq B(0,M)\}$$

$\underline{f}'_M(x;d)$ is called the subderivative of f at x in d ith respect to M, $\overline{f}'_M(x;d)$ is called the superderivative of f at x in d with respect to M, in short, $\underline{f}'_M(x;\cdot)$ the M-subderivative of f at x and $\overline{f}'_M(x;\cdot)$ the M-superderivative of f at x. Obviously, one has

$$f'(x;d) = \underline{f}'_M(x;d) - \overline{f}'_M(x;d)$$

Maybe, the problem on the relationship between $\lim_{M\to\infty}\underline{f}'_M(x;\cdot)$ and $\underline{f}'(x;\cdot)$, or $\lim_{M\to\infty}\overline{f}'_M(x;\cdot)$ and $\overline{f}'(x;\cdot)$ would be quite interesting, we will discuss this issue latter.

Theorem 3.2. *$\underline{f}'_M(x;\cdot)$ and $\overline{f}'_M(x;\cdot)$ are positively homogenous and continuous in directions.*

Proof. Given $Df(x)\in\mathcal{D}_M f(x)$, one has $\underline{f}'_M(x;\cdot)\geq 0$ and $\overline{f}'_M(x;\cdot)\geq 0$ because of (40) and (41). The positive homogenousness of $\underline{f}'_M(x;\cdot)$ and $\overline{f}'_M(x;\cdot)$ is obvious and we only need to prove the continuity of them with respect to direction. Now we prove the continuity of $\underline{f}'_M(x;\cdot)$ and the continuity property of $\overline{f}'_M(x;\cdot)$ can be demonstrated in a similar way. For simplicity, we denote $M^+(Df(x)) = (\underline{\partial}+\overline{\partial})f(x)$ and $M^-(Df(x)) = (\overline{\partial}-\overline{\partial})f(x)$. Thus $\underline{f}'_M(x;\cdot)$ can be rewritten as

$$\underline{f}'_M(x;\cdot) = \inf_{Df(x)\in\mathcal{D}_M f(x)} \delta^*(\cdot\,|\,M^+(Df(x))) + \delta^*(\cdot\,|\,-S) \tag{42}$$

We only need to prove the continuity of the first term, denoted by $\xi(\cdot)$, of the right-hand side of (42). Given $d \in \mathbf{R}^n$, take

$$D_i f(x) \in \mathcal{D}_M f(x), \quad i = 1, 2, \ldots$$

$$\epsilon_i > 0, \quad i = 1, 2, \ldots$$

$$\epsilon_i \downarrow 0, \quad i \to \infty$$

such that

$$\xi(d) > \delta^*(d \,|\, M^+(D_i f(x))) - \epsilon_i$$

Thus, for any i, it holds that

$$\underline{f}'_M(x; d+q) \leq \delta^*(d+q \,|\, M^+(D_i f(x)))$$

and hence

$$\xi(d+q) - \xi(d) \leq \delta^*(q \,|\, M^+(D_i f(x))) + \epsilon_i$$

Since

$$\delta^*(d \,|\, M^+(Df(x))) \leq \delta^*(d+q \,|\, M^+(Df(x))) + \delta^*(-q \,|\, M^+(Df(x)))$$

it follows that

$$\begin{aligned}
\xi(d) - \xi(d+q) \;\leq\; & \inf_{Df(x)\in\mathcal{D}_M f(x)} \delta^*(d \,|\, M^+(Df(x))) - \inf_{Df(x)\in\mathcal{D}_M f(x)} [\delta^*(d \,|\, M^+(Df(x))) - \\
& -\delta^*(-q \,|\, M^+(Df(x)))] \\
\leq\; & \inf_{Df(x)\in\mathcal{D}_M f(x)} \delta^*(d \,|\, M^+(Df(x))) - \inf_{Df(x)\in\mathcal{D}_M f(x)} [\delta^*(d \,|\, M^+(Df(x))) - \\
& - \sup_{Df(x)\in\mathcal{D}_M f(x)} \|\delta^*(\cdot \,|\, M^+(Df(x)))\| \|q\|] \\
\leq\; & \sup_{Df(x)\in\mathcal{D}_M f(x)} \|\delta^*(\cdot \,|\, M^+(Df(x)))\| \|q\| \\
\leq\; & 2M\|q\|
\end{aligned}$$

Thus

$$\begin{aligned}
|\xi(d+q) - \xi(d)| &\leq \max\{\delta^*(q \,|\, M^+(D_i f(x))) + \epsilon_i, 2M\|q\|\} \\
&\leq 2M\|q\| + \epsilon_i
\end{aligned}$$

holds for every $i \geq 1$ and in turn,

$$|\xi(d+q) - \xi(d)| \leq 2M\|q\|, \quad \forall\, q \in \mathbf{R}^n$$

which implies the Lipschitz continuity of $\underline{f}'_M(x; \cdot)$ with respect to direction. The proof is completed. □

Corollary 3.3. *$\underline{f}'_M(x; \cdot)$ and $\overline{f}'_M(x; \cdot)$ are Lipschitz continuous in direction.* □

Taking a monotonic sequence $\{M_i\}_1^\infty \uparrow \infty$, one obtains a monotonically decreasing sequence $\{\underline{f}'_{M_i}(x;\cdot)\}_1^\infty \downarrow$, so does $\{\overline{f}'_{M_i}(x;\cdot)\}_1^\infty \downarrow$. It follows the monotonicity of the sequences that the limit $\lim_{i\to\infty} \underline{f}'_{M_i}(x;\cdot)$ exists. It is obvious that for any fixed $h \in \mathbf{R}^n$, $\underline{f}'_{M_i}(x;h) \geq \underline{f}'(x;h)$ for any i and hence one has $\underline{f}'_{M_i}(x;h) > \underline{f}'(x;h) - \epsilon$ for any $\epsilon > 0$. On the other hand, for the fixed h and any $\epsilon > 0$ there exist sequences $\{M^+(D_i f(x)) - S\}_1^\infty$ and $\{M_i > 0\}_1^\infty \uparrow \infty$, and an i_0 such that

$$[\underline{\partial}_i, \overline{\partial}_i] f(x) \in \mathcal{D}_{M_i} f(x), \quad i \geq i_0 \tag{42}$$

$$\{\delta^*(h \,|\, ((\underline{\partial}_i + \overline{\partial}_i) f(x) - S)\}_1^\infty \downarrow \underline{f}'(x;h) \tag{43}$$

$$\delta^*(h \,|\, ((\underline{\partial}_i + \overline{\partial}_i) f(x) - S) < \underline{f}'(x;h) + \epsilon, \quad i \geq i_0 \tag{44}$$

Since $\underline{f}'_{M_i}(x;h) \leq \delta^*(h \,|\, (\underline{\partial}_i + \overline{\partial}_i) f(x) - S)$, it follows that

$$\underline{f}'_{M_i}(x;h) < \underline{f}'(x;h) + \epsilon, \quad i \geq i_0 \tag{45}$$

Therefore, one obtains

$$|\underline{f}'_{M_i}(x;h) - \underline{f}'(x;h)| < \epsilon, \quad i \geq i_0 \tag{46}$$

which leads to

$$\lim_{M\to\infty} \underline{f}'_M(x;h) = \underline{f}'(x;h), \quad \forall\, h \in \mathbf{R}^n \tag{47}$$

in terms of the monotonicity of $\underline{f}'_M(x;h)$ with respect to $M > 0$, i.e., $\underline{F}'_M$ converges to $\underline{f}'$ in pointwise. The same statement can be used to $\overline{f}'_M(x;h)$ and we have

$$\lim_{M\to\infty} \overline{f}'_M(x;h) = \overline{f}'(x;h), \quad \forall\, h \in \mathbf{R}^n \tag{48}$$

Summarizing the words given above, we have the following theorem.

Theorem 3.4. *$\underline{f}'_M$ and $\overline{f}'_M$ converge to $\underline{f}'$ and $\overline{f}'$ with respect to $M > 0$, respectively.* □

Theorem 3.4 has pointed out that $\underline{f}'_M(x;\cdot)$ and $\overline{f}'_M(x;\cdot)$ are continuous for every fixed M. If $\underline{f}'_M(x;\cdot)$ ($\overline{f}'_M(x;\cdot)$) is continuous uniformly to $M \in [c,\infty)$, $c > 0$ at a point, say h, then $\underline{f}'(x;\cdot)$ ($\overline{f}'(x;\cdot)$) is continuous at h.

Let Ω be a set of $\mathbf{R}^n$, $0 \in \Omega$, the function $|\cdot|_\Omega$ be defined by

$$|x|_\Omega = \inf\{\lambda > 0 \,|\, x \in \lambda\Omega\} \tag{49}$$

(define $\inf = +\infty$) is called the (Minkowski) function of set Ω. The following lemma, due to Rubinov and Yagubov (1986), plays a very important role in the sequel discussion.

Lemma 3.5. *Let ϕ be a function defined on $\mathbf{R}^n$. The following propositions are then equivalent:*

(a) *the function ϕ is positively homogeneous, nonnegative and continuous;*

(b) *ϕ coincides with the gauge of a star-shaped set Ω, where $\Omega = \{x \,|\, \phi(x) \leq 1\}$.*

Theorem 3.6. *$\underline{f}'_M(x;\cdot)$ and $\overline{f}'_M(x;\cdot)$ can be expressed as*

$$\underline{f}'_M(x;\cdot) = |\cdot|_{\underline{\Omega}_M}, \quad \overline{f}'_M(x;\cdot) = |\cdot|_{\overline{\Omega}_M} \tag{50}$$

where

$$\underline{\Omega}_M = cl \bigcup_{Df(x)\in\mathcal{D}_M f(x)} (M^+(Df(x)) - S)^\circ, \quad \overline{\Omega}_M = cl \bigcup_{Df(x)\in\mathcal{D}_M f(x)} (M^-(Df(x)) + S)^\circ \tag{51}$$

where polar operation A° of A is defined by $A^\circ = \{y \in \mathbf{R}^n \mid \delta^*(y \mid A) \leq 1\}$.

Proof. We only prove the first equality of (50) and the other equality can be obtained in a similar way. Obviously, $M^+(Df(x)) - S$ is closed and contains the origin for all $Df(x) \in \mathcal{D}f(x)$. In consequence, the support function $\delta^*(\cdot \mid M^+(Df(x)) - S)$ is the gauge function of $(M^+(Df(x)) - S)^\circ$ in terms of [13], i.e.,

$$\delta^*(\cdot \mid M^+(Df(x)) - S) = |\cdot|_{(M^+(Df(x))-S)^\circ}$$

According to Th. 3.2, $\underline{f}'_M(x;\cdot)$ is nonnegative, positively homogeneous and continuous in direction. By virtue of Lemma 3.3 and Proposition 2 of Rubinov & Yagubov (1986), there exists a star-shaped set $\underline{\Omega}_M$ such that

$$\inf_{Df(x)\in\mathcal{D}_M f(x)} |\cdot|_{(M^+(Df(x))-S)^\circ} = |\cdot|_{\underline{\Omega}_M}$$

where $\underline{\Omega}_M$ is defined by (51). This shows that the first equality of (50) is correct. The proof is completed. □

The pair $[\underline{\Omega}_M, \overline{\Omega}_M]$ of sets given in the theorem above is called a star kernel of f at x and $\underline{\Omega}_M$, $\overline{\Omega}_M$ are called sub-star kernel and super-star kernel, respectively. If C_1 and C_2 is defined by

$$\bigcup_{0\leq\lambda\leq 1} \{(1-\lambda)C_1 \cap \lambda C_2\} \tag{52}$$

denoted by $C_1\#C_2$, see [13] or $C_1 \oplus C_2$, see [15]. The symbols "#" and "$\oplus$" are regarded as the same in this paper. Since if $0 \in C_1 \cap C_2$, it holds

$$(C_1 + C_2)^\circ = C_1^\circ \# C_2^\circ \tag{53}$$

in terms of Rubinov et al. (1986). We have the following corollary.

Corollary 3.7. *We have the following equality*

$$\overline{\Omega}_M = cl \bigcup_{Df(x)\in\mathcal{D}_M f(x)} ((M^+(Df(x)))^\circ \# S^\circ) \tag{54}$$

It is easy to be seen that $\underline{\Omega}_{M_1} \subseteq \underline{\Omega}_{M_2}$ ($\overline{\Omega}_{M_1} \subseteq \overline{\Omega}_{M_2}$) if $M_1 \leq M_2$, in other words, $\underline{\Omega}_M \subseteq \underline{\Omega}_M$ are monotonically increasing as M is increasing. Note that

$$M^+(Df(x)) - S \subseteq \underline{S}_M - S, \quad M^-(Df(x)) + S \subseteq \overline{S}_M + S \tag{55}$$

for any $Df(x) \in \mathcal{D}_M f(x)$ and $M > 0$ such that $\mathcal{D}_M f(x) \neq \emptyset$, where

$$\underline{S}_M = \bigcap_{Df(x)\in\mathcal{D}_M f(x)} M^+(Df(x)), \quad \overline{S}_M = \bigcap_{Df(x)\in\mathcal{D}_M f(x)} M^-(Df(x))$$

Therefore, for M large enough such that $\mathcal{D}f(x) \neq \emptyset$, one has that $\underline{\Omega}_M$ and $\overline{\Omega}_M$ are bounded above by $(\underline{S} + S)^\circ$ and $(\overline{S} + S)^\circ$, respectively, in the preordering relationship defined by inclusion relationship.
Let $f-1$ and $f-2$ be quasidifferentiable on $\mathbf{R}^n$. Suppose $\mathcal{D}_M f_1(x) \neq \emptyset$ and $\mathcal{D}_M f_2(x) \neq \emptyset$. Define

$$\underline{\Omega}_M(f_1+f_2)(x) = \text{cl} \bigcup_{\substack{Df_1(x)\in\mathcal{D}_M f_1(x)\\ Df_2(x)\in\mathcal{D}_M f_2(x)}} ((M^+(Df_1(x)+Df_2(x)) - S_1 - S_2)^\circ \tag{56}$$

where

$$S_1 = \text{co}\{\{0\}, \bigcap_{Df_1(x)\in\mathcal{D}f(x)} M^+(Df_1(x))\} \quad S_2 = \text{co}\{\{0\}, \bigcap_{Df_2(x)\in\mathcal{D}f(x)} M^+(Df_2(x))\}$$

and $S(f_1(x)$ and $S(f_2(x))$ are used from time to time instead of S_1 and S_2 when it is better to be specified more clearly. According to the definition (56), one has that

$$\begin{aligned}
\underline{\Omega}_M(f_1+f_2)(x) &= \text{cl} \bigcup_{\substack{Df_1(x)\in\mathcal{D}_M f_1(x)\\ Df_2(x)\in\mathcal{D}_M f_2(x)}} (M^+(Df_1(x)) - S_1)^\circ \# (M^+(Df_2(x)) - S_2)^\circ \\
&= \text{cl} \bigcup_{\substack{Df_1(x)\in\mathcal{D}_M f_1(x)\\ Df_2(x)\in\mathcal{D}_M f_2(x)}} \bigcup_{0\le\alpha\le 1} [(1-\alpha)(M^+(Df_1(x)) - S_1)^\circ \cap \\
&\qquad \cap \alpha(M^+(Df_2(x)) - S_2)^\circ]
\end{aligned}$$

It follows from the definition of $\underline{\Omega}_M(\cdot)$ and the operation "#" (or " $\oplus$ " in the sense of Rubinov & Yagubov (1986)) that

$$\underline{\Omega}_M(f_1+f_2)(x) = \underline{\Omega}_M(f_1(x)) \# \underline{\Omega}_M(f_2(x)) \tag{57}$$

Likewise, it also can be proved that

$$\overline{\Omega}_M(f_1+f_2)(x) = \overline{\Omega}_M(f_1(x)) \# \overline{\Omega}_M(f_2(x)) \tag{58}$$

For f multiplied bt a scalar, say $\alpha \in \mathbf{R}^1$, we have that

$$\Omega_M(\alpha f)(x) = |\alpha| \Omega_M((\text{sign}\alpha) f(x) \tag{59}$$

where $\Omega_M = [\underline{\Omega}_M, \overline{\Omega}_M]$, and $\alpha\Omega_M$ is defined by

$$\alpha\Omega_M = \begin{cases} [\alpha\underline{\Omega}_M, \alpha\overline{\Omega}_M], & \alpha \ge 0 \\ [|\alpha|\overline{\Omega}_M, |\alpha|\underline{\Omega}_M], & \alpha < 0 \end{cases} \tag{60}$$

Formulae (57)-(60) can be used to deduce operations $\Omega_M(f_1 f_2), \Omega_M(f_2/f_1)$ and $\Omega_M(\max_{i\in I} f_i(x))$, where Ω_M is understood as the same as mentioned above. We list some of these operations without demonstration.

$$\Omega_M(f_1 f_2)(x) = f_1(x)\Omega_M(f_2(x))\# f_2(x)\Omega_M(f_1(x)) \tag{61}$$

$$\Omega_M(f_1/f_2)(x) = \frac{f_2(x)\Omega_M(f_1(x))\#(-f_1(x))\Omega_M(f_2(x))}{f_2^2(x)}, \quad f_2(x) \neq 0 \tag{62}$$

$$\Omega_M(\max_{i\in I} f_i(x)) = [\bigcap_{i\in I(x)} \underline{\Omega}_M(f_i(x))(\sum_{j\in I(x)\setminus\{i\}} \#)\overline{\Omega}_M(f_j(x)), (\sum_{i\in I(x)} \#)\overline{\Omega}_M(f_i(x))] \tag{63}$$

where $I(x) = \{i \in I \mid f_i = \max_{j\in I} f_j(x)\}$. From the definitions of $\mathcal{D}_M f(x)$ and $M^+(Df(x))$ and $M^-(Df(x))$ it follows that the limits

$$\underline{\Omega} = \lim_{M\to\infty} \underline{\Omega}_M = \lim_{M\to\infty} \mathrm{cl} \bigcup_{Df(x)\in\mathcal{D}_M f(x)} (M^+(Df(x)) - S)^\circ \tag{64}$$

$$\overline{\Omega} = \lim_{M\to\infty} \overline{\Omega}_M = \lim_{M\to\infty} \mathrm{cl} \bigcup_{Df(x)\in\mathcal{D}_M f(x)} (M^-(Df(x)) + S)^\circ \tag{65}$$

exist, where the limit is defined by

$$\lim_{M\to\infty} A_M = \liminf_{M\to\infty} A_M = \limsup_{M\to\infty} A_M \tag{66}$$

and the definitions of liminf and limsup are defined by the way similar to the ones given in [14].
Taking the limit to two sides in (57), one has

$$\underline{\Omega}(f_1 + f_2)(x) = \underline{\Omega} f_1(x)\#\underline{\Omega} f_2(x) \tag{67}$$

Likewise one has that

$$\overline{\Omega}(f_1 + f_2)(x) = \overline{\Omega} f_1(x)\#\overline{\Omega} f_2(x) \tag{68}$$

$$\Omega(\alpha f)(x) = |\alpha|(\mathrm{sign}\alpha) f)(x) \tag{69}$$

$$\Omega(f_1 f_2)(x) = f_1(x)\Omega(f_2(x))\# f_2(x)\Omega(f_1(x)) \tag{70}$$

$$\Omega(f_1/f_2)(x) = \frac{f_2(x)\Omega(f_1(x))\#(-f_1(x))\Omega(f_2(x))}{f_2^2(x)} \tag{71}$$

$$\Omega(\max_{i\in I} f_i(x)) = [\bigcap_{i\in I(x)} \underline{\Omega}(f_i(x))(\sum_{j\in I(x)\setminus\{i\}} \#)\overline{\Omega}(f_j(x)), (\sum_{i\in I(x)} \#)\overline{\Omega}(f_i(x))] \tag{72}$$

where $\Omega = [\underline{\Omega}, \overline{\Omega}]$ and the scalar multiplication is defined by (60).
According to [15], we have the following theorem.

Theorem 3.8. *Assume $[\underline{\Omega}_M, \overline{\Omega}_M]$ is a star kernel of f at x, A and B are star shaped sets. Then $f'(x;\cdot)$ can be expressed*

$$f'(x;\cdot) = |\cdot|_A - |\cdot|_B \tag{73}$$

if and only if

$$A\#\overline{\Omega}_M = B\#\underline{\Omega}_M \tag{74}$$

Let f be convex in $\mathbf{R}^n$. If $\partial f(x) \neq \emptyset$, then

$$\underline{\Omega}_M = (\partial f(x) - S)^\circ, \quad \overline{\Omega}_M = S^\circ$$

where $S = \text{co}\{\{0\}, \partial f(x)\}$ and $M > 0$ such that $\partial f(x) \in MB(0,1)$. Obviously S is the umbra of $\partial f(x)$ with respect to the origin. In fact, one has that

$$\begin{aligned} \partial f(x) - S &\subset M^+(Df(x)) - S \\ S &\subset M^-(Df(x)) + S, \quad \forall\, Df(x) \in \mathcal{D}f(x) \end{aligned}$$

and hence

$$\bigcup_{Df(x)\in\mathcal{D}f(x)} (M^+(Df(x)) - S)^\circ \subset (\partial f(x) - S)^\circ = \underline{\Omega}_M$$

$$\bigcup_{Df(x)\in\mathcal{D}f(x)} (M^-(Df(x)) + S)^\circ \subset S^\circ = \overline{\Omega}_M$$

Suppose f is concave and $\overline{\partial}_0 f(x) \neq \emptyset$. It has been proved that $\underline{S} = \overline{\partial}_0 f(x)$ and

$$[\overline{\partial}_0 f(x), \overline{\partial}_0 f(x) - \overline{\partial}_0 f(x)] \in \mathcal{D}f(x) \tag{75}$$

and

$$\overline{\partial}_0 f(x) \subset M^+(Df(x)), \quad \forall\, Df(x) \in \mathcal{D}f(x) \tag{76}$$

and hence

$$\begin{aligned} &\overline{\partial}_0 f(x) - S \subset M^+(Df(x)) - S \\ &(\overline{\partial}_0 - \overline{\partial}_0) f(x) + S \subset M^-(Df(x)) + S \end{aligned} \tag{77}$$

Therefore,

$$\begin{aligned} &\textstyle\bigcup_{Df(x)\in\mathcal{D}f(x)} (M^+(Df(x)) - S)^\circ \subset (\overline{\partial}_0 f(x) - S)^\circ \\ &\textstyle\bigcup_{Df(x)\in\mathcal{D}f(x)} (M^-(Df(x)) + S)^\circ \subset ((\overline{\partial}_0 - \overline{\partial}_0) f(x) + S)^\circ \end{aligned} \tag{78}$$

It follows from (78) that for a concave function the sub- and super-star kernells at x could be given by

$$\underline{\Omega}_M = (\overline{\partial}_0 f(x) - S)^\circ, \quad \overline{\Omega}_M = ((\overline{\partial}_0 - \overline{\partial}_0) f(x) + S)^\circ \tag{79}$$

(where $M > 0$ satisfying $\mathcal{D}_M f(x) \neq \emptyset$) which are both convex and star-shaped.

4. Star Differentials

It has been seen from the last section that for any $Df(x) \in \mathcal{D}f(x)$, one has that

$$(0,0) \in [M^+(Df(x)) - S, M^-(Df(x)) + S] \in \mathcal{D}f(x) \tag{80}$$

This leads to considering a special subclass of quasidifferntials that contain the origin in both subdifferentials and superdifferentials. We denote by $\mathcal{D}^0 f(x)$ this class containing the origin.

Given a positive scalar $M > 0$. We might define $\mathcal{D}^0_M f$, $\underline{f}'_M, \overline{f}'_M, \underline{S}_M(f)$ and $\overline{S}_M(f)$ as follows.

$$\mathcal{D}^0_M f(x) = \{Df(x) \in \mathcal{D}f(x) \,|\, 0 \in \underline{\partial} f(x) \cap \overline{\partial} f(x),\, \underline{\partial} f(x) \cup \overline{\partial} f(x) \subseteq B(0,M)\} \tag{81}$$

$$\underline{f}'_M(x;\cdot) = \inf_{Df(x)\in\mathcal{D}^0_M f(x)} \delta^*(\cdot\,|\,\underline{\partial} f(x)), \quad \overline{f}'_M(x;\cdot) = \inf_{Df(x)\in\mathcal{D}^0_M f(x)} \delta^*(\cdot\,|\,\overline{\partial} f(x)) \tag{82}$$

$$\underline{S}_M(f(x)) = \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x), \quad \overline{S}_M(f(x)) = \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \overline{\partial} f(x) \tag{83}$$

Similar to the proof of Th. 3.2, we can prove both $\underline{f}'_M(x;\cdot)$ and $\overline{f}'_M(x;\cdot)$ are nonnegative-valued, positively homogeneous and continuous. $\underline{f}'_M(x;\cdot)$ and $\overline{f}'_M(x;\cdot)$ can be expressed as follows

$$\underline{f}'_M(x;\cdot) = |\cdot|_{\underline{\Omega}_M}, \quad \overline{f}'_M(x;\cdot) = |\cdot|_{\overline{\Omega}_M} \tag{84}$$

where

$$\underline{\Omega}_M = \mathrm{cl} \bigcup_{Df(x)\in\mathcal{D}^0_M f(x)} (\underline{\partial} f(x))^\circ, \quad \overline{\Omega}_M = \mathrm{cl} \bigcup_{Df(x)\in\mathcal{D}^0_M f(x)} (-\overline{\partial} f(x))^\circ \tag{85}$$

Lemma 4.1. *$\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$ holds iff $[\underline{S}_M(f(x)), \overline{S}_M(f(x))] \in \mathcal{D}^0_M f(x)$.*

Proof. Since $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$, there exists a nonempty convex set $B \subset B$ such that $[\underline{S}_M(f(x)), N] \in \mathcal{D}^0_M f(x)$. Thus one has

$$f'(x;\cdot) = \delta^*(\cdot\,|\,\underline{S}_M(f(x))) - \delta^*(\cdot\,|\,-B)$$

Since

$$f'(x;\cdot) = \underline{f}'_M(x;\cdot) - \overline{f}'_M(x;\cdot)$$

one has, for each $Df(x) \in \mathcal{D}^0_M f(x)$, that

$$\underline{S}_M(f(x)) \subseteq \underline{\partial} f(x), \quad \underline{S}_M(f(x)) - \overline{\partial} f(x) = \underline{\partial} f(x) - B$$

which, by Minkowski duality, implies

$$\delta^*(\cdot\,|\,-\overline{\partial} f(x)) \geq \delta^*(\cdot\,|\,-B)$$

and thus $B \subseteq \overline{\partial} f(x)$ and inturn,

$$B \subseteq \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \overline{\partial} f(x)$$

Thus we have $B = \overline{S}_M(f(x))$. The proof is completed. □

The above lemma implies that if $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$ then $[\underline{S}_M(f(x)), \overline{S}_M(f(x)]$ is a smallest quasidifferential in the set $\mathcal{D}^0_M f(x)$.

Corollary 4.2. *If $\overline{S}_M(f(x)) \in \overline{\mathcal{D}}^0_M f(x)$, then $[\underline{S}_M(f(x)), \overline{S}_M(f(x)]$ is a smallest quasi-differential in the set $\mathcal{D}^0_M f(x)$.* □

Theorem 4.3. *$\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$ if and only if one of the following conditions holds*
(i) For every $d \in \mathbf{R}^n$, the following inequalities are valid

$$\begin{aligned} &\underline{S}_M(f(x))(d) \bigcap \textstyle\bigcup_{\underline{\partial} f(x)\in\underline{\mathcal{D}}^0_M f(x)} \underline{\partial} f(x)(d) \neq \emptyset \\ &(-\overline{S}_M(f(x)))(d) \bigcap \textstyle\bigcup_{\overline{\partial} f(x)\in\overline{\mathcal{D}}^0_M f(x)} (-\overline{\partial} f(x))(d) \neq \emptyset \end{aligned} \tag{86}$$

where $A(d) = \partial\delta^*(d\,|\,A)$, A is a nonempty convex compact set of $\mathbf{R}^n$.
(ii) For a set $T \subseteq \mathbf{R}^n$ satisfying

$$\bigcup_{u\in T} N(u, \underline{S}_M(f(x))) = \mathbf{R}^n, \quad \bigcup_{u\in T} N(u, \overline{S}_M(f(x))) = \mathbf{R}^n \tag{87}$$

it follows, for each $u \in T$, that

$$\begin{aligned} N(u, \underline{S}_M(f(x))) &= \textstyle\bigcup_{\underline{\partial} f(x)\in \underline{\mathcal{D}}^0_M f(x)} N(u, \underline{\partial} f(x)) \\ N(u, -\overline{S}_M(f(x))) &= \textstyle\bigcup_{\overline{\partial} f(x)\in \overline{\mathcal{D}}^0_M f(x)} N(u, -\overline{\partial} f(x)) \end{aligned} \tag{88}$$

Proof. It is easy to be seen that $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$ implies either of (86) or (88). We first demonstrate that if (88) is valid then $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$. For any $d \in \mathbf{R}^n$ there exist $u \in \underline{S}_M(f(x))(d)$ and $\underline{\partial} f(x) \in \underline{\mathcal{D}}^0_M f(x)$ such that $u \in \underline{\partial} f(x)(d)$. Thus

$$\underline{f}'_M(x; d) \geq \delta^*(d\,|\,\underline{S}_M(f(x))) =< u, d >= \delta^*(d\,|\,\underline{\partial} f(x)) \geq \underline{f}'_M(x; d)$$

Likewise, there exist $v \in (-\overline{S}_M(f(x)))(d)$ and $\overline{\partial} f(x) \in \overline{\mathcal{D}}^0_M f(x)$ such that $v \in (-\overline{\partial} f(x))(d)$ and hence

$$\overline{f}'_M(x; d) \geq \delta^*(d\,|-\overline{S}_M(f(x))) =< v, d >= \delta^*(d\,|-\overline{\partial} f(x)) \geq \overline{f}'_M(x; d)$$

The above lines imply that $\underline{f}'_M(x; d) = \delta^*(d\,|\,\underline{S}_M(f(x)))$ and $\overline{f}'_M(x; d) = \delta^*(d\,|-\overline{S}_M(f(x)))$. In view of the convex compactness of $\underline{S}_M(f(x))$ and $\overline{S}_M(f(x))$, one has $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$.
We now turn our attention to proving that if (ii) is valid then $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$. For every $d \in \mathbf{R}^n$ there exists $u \in T$ such that $d \in N(u, \underline{S}_M(f(x)))$ and from (ii), there exists $\underline{\partial} f(x) \in \underline{\mathcal{D}}^0_M f(x)$ such that $d \in N(u, \underline{\partial} f(x))$. Then one has

$$\underline{f}'_M(x; d) \geq \delta^*(d\,|\,\underline{S}_M(f(x))) =< u, d >= \delta^*(d\,|\,\underline{\partial} f(x)) \geq \underline{f}'_M(x; d)$$

Likewise, there exist $v \in T$ such that $d \in N(v, -\overline{S}_M(f(x)))$ and $\overline{\partial} f(x) \in \overline{\mathcal{D}}^0_M f(x)$ such that $d \in N(v, -\overline{\partial} f(x))$ and hence

$$\overline{f}'_M(x; d) \geq \delta^*(d\,|-\overline{S}_M(f(x))) =< v, d >= \delta^*(d\,|-\overline{\partial} f(x)) \geq \overline{f}'_M(x; d)$$

Thus we have that $\underline{f}'_M(x; d) = \delta^*(d\,|\,\underline{S}_M(f(x)))$ and $\overline{f}'_M(x; d) = \delta^*(d\,|-\overline{S}_M(f(x)))$. Since $\underline{S}_M(f(x))$ and $\overline{S}_M(f(x))$ are both nonempty convex and compact, one has $\underline{S}_M(f(x)) \in \underline{\mathcal{D}}^0_M f(x)$. The proof is completed. □

5. Star Differentiable Functions

In this section, a differential of a function at a point, where the function is directionally differentiable at the point and the directional derivative is continuous in direction, is investigated based on results due to Rubinov and Yagubov (1986). Suppose $f : \mathbf{R}^n \longrightarrow \mathbf{R}^1$ is directionally differentiable at x and $f'(x; \cdot)$ is continuous in direction. Then there exists a pair of star-shaped sets U and V, (U, V) such that

$$f'(x; d) = |d|_U - |d|_V, \quad \forall d \in \mathbf{R}^n \tag{89}$$

see Rubinov and Yagubov (1986), where $|\cdot|_\Omega$ denotes the gauge function of Ω (or with respect to Ω) and Ω is a star-shaped set in $\mathbf{R}^n$. Such a pair may be called a star differential, in the sense of Rubinov and Yagubov, of f at x, denoted by $df(x) = [\underline{d}f(x), \overline{d}f(x)]$. It is not unique similar to the case for quasidifferentiable functions. Some rules for algebraic operations over functions and corresponding differentials are also given in Rubinov and Yagubov (1986) that can be listed as follows:

a) The following equalities are valid

$$\underline{d}(f_1 + f_2)(x) = \underline{d}f_1(x)\#\underline{d}f_2(x)$$

$$\overline{d}(f_1 + f_2)(x) = \overline{d}f_1(x)\#\overline{d}f_2(x)$$

b) We have the following formula

$$d(f_1 f_2)(x) = f_1(x) \odot df_2(x)\#f_2(x) \odot df_1(x)$$

where $d(\cdot) = [\underline{d}(\cdot), \overline{d}(\cdot)]$ and

$$[A_1, B_1]\#[A_2, B_2] = [A_1\#A_2, B_1\#B_2] \tag{90}$$

$$\alpha[A, B] = \begin{cases} [A/\alpha, B/\alpha], & \alpha > 0 \\ [\mathbf{R}^n, \mathbf{R}^n], & \alpha = 0 \\ [B/|\alpha|, A/|\alpha|], & \alpha < 0 \end{cases} \tag{91}$$

c) For star differentiable functions $f_1, \ldots, f_m$, $f = \max_{1\le i\le m} f_i$ is also star differentiable and its star differential $df(x)$ can be expressed as

$$df(x) = [\bigcap_{k=1}^{m}(\underline{d}f_k(x)\#(\sum_{i\ne k}\#)\overline{d}f_i(x)), (\sum_{i=1}^{m}\#)\overline{d}f_i(x)] \tag{92}$$

d) If $f_i, i = 1, \ldots, m$ are star differentiable at x, then $\phi(x) = \min_{1\le i\le m} f_i(x)$ is also star differentiable at x, and

$$d\phi(x) = [(\sum_{i=1}^{m})\underline{d}f_i(x), \bigcap_{k=1}^{m}(\overline{d}f_k(x)\#(\sum_{i\ne k}\#)\underline{d}f_i(x)] \tag{93}$$

It is also pointed out that if f is quasidifferentiable at x then

$$\underline{d}f(x) = (\underline{\partial}f(x))^\circ, \quad \overline{d}f(x) = (-\overline{\partial}f(x))^\circ$$

for $[\underline{\partial}f(x), \overline{\partial}f(x)] \in \mathcal{D}^0 f(x)$, where $\mathcal{D}^0 f(x) = \{Df(x) \in \mathcal{D}f(x) \,|\, (0,0) \in Df(x)\}$. If there exists a pair of star-shaped sets U and V for a function f at point x such that (89) holds, then the function f might be said to be star differentiable at x. It is to be seen that if f at x is star differentiable then $f'(x;\cdot)$ is continuous in direction.

Let (U_1, V_1) and (U_2, V_2) are pairs of star-shaped sets. They are said to be equivalent iff

$$U_1\#V_2 = U_2\#V_1 \tag{94}$$

denoted by $(U_1, V_1) \sim (U_2, V_2)$, see [15]. Suppose f is star differentaible at x and let $\mathcal{D}_S f(x)$ denote the set of all star differentials of f at x. Then $(U_1, V_1), (U_2, V_2) \in \mathcal{D}_S f(x)$ if and only if

$$(U_1, V_1) \sim (U_2, V_2) \tag{95}$$

Suppose that f is directionally differentiable at x and $f'(x;\cdot)$ is continuous in direction. Define

$$f'_+(x;\cdot) = \max\{f'(x;\cdot),0\}, \quad f'_-(x;\cdot) = \max\{-f'(x;\cdot),0\} \tag{96}$$

then $f'_+(x;\cdot)$ and $f'_-(x;\cdot)$ are positively homogeneous and nonnegative in direction. Obviously

$$f'(x;\cdot) = f'_+(x;\cdot) - f'_-(x;\cdot) \tag{97}$$

Let

$$\Omega_+ = \{y \,|\, f'_+(x;y) \le 1\}, \quad \Omega_- = \{y \,|\, f'_-(x;y) \le 1\} \tag{98}$$

Theorem 5.1. $[\Omega_+,\Omega_-] \in \mathcal{D}_S f(x)$.

Proof. In view of Lemma 3.3 or Th.1 of [15], we have that

$$f'_+(x;\cdot) = |\cdot|_{\Omega_+}, \quad f'_-(x;\cdot) = |\cdot|_{\Omega_-}$$

and $f'(x;\cdot) = |\cdot|_{\Omega_+} - |\cdot|_{\Omega_-}$, i.e., $[\Omega_+,\Omega_-] \in \mathcal{D}_S f(x)$. □

It is reasonable to regard $[\Omega_+,\Omega_-]$ as the representative of $\mathcal{D}_S f(x)$. Obviously, it is determined uniquely. Since (97) is the smallest decopmosition of (89), it follows that the star differential given by (98) is the smallest (in the sense of anti-inclusion) one of f at x. □

Theorem 5.2. *If f is quasidifferentiable at x then $[\underline{\Omega}_M,\overline{\Omega}_M] \in \mathcal{D}_S f(x)$ where $[\underline{\Omega}_M,\overline{\Omega}_M]$ is defined by (51) or (85), and*

$$\underline{\Omega}_M \subseteq \Omega_+, \quad \overline{\Omega}_M \subseteq \Omega_- \tag{99}$$

Proof. Since $|\cdot|_{\underline{\Omega}_M} \ge f'(x;\cdot)$ and $|\cdot|_{\overline{\Omega}_M} \ge -f'(x;\cdot)$, we have $|\cdot|_{\underline{\Omega}_M} \ge f'_+(x;\cdot) = |\cdot|_{\Omega_+}$ and $|\cdot|_{\overline{\Omega}_M} \ge f'_-(x;\cdot) = |\cdot|_{\Omega_-}$. In view of the properties of gauge function, we obtain (99). □

6. On Penot Differentials

Assume that f is quasidifferentiable at x, define

$$\underline{\partial}^{\ge}_M f(x) = \{y \,|\, <y,z> \le \underline{f}'_M(x;z), \forall\, z \in \mathbf{R}^n\} \tag{100}$$

$$\overline{\partial}^{\ge}_M f(x) = \{y \,|\, <y,z> \le \overline{f}'_M(x;z), \forall\, z \in \mathbf{R}^n\} \tag{101}$$

where $\underline{f}'_M$ and $\overline{f}'_M$ are defined by (40) and (41) or (82). We call $\underline{\partial}^{\ge}_M f(x)$ and $\overline{\partial}^{\ge}_M f(x)$ the sub-Penot differential of f at x and the super-Penot differential of f at x, respectively.

Lemma 6.1. *$\underline{\partial}^{\ge}_M f(x)$ and $\overline{\partial}^{\ge}_M f(x)$ are nonempty convex compact sets containing $(0,0)$.* □

Lemma 6.2. *$\delta^*(\cdot\,|\,\underline{\partial}^{\ge}_M f(x))$ ($\delta^*(\cdot\,|\,\overline{\partial}^{\ge}_M f(x))$) is the greatest sublinear function majorized by $\underline{f}'_M(x;\cdot)$ ($\overline{f}'_M(x;\cdot)$).* □

Theorem 6.3. *For $\mathcal{D}^0_M f(x)$, one has that*

$$\underline{\partial}^{\ge}_M f(x) = \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x), \quad \overline{\partial}^{\ge}_M f(x) = \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \overline{\partial} f(x) \tag{102}$$

Proof. We only prove the first equality of (102). For each $Df(x) \in \mathcal{D}^0_M f(x)$, one has

$$\delta^*(\cdot\,|\,\underline{\partial} f(x)) \geq \delta^*(\cdot\,|\bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x))$$

i.e.,

$$\underline{f}'_M(x;\cdot) \geq \delta^*(\cdot\,|\bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x))$$

In view of Lemma 6.2, we have

$$\delta^*(\cdot\,|\,\underline{\partial}^{\geq}_M f(x)) \geq \delta^*(\cdot\,|\bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x))$$

and by Minkowski duality, one has

$$\underline{\partial}^{\geq}_M f(x) \supseteq \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x)$$

Since $\delta^*(\cdot\,|\,\underline{\partial} f(x)) \geq \delta^*(\cdot\,|\,\underline{\partial}^{\geq}_M f(x))$ for any $Df(x) \in \mathcal{D}^0_M f(x)$ we obtain $\underline{\partial} f(x)) \supseteq \underline{\partial}^{\geq}_M f(x))$ for any $Df(x) \in \mathcal{D}^0_M f(x)$. Thus $\underline{\partial}^{\geq}_M f(x) = \bigcap_{Df(x)\in\mathcal{D}^0_M f(x)} \underline{\partial} f(x)$. The proof is completed. □

Theorem 6.4. *For $\underline{\Omega}_M$ and $\overline{\Omega}_M$ defined by (51) or (85), it holds*

$$co\underline{\Omega}_M = (\underline{\partial}^{\geq}_M f(x))^\circ, \quad co\overline{\Omega}_M = (\overline{\partial}^{\geq}_M f(x))^\circ \tag{103}$$

Proof. We only prove the first equality of (103) when $\underline{\Omega}_M$ is defined by (51) and other equalities can be demonstrated by a similar way.
Since $\underline{f}'_M(x;\cdot) = |\cdot|_{\underline{\Omega}_M} \geq |\cdot|_{(\underline{\partial}^{\geq}_M f(x))^\circ}$, we have

$$\underline{\Omega}_M \subseteq (\underline{\partial}^{\geq}_M f(x))^\circ \tag{104}$$

Noting that $\mathrm{co}\underline{\Omega}_M$ is star-shaped and $\mathrm{co}\underline{\Omega}_M \supseteq \underline{\Omega}_M$, one has

$$|\cdot|_{\mathrm{co}\underline{\Omega}_M} \leq |\cdot|_{\underline{\Omega}_M} = \underline{f}'_M(x;\cdot)$$

In view of Lemma 6.2, one has the following inequalities

$$\delta^*(\cdot\,|\,(\mathrm{co}\underline{\Omega}_M)^\circ) \leq \delta^*(\cdot\,|\,\underline{\partial}^{\geq}_M f(x))$$

$$|\cdot|_{\mathrm{co}\underline{\Omega}_M} \leq \delta^*(\cdot\,|\,\underline{\partial}^{\geq}_M f(x)) = |\cdot|_{(\underline{\partial}^{\geq}_M f(x))^\circ}$$

Thus $\mathrm{co}\underline{\Omega}_M \supseteq (\underline{\partial}^{\geq}_M f(x))^\circ$. Combining this conclusion with (104), we obtain the first equality of (103). □

Defining

$$\partial^{\geq} f_+(x) = \{y \,|\, < y, z > \leq f'_+(x;z),\, \forall\, z \in \mathbf{R}^n\} \tag{105}$$

$$\partial^{\geq} f_-(x) = \{y \,|\, < y, z > \leq f'_-(x;z),\, \forall\, z \in \mathbf{R}^n\} \tag{106}$$

$$\underline{d} f(x) = \{y \,|\, f'_+(x;y) \leq 1\}, \quad \overline{d} f(x) = \{y \,|\, f'_-(x;y) \leq 1\} \tag{107}$$

we have the following proposition:

Proposition 6.5. *The following equalities are valid*

$$co\underline{d}f(x) = (\partial^{\geq} f_+(x))^\circ, \quad co\overline{d}f(x) = (\partial^{\geq} f_-(x))^\circ \tag{108}$$

□

For one dimensional case, a star shaped set is also convex, we have $\underline{\Omega}_M = (\underline{\partial}^{\geq}_M f(x))^\circ$, and $\underline{f}'_M(x;\cdot) = \delta^*(\cdot \,|\, \underline{\partial}^{\geq}_M f(x))$ and $\overline{f}'_M(x;\cdot) = \delta^*(\cdot \,|\, \overline{\partial}^{\geq}_M f(x))$. We obtain the same result as (12), also see [7].

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