

Non Maximal Cyclically Monotone Graphs and Construction of a Bipotential for the Coulomb's Dry Friction Law

Marius Buliga

*"Simion Stoilow" Institute of Mathematics of the Romanian Academy,
PO BOX 1-764, 014700 Bucharest, Romania
Marius.Buliga@imar.ro*

Géry de Saxcé

*Laboratoire de Mécanique de Lille, UMR CNRS 8107,
Université des Sciences et Technologies de Lille, Bâtiment Boussinesq,
Cité Scientifique, 59655 Villeneuve d'Ascq cedex, France
gergy.desaxce@univ-lille1.fr*

Claude Vallée

*Laboratoire de Mécanique des Solides, UMR CNRS 6610,
UFR SFA-SP2MI, Bd M. et P. Curie, téléport 2, BP 30179,
86962 Futuroscope-Chasseneuil cedex, France
vallee@lms.univ-poitiers.fr*

Received: March 14, 2008

Revised manuscript received: January 12, 2009

We show a surprising connexion between a property of the inf convolution of a family of convex lsc functions and the fact that the intersection of maximal cyclically monotone graphs is the critical set of a bipotential.

We then extend the results from [4] to bipotentials convex covers, generalizing the notion of a bi-implicitly convex lagrangian cover.

As an application we prove that the bipotential related to Coulomb's friction law is related to a specific bipotential convex cover with the property that any graph of the cover is non maximal cyclically monotone.

1991 Mathematics Subject Classification: 49J53, 49J52, 26B25

1. Introduction

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{R}$. We shall suppose that X, Y have topologies compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

To any convex and lsc function $\phi : X \rightarrow \bar{\mathbb{R}}$ we associate a function called **separable**

bipotential $b : X \times Y \rightarrow \bar{\mathbb{R}}$, defined by the formula:

$$b(x, y) = \phi(x) + \phi^*(y)$$

(for the general notion of a bipotential see Definition 2.4). Here the function $\phi^* : Y \rightarrow \bar{\mathbb{R}}$ is the Fenchel conjugate of ϕ , defined by the expression (3).

The function b is obviously bi-convex and lsc in each argument. By Fenchel inequality we have

$$b(x, y) \geq \langle x, y \rangle. \quad (1)$$

Also the following string of equivalences is true:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y \rangle \quad (2)$$

which is just another way of writing the well known string of equivalences

$$y \in \partial \phi(x) \iff x \in \partial \phi^*(y) \iff \phi(x) + \phi^*(y) = \langle x, y \rangle.$$

In mechanics subgradient inclusions $y \in \partial \phi(x)$ are related to **associated** constitutive laws [7]. There exist also **non-associated** constitutive laws which take the form $y \in \partial b(\cdot, y)(x)$ for functions b of two variables, convex and lsc in each argument and satisfying (1), (2), which are called bipotentials. Such an approach to non-associated constitutive laws has been first proposed in [13]. Examples of such non associated constitutive laws are: non-associated Drucker-Prager [15] and Cam-Clay models in soil mechanics [16], cyclic Plasticity ([14], [2]) and Viscoplasticity [8] of metals with non linear kinematical hardening rule, Lemaitre's damage law [1], the coaxial laws ([17], [20]).

Of special interest to us is the formulation in terms of bipotentials of the Coulomb's friction law [13], [14], [3], [5], [6], [9], [15], [18], [10].

In [4] we solved two key problems: (a) when the graph of a given multivalued operator can be expressed as the set of critical points of a bipotentials, and (b) a method of construction of a bipotential associated (in the sense of point (a)) to a multivalued, typically non monotone, operator. The main tool was the notion of **convex lagrangian cover** of the graph of the multivalued operator, and a related notion of implicit convexity of this cover.

The results of [4] apply only to bi-convex, bi-closed graphs (for short BB-graphs) admitting at least one convex lagrangian cover by **maximal cyclically monotone graphs**. This is a rather large class of graph of multivalued operators but important applications to the mechanics, such as the bipotential associated to contact with friction [13], are not in this class.

This paper is dedicated to the extension of the method presented in [4] to a more general class of BB-graphs. This is done in two steps. In the first step we prove Theorem 3.1, the main result of this paper. The result is that the intersection of two maximal cyclically monotone graphs is the critical set of a strong bipotential if and only if a condition formulated in terms of the inf convolution of a family of convex lsc functions is true. In the second step we extend the main result of [4] by replacing the notion of convex lagrangian cover with the one of a **bipotential convex cover** (Definition 4.2). In this way we are able to apply our results to the bipotential for the Coulomb's friction law.

2. Notations and Definitions

For any convex and closed set $A \subset X$, its indicator function, χ_A , is defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}$$

The indicator function is convex and lsc.

We use the notation: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Given a function $\phi : X \rightarrow \bar{\mathbb{R}}$, the conjugate $\phi^* : Y \rightarrow \bar{\mathbb{R}}$ is defined by:

$$\phi^*(y) = \sup \{ \langle y, x \rangle - \phi(x) \mid x \in X \}. \quad (3)$$

The conjugate is always convex and lsc.

We denote by $\Gamma(X)$ the class of convex and lsc functions $\phi : X \rightarrow \bar{\mathbb{R}}$. The class of convex and lsc functions $\phi : X \rightarrow \bar{\mathbb{R}}$ with non-empty epigraph is denoted by $\Gamma_0(X)$.

The effective domain of a function $\phi : X \rightarrow \bar{\mathbb{R}}$ is $\text{dom } \phi = \{x \in X : \phi(x) < +\infty\}$.

For X real locally convex topological vector space and $f, g : X \rightarrow \bar{\mathbb{R}}$ the inf-convolution of f and g at $x \in X$ is defined by:

$$f \square g(x) = \inf_{x_1+x_2=x} [f(x_1) + g(x_2)]. \quad (4)$$

The subgradient of a function $\phi : X \rightarrow \bar{\mathbb{R}}$ at a point $x \in X$ is the (possibly empty) set:

$$\partial\phi(x) = \{u \in Y \mid \forall z \in X \langle z - x, u \rangle \leq \phi(z) - \phi(x)\}.$$

In a similar way is defined the subgradient of a function $\psi : Y \rightarrow \bar{\mathbb{R}}$ in a point $y \in Y$, as the set:

$$\partial\psi(y) = \{v \in X \mid \forall w \in Y \langle v, w - y \rangle \leq \psi(w) - \psi(y)\}.$$

Definition 2.1 (The graph of a multivalued operator). $A : X \rightarrow 2^Y$ is the set

$$\text{Gr}(A) = \{(x, y) \in X \times Y \mid y \in A(x)\}.$$

Any subset $M \subset X \times Y$ is the graph of a operator $A : X \rightarrow 2^Y$. Associated to M is the multivalued operator

$$X \ni x \mapsto m(x) = \{y \in Y \mid (x, y) \in M\}.$$

The **dual** operator is given by

$$Y \ni y \mapsto m^*(y) = \{x \in X \mid (x, y) \in M\}.$$

The **domain** of the graph M (or the effective domain of the associated operator m) is the set $\text{dom}(M) = \{x \in X \mid m(x) \neq \emptyset\}$.

The **image** of the graph M is the set $\text{im}(M) = \{y \in Y \mid m^*(y) \neq \emptyset\}$.

For any $\phi \in \Gamma(X)$ we shall denote by $M(\phi)$ the graph:

$$M(\phi) = \{(x, y) \in X \times Y \mid \phi(x) + \phi^*(y) = \langle x, y \rangle\}. \quad (5)$$

The operator associated to the graph $M(\phi)$ is $\partial\phi$. The dual operator associated to $M(\phi)$ is $\partial\phi^*$ (the subgradient of the Legendre-Fenchel dual of ϕ).

Definition 2.2. A graph M is **cyclically monotone** if for all integer $m > 0$ and any family of couples $(x_j, y_j) \in M, j = 0, 1, \dots, m$,

$$\langle x_0 - x_m, y_m \rangle + \sum_{k=1}^m \langle x_k - x_{k-1}, y_{k-1} \rangle \leq 0. \quad (6)$$

A cyclically monotone graph M is **maximal** if it does not admit a strict prolongation which is cyclically monotone.

By reindexing the couples, we easily recast the previous inequality as

$$\langle x_m, y_0 - y_m \rangle + \sum_{k=1}^m \langle x_{k-1}, y_k - y_{k-1} \rangle \leq 0, \quad (7)$$

fact which shows that the graphs of a law and of its dual law are simultaneously cyclically monotone. Rockafellar [12], Theorem 24.8 (see also Moreau [11], Proposition 12.2) proved a Theorem that can be stated as:

Theorem 2.3. *Given a graph M , there exist a potential $\phi \in \Gamma_0(X)$ such that $M \subset Gr(\partial\phi)$ if and only if M is cyclically monotone. The potential ϕ is unique up to an additive constant and it is defined by*

$$\phi(x) = \sup \left\{ \langle x - x_m, y_m \rangle + \sum_{k=1}^m \langle x_k - x_{k-1}, y_{k-1} \rangle \right\} + \phi(x_0), \quad (8)$$

where x_0 and $\phi(x_0)$ are arbitrarily fixed and the ‘sup’ is extended to any $m > 0$ and to any couples $(x_k, y_k) \in M, k = 1, 2, \dots, m$.

Because the dual law is also cyclically monotone, we can apply once again the construction of the previous Theorem, giving the function

$$\psi(y) = \sup \left\{ \langle x_m, y - y_m \rangle + \sum_{k=1}^m \langle x_{k-1}, y_k - y_{k-1} \rangle \right\} + \psi(y_0), \quad (9)$$

such that $M \subset Gr(\partial\psi^*)$. Excepted when M is maximal, ϕ and ψ^* are in general distinct function, as it will be seen further in the application.

Definition 2.4. A **bipotential** is a function $b : X \times Y \rightarrow \bar{\mathbb{R}}$, with the properties:

- (a) b is convex and lower semicontinuous in each argument;
- (b) for any $x \in X, y \in Y$ we have $b(x, y) \geq \langle x, y \rangle$;

(c) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y \rangle. \quad (10)$$

The **graph** of b is

$$M(b) = \{(x, y) \in X \times Y \mid b(x, y) = \langle x, y \rangle\}. \quad (11)$$

Particular cases of bipotentials are separable ones, described in the introduction. Many other non separable bipotentials exist.

We introduce next the notion of a strong bipotential. Conditions (B1S) and (B2S) appear as relations (51), (52) in [10].

Definition 2.5. A function $b : X \times Y \rightarrow \bar{\mathbb{R}}$ is a **strong bipotential** if it satisfies the conditions:

- (a) b is convex and lower semicontinuous in each argument;
- (B1S) for any $y \in Y$ $\inf \{b(z, y) - \langle z, y \rangle : z \in X\} \in \{0, +\infty\}$;
- (B2S) for any $x \in X$ $\inf \{b(x, p) - \langle x, p \rangle : p \in Y\} \in \{0, +\infty\}$.

Proposition 2.6. Any strong bipotential is a bipotential.

Proof. Let b be a strong bipotential. From (B1S) and (B2S) we have to prove conditions (b) and (c) of Definition 2.4.

Remark first that any of the two conditions (B1S) and (B2S) implies (b). All is left to prove is (c).

For this take $x \in X$, $y \in Y$ such that $y \in \partial b(\cdot, y)(x)$. This is equivalent with: x is a minimizer of the function $z \in X \mapsto (b(z, y) - \langle z, y \rangle)$. But according to (B1S) the minimum of this function is equal to 0. Therefore $b(x, y) = \langle x, y \rangle$. We proved that $y \in \partial b(\cdot, y)(x) \implies b(x, y) = \langle x, y \rangle$. The inverse implication is trivial, thus we have an equivalence.

In the same, using (B2S) we prove that $x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y \rangle$. The condition (c) is therefore satisfied and b is a bipotential. $\square$

Any separable bipotential is a strong bipotential. Indeed, for any convex lsc $\phi : X \rightarrow \bar{\mathbb{R}}$ we have:

$$\inf \{\phi(z) + \phi^*(y) - \langle z, y \rangle : z \in X\} \in \{0, +\infty\}$$

by the definition (3) of ϕ^* . The notion of a strong bipotential (introduced in relations (51), (52) [10]) is motivated also by the fact that all bipotentials considered in applications in mechanics are in fact strong bipotentials.

Not all bipotentials are strong bipotentials. Consider for example $X = Y = \mathbb{R}$, with the duality $\langle x, y \rangle = xy$, and $b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$b(x, y) = |x| (e^{-y} + 1) + xy.$$

This is bipotential which is not a strong bipotential, and $M(b) = \{0\} \times \mathbb{R}$. Indeed, b is convex and lsc in each argument and $b(x, y) \geq xy$ for any $x, y \in \mathbb{R}$. It is easy to check

that $y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff x = 0$. But $x = 0$ is equivalent with $b(x, y) = xy$, therefore b is a bipotential. Nevertheless, this is not a strong bipotential. Indeed, for $x \neq 0$ we have

$$\inf \{b(x, p) - \langle x, p \rangle : p \in \mathbb{R}\} = |x| \notin \{0, +\infty\}.$$

3. Bipotentials for cyclically monotone graphs

Maximal cyclically monotone graphs are critical sets of separable bipotentials.

The following Theorem shows that there exist bipotentials b with the property that $M(b)$ is a **cyclically monotone, but not maximal** set. In this section we show a surprising connection between bipotentials and the inf convolution operation.

Theorem 3.1. *Let b_1 and b_2 be separable bipotentials associated respectively to the convex and lsc functions $\phi_1, \phi_2 : X \rightarrow \bar{\mathbb{R}}$, that is*

$$b_i(x, y) = \phi_i(x) + \phi_i^*(y)$$

for any $i = 1, 2$ and $(x, y) \in X \times Y$. Consider the following assertions:

- (i) $b = \max(b_1, b_2)$ is a strong bipotential.
- (ii') For any $y \in \text{dom } \phi_1^* \cap \text{dom } \phi_2^*$ and for any $\lambda \in [0, 1]$ we have

$$(\lambda \phi_1 + (1 - \lambda) \phi_2)^*(y) = \lambda \phi_1^*(y) + (1 - \lambda) \phi_2^*(y). \quad (12)$$

- (ii'') For any $x \in \text{dom } \phi_1 \cap \text{dom } \phi_2$ and for any $\lambda \in [0, 1]$ we have

$$(\lambda \phi_1^* + (1 - \lambda) \phi_2^*)^*(x) = \lambda \phi_1(x) + (1 - \lambda) \phi_2(x). \quad (13)$$

Then the point (i) is equivalent with the conjunction of (ii'), (ii''), (for short: (i) $\iff$ ((ii') AND (ii''))).

Remark 3.2. If b_1, b_2 are separable bipotentials and $b = \max(b_1, b_2)$ is a bipotential then $M(b) = M(b_1) \cap M(b_2)$, therefore $M(b)$ is the intersection of two maximal cyclically monotone graphs.

Proof. Before we begin to prove the equivalence let us remark that for any $x \in X$ and for any $y \in Y$, the functions $b(\cdot, x)$ and $b(x, \cdot)$ are convex and lsc as superior envelopes of such functions. Also, for any $(x, y) \in X \times Y$ we have $b_1(x, y) \geq \langle x, y \rangle$ and $b_2(x, y) \geq \langle x, y \rangle$, therefore

$$b(x, y) = \max(b_1(x, y), b_2(x, y)) \geq \langle x, y \rangle.$$

Let $M(b) \subset X \times Y$ be the set of pairs (x, y) with the property $b(x, y) = \langle x, y \rangle$. If $(x, y) \in M(b)$, then

$$\langle x, y \rangle \leq b_i(x, y) \leq b(x, y) = \langle x, y \rangle \quad (i = 1, 2)$$

which proves that $(x, y) \in M(b_1) \cap M(b_2)$. Conversely, if $(x, y) \in M(b_1) \cap M(b_2)$ then $\langle x, y \rangle = b_1(x, y) = b_2(x, y) = b(x, y)$, therefore $(x, y) \in M(b)$. In conclusion $M(b) = M(b_1) \cap M(b_2)$.

Thus the equivalence we have to prove becomes:

- (I) the condition (B1S) from Definition 2.5 is equivalent with (ii'),
- (II) the condition (B2S) from Definition 2.5 is equivalent with (ii'').

These two equivalences have similar proofs. We shall give the proof of the first equivalence.

The function b admits the following characterization:

$$b(x, y) = \max_{\lambda \in [0,1]} \{ \lambda b_1(x, y) + (1 - \lambda) b_2(x, y) \}.$$

For $\lambda \in [0, 1]$ denote by $b^\lambda(x, y) = \lambda b_1(x, y) + (1 - \lambda) b_2(x, y)$. For any $y \in Y$ such that $\phi_1^*(y) < +\infty$, $\phi_2^*(y) < +\infty$ define the set

$$C(y) = \{z \in X : b_1(z, y) < +\infty, b_2(z, y) < +\infty\}$$

and remark that $C(y) \subset X$ is a convex set. In fact $C(y) = \text{dom } \phi_1 \cap \text{dom } \phi_2$, therefore we may drop the y argument and write C instead of $C(y)$.

Consider then the function $f(\cdot, \cdot, y) : C \times [0, 1] \rightarrow \mathbb{R}$ given by $f(z, \lambda, y) = \langle z, y \rangle - b^\lambda(z, y)$. This function is affine and continuous in λ , $[0, 1]$ is a compact convex subset of the vector space $\mathbb{R}$. Also, this function is concave and upper semicontinuous in $z \in C$. Therefore we are in position to apply the minimax Theorem of Sion [19] and deduce that:

$$\min_{\lambda \in [0,1]} \sup_{z \in C} f(z, \lambda, y) = \sup_{z \in C} \min_{\lambda \in [0,1]} f(z, \lambda, y). \quad (14)$$

Let us compute the terms of the equality (14). We have:

$$\begin{aligned} A &= \min_{\lambda \in [0,1]} \sup_{z \in C} f(z, \lambda, y) = \min_{\lambda \in [0,1]} \sup_{z \in C} \{ \langle z, y \rangle - b^\lambda(z, y) \} \\ &= \min_{\lambda \in [0,1]} (\lambda \phi_1 + (1 - \lambda) \phi_2)^*(y) - \lambda \phi_1^*(y) - (1 - \lambda) \phi_2^*(y) \end{aligned}$$

For the other term of the equality (14) we have:

$$B = \sup_{z \in C} \min_{\lambda \in [0,1]} f(z, \lambda, y) = \sup_{z \in C} \{ \langle z, y \rangle - b(z, y) \}.$$

We have $A = B$ thus (14) is equivalent with:

$$\sup_{z \in C} \{ \langle z, y \rangle - b(z, y) \} = \min_{\lambda \in [0,1]} (\lambda \phi_1 + (1 - \lambda) \phi_2)^*(y) - \lambda \phi_1^*(y) - (1 - \lambda) \phi_2^*(y) \quad (15)$$

Suppose that b is a strong bipotential and let $y \in Y$ such that $\phi_1^*(y) < +\infty$, $\phi_2^*(y) < +\infty$. This implies, by (B1S), that

$$\sup_{z \in C} \{ \langle z, y \rangle - b(z, y) \} = 0.$$

By (15) we deduce that

$$\min_{\lambda \in [0,1]} (\lambda \phi_1 + (1 - \lambda) \phi_2)^*(y) - \lambda \phi_1^*(y) - (1 - \lambda) \phi_2^*(y) = 0.$$

But in general we have

$$\max_{\lambda \in [0,1]} (\lambda \phi_1 + (1 - \lambda) \phi_2)^*(y) - \lambda \phi_1^*(y) - (1 - \lambda) \phi_2^*(y) \leq 0$$

which comes from the equality true for any $\lambda \in [0, 1]$:

$$\langle z, y \rangle - \lambda \phi_1(z) - (1 - \lambda) \phi_2(z) = \lambda (\langle z, y \rangle - \phi_1(z)) + (1 - \lambda) (\langle z, y \rangle - \phi_2(z)).$$

We find therefore that for any $\lambda \in [0, 1]$

$$(\lambda \phi_1 + (1 - \lambda) \phi_2)^*(y) - \lambda \phi_1^*(y) - (1 - \lambda) \phi_2^*(y) = 0.$$

Conversely, let $y \in Y$. If $\phi_1^*(y) = +\infty$ or $\phi_2^*(y) = +\infty$ then we have:

$$\inf \{b(z, y) - \langle z, y \rangle : z \in X\} = +\infty.$$

Suppose that $\phi_1^*(y) < +\infty$ and $\phi_2^*(y) < +\infty$. From (ii') and (15) we deduce that

$$\inf \{b(z, y) - \langle z, y \rangle : z \in X\} = 0.$$

The second equivalence has a similar proof. $\square$

It is easy to construct examples of separable bipotentials $b_i(x, y) = \phi_i(x) + \phi_i^*(y)$, $i = 1, 2$, such that $b = \max \{b_1, b_2\}$ is not a bipotential. For this let us take $X = Y = \mathbb{R}$ with the duality given by the product, and let us choose ϕ_1, ϕ_2 smooth (for example C^1) convex functions defined on $\mathbb{R}$ with values in $\mathbb{R}$. Then $M(b_1) = Gr(\phi_1')$ and $M(b_2) = Gr(\phi_2')$, therefore $M(b)$ is the intersection of the graphs of ϕ_1' and ϕ_2' . In general $M(b)$ is not bi-convex, because it is just an intersection of graphs of increasing continuous functions. For example, if $\phi_1(x) = x^4/4$ and $\phi_2(x) = x^2/2$ then $M(b) = \{(1, 1), (0, 0), (-1, -1)\}$, which is not bi-convex.

The conditions (ii'), (ii'') from Theorem 3.1 imply relations which can be expressed with the help of inf convolutions. Let us examine condition (ii''); the same arguments can be used for the symmetric condition (ii'). Consider $\phi_1, \phi_2 \in \Gamma_0(X)$. For any $\lambda \in (0, 1)$ we introduce two functions defined on X by:

$$f_{1,\lambda}(x) = \lambda \phi_1\left(\frac{1}{\lambda}x\right), \quad f_{2,\lambda}(x) = (1 - \lambda) \phi_2\left(\frac{1}{1 - \lambda}x\right).$$

Then condition (ii'') implies that for any $x \in \text{dom } \phi_1 \cap \text{dom } \phi_2$ and for any $\lambda \in (0, 1)$ we have

$$f_{1,\lambda}(\lambda x) + f_{2,\lambda}((1 - \lambda)x) = f_{1,\lambda} \square f_{2,\lambda}(x). \quad (16)$$

Indeed, we have

$$(f_{1,\lambda}^* + f_{2,\lambda}^*)^*(x) = (\lambda \phi_1^* + (1 - \lambda) \phi_2^*)^*(x).$$

The space X is locally convex therefore

$$(f_{1,\lambda}^* + f_{2,\lambda}^*)^*(x) \leq f_{1,\lambda} \square f_{2,\lambda}(x).$$

Therefore (ii'') implies that

$$f_{1,\lambda}(\lambda x) + f_{2,\lambda}((1 - \lambda)x) \leq f_{1,\lambda} \square f_{2,\lambda}(x)$$

which, by the definition (4) of the inf convolution operation, is equivalent with (16).

This is leading us to the following corollary of Theorem 3.1.

Corollary 3.3. *Let $\phi_1, \phi_2 \in \Gamma_0(X)$ such that*

$$b(x, y) = \max(\phi_1(x) + \phi_1^*(y), \phi_2(x) + \phi_2^*(y))$$

is a strong bipotential. Then, with the previous notations, for any $x \in \text{dom } \phi_1 \cap \text{dom } \phi_2$ and for any $\lambda \in (0, 1)$ we have

$$\partial(f_{1,\lambda} \square f_{2,\lambda})(x) = \partial\phi_1(x) \cap \partial\phi_2(x).$$

Proof. By Theorem 3.1, if b is a strong bipotential then (ii'') is true. By previous reasoning this implies the relation (16). We apply then Lemma 2.6, Lemma 2.7, [21] and we obtain that

$$\partial(f_{1,\lambda} \square f_{2,\lambda})(x) = \partial f_{1,\lambda}(\lambda x) \cap \partial f_{2,\lambda}((1 - \lambda)x).$$

From the definition of the functions $f_{1,\lambda}, f_{2,\lambda}$ we obtain the conclusion. $\square$

The following interesting question has been suggested by the anonymous referee: can the maximum of two separable bipotentials be a non strong bipotential?

4. Bipotential convex covers

Let $Bp(X, Y)$ be the set of all bipotentials $b : X \times Y \rightarrow \bar{\mathbb{R}}$. We shall need the following Definition concerning implicitly convex functions.

Definition 4.1. Let Λ be an arbitrary non empty set and V a real vector space. The function $f : \Lambda \times V \rightarrow \bar{\mathbb{R}}$ is **implicitly convex** if for any two elements $(\lambda_1, z_1), (\lambda_2, z_2) \in \Lambda \times V$ and for any two numbers $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ there exists $\lambda \in \Lambda$ such that

$$f(\lambda, \alpha z_1 + \beta z_2) \leq \alpha f(\lambda_1, z_1) + \beta f(\lambda_2, z_2).$$

In the following Definition we generalize the notion of a **bi-implicitly convex lagrangian cover**, Definitions 4.1 and 6.6, [4].

Definition 4.2. A **bipotential convex cover** of the non empty set M is a function $\lambda \in \Lambda \mapsto b_\lambda$ from Λ with values in the set $Bp(X, Y)$, with the properties:

- (a) The set Λ is a non empty compact topological space,
- (b) Let $f : \Lambda \times X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$f(\lambda, x, y) = b_\lambda(x, y).$$

Then for any $x \in X$ and for any $y \in Y$ the functions $f(\cdot, x, \cdot) : \Lambda \times Y \rightarrow \bar{\mathbb{R}}$ and $f(\cdot, \cdot, y) : \Lambda \times X \rightarrow \bar{\mathbb{R}}$ are lower semi continuous on the product spaces $\Lambda \times Y$ and respectively $\Lambda \times X$ endowed with the standard topology,

- (c) We have $M = \bigcup_{\lambda \in \Lambda} M(b_\lambda)$.
- (d) with the notations from point (b), the functions $f(\cdot, x, \cdot)$ and $f(\cdot, \cdot, y)$ are implicitly convex in the sense of Definition 4.1.

Several remarks are in order.

Remark 4.3. A bipotential convex cover $\lambda \in \Lambda \mapsto b_\lambda$ such that for any $\lambda \in \Lambda$ the bipotential b_λ is separable is a bi-implicitly convex lagrangian cover. For such covers the sets $M(b_\lambda)$ are **maximal cyclically monotone** for any $\lambda \in \Lambda$.

Remark 4.4. In general bipotential convex covers are **not lagrangian** (see Remark 6.1, [4] for a justification of the "lagrangian" term). In the language of convex analysis this means that the sets $M(b_\lambda)$ are not supposed to be maximal cyclically monotone.

We shall see in the section concerning the applications to the Coulomb's friction law that there exists bipotential convex covers with the property that for any $\lambda \in \Lambda$ the set $M(b_\lambda)$ is **cyclically monotone but non maximal**. This is done by using bipotential covers constructed with the help of Theorem 3.1.

A bipotential convex cover is in some sense described by the collection $\{b_\lambda : \lambda \in \Lambda\}$. This is shown in the next Proposition.

Proposition 4.5. *Let $\lambda \in \Lambda \mapsto b_\lambda \in Bp(X, Y)$ be a bipotential convex cover and $g : \Lambda \rightarrow \Lambda$ be a continuous, invertible, with continuous inverse, function. Then $\lambda \in \Lambda \mapsto b_{g(\lambda)} \in Bp(X, Y)$ is a bipotential convex cover.*

Proof. This is obvious due to the general fact that if $f : \Lambda \times V \rightarrow \bar{\mathbb{R}}$ is implicitly convex and $g : \Lambda \rightarrow \Lambda$ is a bijection then the function $f' : \Lambda \times V \rightarrow \bar{\mathbb{R}}$, $f'(\lambda, x) = f(g(\lambda), x)$ is implicitly convex. This reflects into the fact that a bipotential convex cover is a notion invariant with respect to **continuous** reparametrizations of Λ (the continuity is needed in order to preserve the lower semi continuity assumptions from point (b) of the Definition 4.2). $\square$

The next Theorem generalizes Theorem 6.7, the main result of [4]. We shall skip its proof because it is just a rephrasing of the proof of Theorem 6.7, [4].

Theorem 4.6. *Let $\lambda \mapsto b_\lambda$ be a bipotential convex cover of the graph M and $b : X \times Y \rightarrow \bar{\mathbb{R}}$ defined by*

$$b(x, y) = \inf \{b_\lambda(x, y) \mid \lambda \in \Lambda\}. \quad (17)$$

Then b is a bipotential and $M = M(b)$.

5. Application: Coulomb's law of dry friction contact

This is a typical example of what is called a non associated constitutive law in mechanics. Despite of its rather complex structure, it is worthwhile to have interest in it because of its importance in many practical problems.

We shall not discuss here the phenomenal and experimental aspects but only the mathematical modeling with respect to the bipotential theory. To be short, the space $X = \mathbb{R}^3$ is the one of relative velocities between points of two bodies, and the space Y , identified also to $\mathbb{R}^3$, is the one of the contact reaction stresses. The duality product is the usual scalar product. We put

$$(x_n, x_t) \in X = \mathbb{R} \times \mathbb{R}^2, \quad (y_n, y_t) \in Y = \mathbb{R} \times \mathbb{R}^2,$$

where x_n is the gap velocity, x_t is the sliding velocity, y_n is the contact pressure and y_t is minus the friction stress. The friction coefficient is $\mu > 0$. The graph of the law

of unilateral contact with Coulomb's dry friction is defined as the union of three sets, respectively corresponding to the 'body separation', the 'sticking' and the 'sliding'.

$$M = \{(x, 0) \in X \times Y \mid x_n < 0\} \cup \{(0, y) \in X \times Y \mid \|y_t\| \leq \mu y_n\} \quad (18)$$

$$\cup \left\{ (x, y) \in X \times Y \mid x_n = 0, x_t \neq 0, y_t = \mu y_n \frac{x_t}{\|x_t\|} \right\}.$$

It is well known that this graph is not monotone, then not cyclically monotone. As usual, we introduce Coulomb's cone

$$K_\mu = \{(y_n, y_t) \in Y \mid \|y_t\| \leq \mu y_n\},$$

and its conjugate cone

$$K_\mu^* = \{(x_n, x_t) \in X \mid \mu \|x_t\| + x_n \leq 0\}.$$

In particular, we have

$$K_0 = \{(y_n, 0) \in Y \mid y_n \geq 0\}, \quad K_0^* = \{(x_n, x_t) \in X \mid x_n \leq 0\}.$$

Now, we define some sets useful in the sequel. Let us consider $p > 0$ and the closed convex disc obtained by cutting Coulomb's cone at the level $y_n = p$

$$D(p) = \{y_t \in \mathbb{R}^2 \mid \|y_t\| \leq \mu p\}.$$

Therefore, for each value of $p > 0$, we define a set of 'sticking couples'

$$M_p^{(a)} = \{(0, (p, y_t)) \in X \times Y \mid y_t \in D(p)\},$$

and a set of 'sliding couples'

$$M_p^{(s)} = \{((0, x_t), (p, y_t)) \in X \times Y \mid \|y_t\| = \mu p, \exists \lambda > 0, x_t = \lambda y_t\}.$$

So, we can cover the graph M by the set of following subgraphs parameterized by $p \in [0, +\infty]$

- (a) $M_p = M_p^{(a)} \cup M_p^{(s)}, p \in (0, +\infty),$
- (b) $M_0 = \{(x, 0) \in X \times Y \mid x_n \leq 0\},$
- (c) $M_{+\infty} = \emptyset$, by convention.

All these subgraphs are cyclically monotone but none of them is maximal. Let us construct by Rockafellar's Theorem the corresponding associated functions ϕ_p and ψ_p such that $x_0 = 0$ and $\phi_p(0) = \psi_p(y_0) = 0$. For $p \in (0, +\infty)$, the computations give

$$\phi_p(x) = px_n + \mu p \|x_t\|, \quad \psi_p(y) = \chi_{D(p)}(y_t).$$

Their Legendre-Fenchel duals are

$$\phi_p^*(y) = \chi_{\{p\}}(y_n) + \chi_{D(p)}(y_t), \quad \psi_p^*(x) = \mu p \|x_t\| + \chi_{\{0\}}(x_n).$$

For $p = 0$, we obtain

$$\phi_0(x) = 0, \quad \psi_0(y) = \chi_{K_0}(y).$$

Their Legendre-Fenchel duals are

$$\phi_0^*(y) = \chi_{\{0\}}(y), \quad \psi_0^*(x) = \chi_{K_0^*}(x).$$

For fixed p , define the bipotentials $b_{i,p}$, $i = 1, 2$, by:

$$b_{1,p}(x, y) = \phi_p(x) + \phi_p^*(y),$$

$$b_{2,p}(x, y) = \psi_p^*(x) + \psi_p(y).$$

As an application of Theorem 3.1 we obtain that $b_p = \max\{b_{1,p}, b_{2,p}\}$ is a bipotential. Indeed, we shall check only the point (ii') from Theorem (3.1) (the point (ii'') is true by a similar computation). For $\lambda \in [0, 1]$ and $p \neq 0$ we have:

$$\lambda\phi_p(x) + (1 - \lambda)\psi_p^*(x) = \chi_{\{0\}}(x_n) + \mu p \parallel x_t \parallel$$

therefore we get

$$(\lambda\phi_p(x) + (1 - \lambda)\psi_p^*(x))^*(y) = \chi_{D(p)}(y_t).$$

Also, by computation we obtain:

$$\lambda\phi_p^*(y) + (1 - \lambda)\psi_p(y) = \chi_{\{p\}}(y_n) + \chi_{D(p)}(y_t).$$

If $\phi_p^*(y) < +\infty$, $\psi_p(y) < +\infty$ then in particular $y_n = p$ and we obtain (12) as an equality $0 = 0$. All other cases, involving $\lambda = 1$ or $p = 0$ are solved in the same way.

The bipotential b_p has the expression:

$$b_p(x, y) = \mu p \parallel x_t \parallel + \chi_{D(p)}(y_t) + \chi_{\{p\}}(y_n) + \chi_{\{0\}}(x_n), \quad p \in (0, +\infty),$$

$$b_0(x, y) = \chi_{\{0\}}(y) + \chi_{(-\infty, 0]}(x_n).$$

It is easy to check that the function $p \in [0, +\infty] \mapsto b_p$ is a bipotential convex cover, therefore by Theorem 4.6 we obtain a bipotential for the set M . By direct computation, this bipotential, defined as

$$b(x, y) = \inf \{b_p(x, y) : p \in [0, +\infty]\},$$

has the following expression:

$$b(x, y) = \mu y_n \parallel x_t \parallel + \chi_{K_\mu}(y) + \chi_{K_0^*}(x).$$

Therefore, we recover the bipotential previously given in [13].

6. Conclusion

The present approach shows that the bipotential related to Coulomb's friction law is related to a specific bipotential convex cover with the property that any graph of the cover is non maximal cyclically monotone.

Acknowledgements. We express our thanks to the anonymous referee for detecting a flaw in a previous version of the paper, and for several valuable questions and suggestions for improvement of this article. The first author acknowledges partial support from the Romanian Ministry of Education and Research, through the grant CEX06-11-12/2006.

References

- [1] G. Bodovillé: On damage and implicit standard materials, *C. R. Acad. Sci., Paris, Sér. II, Fasc. b, Méc. Phys. Astron.* 327(8) (1999) 715–720.
- [2] G. Bodovillé, G. de Saxcé: Plasticity with non linear kinematic hardening: modelling and shakedown analysis by the bipotential approach, *Eur. J. Mech., A, Solids* 20 (2001) 99–112.
- [3] L. Boussshine, A. Chaaba, G. de Saxcé: Plastic limit load of plane frames with frictional contact supports, *Int. J. Mech. Sci.* 44(11) (2002) 2189–2216.
- [4] M. Buliga, G. de Saxcé, C. Vallée: Existence and construction of bipotentials for graphs of multivalued laws, *J. Convex Analysis* 15(1) (2008) 87–104.
- [5] Z.-Q. Feng, M. Hjiiaj, G. de Saxcé, Z. Mróz: Effect of frictional anisotropy on the quasistatic motion of a deformable solid sliding on a planar surface, *Comput. Mech.* 37 (2006) 349–361.
- [6] J. Fortin, M. Hjiiaj, G. de Saxcé: An improved discrete element method based on a variational formulation of the frictional contact law, *Comput. Geotech.* 29(8) (2002) 609–640.
- [7] B. Halphen, Nguyen Quoc Son: Sur les matériaux standard généralisés, *J. Méc., Paris* 14 (1975) 39–63.
- [8] M. Hjiiaj, G. Bodovillé, G. de Saxcé: Matériaux viscoplastiques et loi de normalité implicites, *C. R. Acad. Sci., Paris, Sér. II, Fasc. b, Méc. Phys. Astron.* 328 (2000) 519–524.
- [9] M. Hjiiaj, Z.-Q. Feng, G. de Saxcé, Z. Mróz: Three dimensional finite element computations for frictional contact problems with non-associated sliding rule, *Int. J. Numer. Methods Eng.* 60(12) (2004) 2045–2076.
- [10] P. Laborde, Y. Renard: Fixed point strategies for elastostatic frictional contact problems, *Math. Methods Appl. Sci.* 31 (2008) 415–441.
- [11] J.-J. Moreau: *Fonctionnelles Convexes*, Istituto Poligrafico e Zecca dello Stato, Rome (2003), pp. 71, 97.
- [12] R.T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970), p. 238.
- [13] G. de Saxcé, Z. Q. Feng: New inequation and functional for contact with friction: the implicit standard material approach, *Int. J. Mech. Struct. Mach.* 19(3) (1991) 301–325.
- [14] G. de Saxcé: Une généralisation de l'inégalité de Fenchel et ses applications aux lois constitutives, *C. R. Acad. Sci., Paris, Sér. II* 314 (1992) 125–129.
- [15] G. de Saxcé, L. Boussshine: On the extension of limit analysis theorems to the non associated flow rules in soils and to the contact with Coulomb's friction, in: *Proc. XI Polish Conference on Computer Methods in Mechanics (Kielce, 1993)*, Vol. 2 (1993) 815–822.
- [16] G. de Saxcé: The bipotential method, a new variational and numerical treatment of the dissipative laws of materials, in: *Proc. 10th Int. Conf. on Mathematical and Computer Modelling and Scientific Computing (Boston, 1995)*.
- [17] G. de Saxcé, L. Boussshine: Implicit standard materials, in: *Inelastic Behaviour of Structures Under Variable Repeated Loads*, D. Weichert, G. Maier (eds.), CISM Courses and Lectures 432, Springer, Wien (2002) 59–76.
- [18] G. de Saxcé, Z.-Q. Feng: The bipotential method: a constructive approach to design the complete contact law with friction and improved numerical algorithms, *Math. Comput. Modelling* 28(4–8) (1998) 225–245.

- [19] M. Sion: On general minimax theorems, *Pac. J. Math.* 8 (1958) 171–176.
- [20] C. Vallée, C. Lerintiu, D. Fortuné, M. Ban, G. de Saxcé: Hill's bipotential, in: *New Trends in Continuum Mechanics*, M. Mihăilescu-Suliciu (ed.), Theta Series in Advanced Mathematics 3, Theta Foundation, Bucharest (2005) 339–351.
- [21] D. Zagrodny: The cancellation law for inf-convolution of convex functions, *Stud. Math.* 110(3) (1994) 271–282.