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Pseudomonotone Diagonal Subdifferential Operators

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Let f be an equilibrium bifunction defined on the product space $\mathbb{X}\times\mathbb{X}$, where $\mathbb{X}$ is a Banach space. If f is locally Lipschitz with respect to the second variable, for every $x \in \mathbb{X}$ we define $T_f(x)$ as the Clarke subdifferential of $f(x,\cdot)$ evaluated at x. This multivalued operator plays a fundamental role for the reformulation of equilibrium problems as variational inequality ones. We analyze additional conditions on f which ensure the D–maximal pseudomonotonicity and the cyclically pseudomonotonicity of T_f. Such results have consequences in terms of the characterization of the set of solutions of a subclass of pseudomonotone equilibrium problems.

Keywords: Equilibrium problem, pseudomonotone bifunction, pseudomonotone operator, diagonal subdifferential

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1. Introduction

Let $\mathbb{X}$ be a Banach space with dual space $\mathbb{X}^*$. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in \mathbb{X}^*$ at $x \in \mathbb{X}$. Consider a function $f : \mathbb{X}\times\mathbb{X} \to \mathbb{R}$ and let $C \subseteq \mathbb{X}$ be a nonempty convex and closed set. The equilibrium problem associated to f over C (in short $\mathrm{EP}(f, C)$) consists in finding $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C;$$

we call f an equilibrium bifunction if $f(x, x) = 0$ for all $x \in \mathbb{X}$. The equilibrium problem unifies and includes as special cases the classical fixed–point problem, the variational inequality problem, the Nash equilibrium problem, the saddle–point problem and minimization problem in mathematical economics and other related issues such as the complementarity problem in mathematical programming [5].

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Starting from the seminal paper [1], some authors studied properties of the multivalued operator $A^f : \mathbb{X} \rightrightarrows \mathbb{X}^*$ defined by $A^f(x) = \partial f(x, \cdot)(x)$ where ∂ denotes the Fenchel subdifferential. Such a map was called diagonal subdifferential operator in [18]. Observe that A^f is not the subdifferential of a fixed function: at each point x it coincides with the subdifferential of a certain function evaluated at x but the functions themselves change with x. Moreover, if $\bar{x}$ solves the variational inequality problem VIP(A^f, C), i.e. there exists $\bar{x}^* \in A^f(\bar{x})$ such that $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0$ for all $y \in C$, then it solves EP(f, C). Furthermore EP(f, C) and VIP(A^f, C) are equivalent if $f(x, \cdot)$ is convex for all $x \in \mathbb{X}$.

It is well known that variational inequality problems are substantially easier to solve when the involved operator is maximal monotone. Thus, the study of conditions under which A^f is maximal monotone has a significant impact on the theory of equilibrium problems. It is easy to show that, if f is monotone, i.e.

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in \mathbb{X}$$

then A^f is a monotone operator (see for instance [7]). In recent years a number of papers (see [6, 11, 15, 18, 21]) have appeared devoted to establish sufficient conditions which guarantee the maximal monotonicity of A^f.

Nevertheless, the assumptions on f of monotonicity and/or convexity with respect to its second argument may be quite restrictive. So it is not surprising that in the last decade a vast literature on generalized convex and generalized monotone equilibrium problems has been developed (see for instance [3, 4, 13, 19, 20] and references therein). Among the various concepts of generalized monotonicity, one of the most used is the pseudomonotonicity. We recall that an equilibrium bifunction f is said pseudomonotone if for all $x, y \in \mathbb{X}$ such that $f(x, y) \geq 0$ then $f(y, x) \leq 0$.

In this paper, by focusing on the class of pseudoconvex and locally Lipschitz equilibrium bifunctions, and taking advantage of the recent results on the structure of pseudomonotone operators [14, 16, 17], we introduce the multivalued operator T_f defined by $T_f(x) = \partial^\circ f(x, \cdot)(x)$ where ∂° is the Clarke subdifferential. This operator collapses into A^f when $f(x, \cdot)$ is convex too. As in the convex case, the variational inequality problem associated to T_f has the same set of solutions of EP(f, C). In fact the equivalence between two equilibrium problems can be translated into the equivalence between the associated diagonal operators. This result suggests to deepen the properties of the operator T_f. We analyze additional conditions on f which ensure its pseudomonotonicity, and furthermore its D–maximal pseudomonotonicity and cyclic pseudomonotonicity. These results have consequences in terms of the characterization of the set of solutions of EP(f, C).

We conclude this introduction fixing our notation. The algebraic interior of a subset C of $\mathbb{X}$ is the defined by

$$\text{core}(C) = \{x \in C \ : \ \forall v \in \mathbb{X}, \exists t_v > 0 \text{ s.t. } x + tv \in C, \ \forall t \in (0, t_v)\}.$$

The set C is called radially open if $C = \text{core}(C)$. If C is a convex subset of $\mathbb{X}$, the normal cone to C at $x \in C$ is

$$N(x, C) = \{x^* \in \mathbb{X}^* \ : \ \langle x^*, y - x \rangle \leq 0, \ \forall y \in C\}.$$

Let $\varphi : \mathbb{X} \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential at x is

$$\partial^\circ \varphi(x) = \left\{ x^* \in \mathbb{X}^* \;:\; \langle x^*, v \rangle \leq \limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{\varphi(x' + tv) - \varphi(x')}{t}, \ \forall v \in \mathbb{X} \right\}.$$

Moreover φ is said to be ∂°–pseudoconvex if for every $x, y \in \mathbb{X}$ and $x^* \in \partial^\circ \varphi(x)$ such that $\langle x^*, y - x \rangle \geq 0$, we have $\varphi(y) \geq \varphi(x)$. We also take the opportunity of recalling the Lebourg's Mean Value Theorem (see [9, Theorem 2.4]).

Theorem 1.1 (Lebourg's Mean Value Theorem). *Let $\varphi : \mathbb{X} \to \mathbb{R}$ be a locally Lipschitz function and $x, y \in \mathbb{X}$. Then there exist $u \in (x, y)$ and $u^* \in \partial^\circ \varphi(u)$ such that*

$$\varphi(y) - \varphi(x) = \langle u^*, y - x \rangle.$$

Given a multivalued operator $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ we denote by $D(T)$ its domain and by $Z(T)$ the set of zeros of T, i.e.

$$Z(T) = \{x \in \mathbb{X} \;:\; 0 \in T(x)\}.$$

The operator T is said pseudomonotone (in the sense of Karamardian [22]) if for all $x, y \in \mathbb{X}$ and $x^* \in T(x)$ such that $\langle x^*, y - x \rangle \geq 0$, then $\langle y^*, x - y \rangle \leq 0$ for all $y^* \in T(y)$. We lastly recall that ([23, Theorem 4.1]) a locally Lipschitz function φ is ∂°–pseudoconvex if and only if $\partial^\circ \varphi$ is pseudomonotone.

2. Equivalent equilibrium problems

We formalize the previously described approach, starting with the definition of the multivalued operator T_f associated to the equilibrium bifunction f. For the sake of simplicity, in all this paper we make the following assumptions about the equilibrium bifunctions:

- f is defined on the whole product space $\mathbb{X} \times \mathbb{X}$,
- for every $x \in \mathbb{X}$ the function $f(x, \cdot)$ is locally Lipschitz,
- for every $x \in \mathbb{X}$ the function $f(x, \cdot)$ is ∂°–pseudoconvex.

Definition 2.1. Given the equilibrium bifunction f, the multivalued operator $T_f : \mathbb{X} \rightrightarrows \mathbb{X}^*$ defined by

$$T_f(x) = \left\{ x^* \in \mathbb{X}^* \;:\; \langle x^*, v \rangle \leq \limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{f(x, x' + tv) - f(x, x')}{t}, \ \forall v \in \mathbb{X} \right\}$$

is called the (Clarke) diagonal subdifferential operator.

Even if $T_f(x) = \partial^\circ f(x, \cdot)(x)$ for all $x \in \mathbb{X}$, the operator T_f is not defined as the Clarke subdifferential operator of a unique function since, for each $x \in \mathbb{X}$, $f(x, \cdot)$ is a different function. Nevertheless, just as with the Clarke subdifferential, the values $T_f(x)$ are convex and weak* compact sets.

This operator is strictly related to the operator A^f introduced in [1]. Since we have supposed that f is defined on the whole space $\mathbb{X} \times \mathbb{X}$, the definition of A^f can be simplified as follows:

$$A^f(x) = \{x^* \in \mathbb{X}^* \ : \ f(x,y) \geq \langle x^*, y-x \rangle, \ \forall y \in \mathbb{X}\}.$$

Of course $A^f(x) = \partial f(x,\cdot)(x)$, where ∂ is the Fenchel subdifferential, and it may be eventually empty. In [6, 15, 18, 21] the authors gave some criteria that guarantee the maximal monotonicity for A^f but all their results require the convexity of $f(x,\cdot)$: under such further assumption $T_f = A^f$.

We start by investigating when two equilibrium problems have the same solution set. This kind of equivalence may be exploited to transform the starting equilibrium problem into a new one with some additional nice properties, which can be used to obtain existence results or to formulate iterative solution methods. The key of this equivalence starts from the following simple result.

Theorem 2.2. *For every equilibrium bifunction f, the sets of solutions of* $\mathrm{EP}(f, C)$ *and* $\mathrm{VIP}(T_f, C)$ *coincide.*

Proof. If $\bar{x}$ solves $\mathrm{EP}(f, C)$ then it minimizes $f(\bar{x}, \cdot)$ on the feasible set C. From the first order optimality condition there exists $\bar{x}^* \in T_f(\bar{x})$ such that $-\bar{x}^* \in N(\bar{x}, C)$. This condition is equivalent to say that $\bar{x}$ solves $\mathrm{VIP}(T_f, C)$. The converse descends immediately from the definition of ∂°–pseudoconvexity. □

In [14], Hadjisavvas introduced an equivalence relation in the class of pseudomonotone operators. More in general, without any assumption of generalized monotonicity, we say that two operators S and T are equivalent, and write $S \sim T$, if $Z(S) = Z(T)$ and for every $x \in \mathbb{X} \setminus Z(S)$,

$$\bigcup_{r>0} rS(x) = \bigcup_{s>0} sT(x).$$

This equivalence is closely related to variational inequalities. Indeed it is easy to show that whenever $T \sim S$, $\bar{x}$ is a solution of $\mathrm{VIP}(T, C)$ if and only if $\bar{x}$ is a solution of $\mathrm{VIP}(S, C)$ for every convex set C. Moreover the converse holds if the operators S and T are pseudomonotone and have weak* compact convex values [17, Theorem 4.2]. From this fact, together with Theorem 2.2, we get the following generalization for equilibrium problems.

Corollary 2.3. *Let f and g be two equilibrium bifunctions such that $T_f \sim T_g$. Then* $\mathrm{EP}(f, C)$ *and* $\mathrm{EP}(g, C)$ *have the same set of solutions for every convex set C. Conversely, if T_f and T_g are pseudomonotone and* $\mathrm{EP}(f, C)$ *and* $\mathrm{EP}(g, C)$ *have the same set of solutions for every straight line segment C, then $T_f \sim T_g$.*

Proof. From Theorem 2.2 the problems $\mathrm{EP}(f, C)$ and $\mathrm{EP}(g, C)$ have the same set of solutions of the variational inequalities $\mathrm{VIP}(T_f, C)$ and $\mathrm{VIP}(T_g, C)$ respectively. Furthermore the operators T_f and T_g have weak* compact convex values and they satisfy all the assumptions of Theorem 4.2 in [17]. Therefore the required implications immediately follow. □

We stress that the proof of the equivalence between T_f and T_g involves the pseudomonotonicity of the diagonal subdifferential operators. The next result furnishes a practical and natural sufficient condition for T_f to be pseudomonotone.

Theorem 2.4. *If f is a pseudomonotone equilibrium bifunction, then T_f is a pseudomonotone operator.*

Proof. Fix $x, y \in \mathbb{X}$ and $x^* \in T_f(x)$ such that $\langle x^*, y - x\rangle \geq 0$. Hence, from the ∂°–pseudoconvexity of $f(x, \cdot)$ we deduce

$$f(x, y) \geq f(x, x) = 0.$$

Since f is pseudomonotone then $f(y, x) \leq 0$. Applying the Lebourg's Mean Value Theorem to the functions $f(y, \cdot)$ on the segment $[x, y]$ we deduce the existence of $z = tx + (1 - t)y$ with $t \in (0, 1)$ and $z^* \in \partial^\circ f(y, \cdot)(z)$ such that

$$0 \leq f(y, y) - f(y, x) = \langle z^*, y - x\rangle = t^{-1}\langle z^*, y - z\rangle.$$

Since the operator $\partial^\circ f(y, \cdot)$ is pseudomonotone, then

$$0 \geq \langle y^*, z - y\rangle = t\langle y^*, x - y\rangle, \quad \forall y^* \in T_f(y)$$

which completes the proof. $\square$

Now we proceed to construct two examples. The former shows that the converse of Theorem 2.4 does not hold. This means that there exist equilibrium problems which are not pseudomonotone but have an equivalent pseudomonotone reformulation as variational inequalities.

Example 2.5. Given the symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$, consider the bifunction $f(x, y) = \langle y, A(y - x)\rangle$ defined on $\mathbb{R}^n \times \mathbb{R}^n$. The function $f(x, \cdot)$ is convex and therefore it fulfills all the assumptions of Theorem 2.4 but the pseudomonotonicity: indeed, for any fixed $u \in \mathbb{R}^n$ different from 0 we have $f(u, -u) = f(-u, u) = 2\langle u, Au\rangle > 0$. Nevertheless the diagonal subdifferential operator coincides with the singleton $T_f(x) = \{Ax\}$. The positive definiteness of A implies the monotonicity (and hence also the pseudomonotonicity) of T_f.

The latter example shows that the pseudomonotonicity alone, without the assumption of ∂°–pseudoconvexity of $f(x, \cdot)$, is not enough to guarantee the pseudomonotonicity of T_f.

Example 2.6. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function which has global minimum at the (unique) point $\bar{x} \in \mathbb{R}^n$ and consider the bifunction $f(x, y) = \varphi(x) - \varphi(y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$. The bifunction f is trivially pseudomonotone and, for every $x, y \in \mathbb{R}^n$ we have $\partial^\circ f(x, \cdot)(y) = -\partial\varphi(y)$. For every $x \in \mathbb{R}^n$, the function $f(x, \cdot)$ is not ∂°–pseudoconvex since it assumes global maximum at $\bar{x}$ and $0 \in \partial^\circ f(x, \cdot)(\bar{x})$. Moreover $T_f = -\partial\varphi$ is not pseudomonotone since $0 \in T_f(\bar{x})$ and, for every $y \neq \bar{x}$ and every $y^* \in T_f(y)$, we have $\langle y^*, \bar{x} - y\rangle \geq \varphi(y) - \varphi(\bar{x}) > 0$.

3. Some properties of the diagonal subdifferential operator

Monotone operators are a very powerful tool in the study of PDE's and variational inequalities. Among monotone operators, the subclass of maximal monotone operators is one of the most studied in Nonlinear Analysis. For instance, maximal monotonicity of T implies that $T(x)$ is convex and weak* closed and the graph of T is sequentially closed in the topological product space $\mathbb{X} \times \mathbb{X}^*$, where $\mathbb{X}$ is endowed with the strong topology and $\mathbb{X}^*$ is endowed with the weak* topology. Such a result is very useful when passing to the limits in approximation arguments. Under reflexivity of $\mathbb{X}$, the maximal monotonicity is fundamental for the study of the properties of the resolvent operator and the Yosida approximation.

Recently, Hadjisavvas [14] introduced the concept of D–maximal pseudomonotonicity exploiting the equivalence relation which has been introduced in the previous section. Using this concept the author established some results regarding continuity and single–valuedness which generalize analogous results known for monotone operators.

Given the previously defined equivalence relation $\sim$, and fixed a multivalued operator T, we denote by

$$\widehat{T}(x) = \bigcup_{S \sim T} S(x)$$

the equivalent maximum element with respect to the order defined by graph inclusion. If we consider the family of pseudomonotone operators only, it is easy to show that $\widehat{T}$ is pseudomonotone and it could be expressed as

$$\widehat{T}(x) = \begin{cases} N(x, L_{T,x}), & \text{if } x \in Z(T) \\ \bigcup_{r>0} rT(x), & \text{if } x \in D(T) \setminus Z(T) \\ \emptyset, & \text{if } x \notin D(T) \end{cases}$$

where

$$L_{T,x} = \{y \in \mathbb{X} \ : \ \exists y^* \in T(y) \text{ s.t. } \langle y^*, x - y\rangle = 0\}.$$

Definition 3.1. A pseudomonotone multivalued operator $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is called D–maximal pseudomonotone if $\widehat{T}$ has no pseudomonotone extension with the same domain, apart from itself.

D–maximal pseudomonotone multivalued operators have some nice properties. If T is D–maximal pseudomonotone, then $\widehat{T}(x)$ is convex for all $x \in D(T)$; moreover the set $Z(T)$ is weakly closed in $D(T)$ and it is convex if $D(T)$ is convex. In order to furnish a criterion for D–maximal pseudomonotonicity, we recall the following definition introduced in [14].

Definition 3.2. The operator $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is said upper sign continuous if for any $x, y \in \mathbb{X}$ one has

$$\inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0, \quad \forall t \in (0,1) \quad \Rightarrow \quad \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0,$$

where $x_t = x + t(y - x)$.

This is a very weak kind of continuity. For instance, if T is upper hemicontinuous (i.e. the restriction of T to every line segment is upper semicontinuous) with respect to the weak* topology, then T is upper sign continuous.

Theorem 3.3 ([14]). *Let T be a pseudomonotone and upper sign continuous operator, $D(T)$ be radially open, and $T(x)$ be weak* compact and convex for all $x \in D(T)$. Then T is D–maximal pseudomonotone.*

The first step of this section is to prove that, under a weak continuity assumption on f, the diagonal subdifferential operator T_f is D–maximal pseudomonotone. We recall the following notion introduced in [3] in the framework of the theory of the existence of solutions of equilibrium problems.

Definition 3.4. An equilibrium bifunction $f : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is said upper sign continuous (with respect to the first argument) if for any $x, y \in \mathbb{X}$ one has

$$f(x_t, y) \geq 0, \quad \forall t \in (0,1) \quad \Rightarrow \quad f(x, y) \geq 0,$$

where $x_t = x + t(y - x)$.

The following result shows that the diagonal subdifferential operator T_f inherits the property of upper sign continuity of the bifunction f.

Theorem 3.5. *If f is an upper sign continuous equilibrium bifunction, then T_f is an upper sign continuous operator.*

Proof. Let $x, y \in \mathbb{X}$ be such that

$$\inf_{x_t^* \in T_f(x_t)} \langle x_t^*, y - x \rangle \geq 0, \quad \forall t \in (0,1),$$

where $x_t = x + t(y - x)$ and, by contradiction suppose that

$$\sup_{x^* \in T_f(x)} \langle x^*, y - x \rangle < 0.$$

Since $\sup_{x^* \in T_f(x)} \langle x^*, y - x \rangle$ is the Clarke directional derivative of $f(x, \cdot)$ at x in the direction $y - x$, it follows that $y - x$ is a descent direction for $f(x, \cdot)$ at x. Then there exists $\varepsilon > 0$ small enough such that $f(x, z) < 0$, where $z = x + \varepsilon(y - x)$. Let $z_s = (1 - s)x + sz = (1 - \varepsilon s)x + \varepsilon s y = x_{\varepsilon s}$. By assumption, for every $s \in (0,1)$

$$\inf_{z_s^* \in T_f(z_s)} \langle z_s^*, z - z_s \rangle = \varepsilon(1 - s) \inf_{z_s^* \in T_f(z_s)} \langle z_s^*, y - x \rangle \geq 0.$$

Since $f(z_s, \cdot)$ is ∂°–pseudoconvex we deduce that $f(z_s, z) \geq f(z_s, z_s) = 0$. But the bifunction f is upper sign continuous and therefore we have the contradiction $f(x, z) \geq 0$. $\square$

The following example shows that the converse of Theorem 3.5 does not hold.

Example 3.6. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} y^2 - (1+x)y + x, & \text{if } x \neq 0 \\ -y, & \text{if } x = 0. \end{cases}$$

Obviously f is an equilibrium bifunction and for every fixed $x \in \mathbb{R}$, the function $f(x,\cdot)$ is locally Lipschitz and ∂°–pseudoconvex since it is convex. Moreover $T_f(x) = \{x-1\}$ is (upper sign) continuous. On the contrary f is not an upper sign continuous bifunction. Indeed, for $x = 0$, $y = 1$ and $x_t = x + t(y-x) = t$, we have $f(x_t, 1) = 0$ for all $t \neq 0$ but $f(0,1) = -1$.

Now we are able to furnish the following result which gives a practical way to show that the diagonal subdifferential operator is D–maximal pseudomonotone.

Corollary 3.7. *If f is a pseudomonotone and upper sign continuous equilibrium bifunction, then T_f is a D–maximal pseudomonotone operator.*

Proof. The result follows from Theorem 3.3. Indeed the pseudomonotonicity of f implies the pseudomonotonicity of T_f (Theorem 2.4), while the upper sign continuity of T_f is a consequence of Theorem 3.5. Lastly, $D(T_f) = \mathbb{X}$ and $T_f(x)$ is weak* compact and convex since it is the Clarke subdifferential of $f(x,\cdot)$. □

The notion of cyclic pseudomonotonicity has been introduced in [12], where classical results of Convex Analysis concerning the cyclic monotonicity of the subdifferential of a convex function were extended to corresponding results for the Clarke-Rockafellar subdifferential of a pseudoconvex function.

Definition 3.8. A multivalued operator $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is said cyclically pseudomonotone if for all $x_1, x_2, \ldots, x_n \in \mathbb{X}$ the following implication holds: if there exists $i \in \{1,2,\ldots,n\}$ and $x_i^* \in T(x_i)$ such that $\langle x_i^*, x_{i+1} - x_i\rangle > 0$, then there exists $j \in \{1,2,\ldots,n\}$ such that $\langle x_j^*, x_{j+1} - x_j\rangle < 0$ for all $x_j^* \in T(x_j)$, where $x_{n+1} = x_1$.

The cyclic pseudomonotonicity can be reformulated in the following equivalent way: the existence of $x_i^* \in T(x_i)$ such that $\langle x_i^*, x_{i+1} - x_i\rangle \geq 0$ for all $i = 1,2,\ldots,n-1$ implies $\langle x_n^*, x_1 - x_n\rangle \leq 0$ for all $x_n^* \in T(x_n)$.

Even if T_f is not the Clarke subdifferential of a function, it is possible to show that is cyclically pseudomonotone if f satisfies a suitable property. We make use of the following definition which has been introduced in [2] in the framework of vector equilibrium problems.

Definition 3.9. An equilibrium bifunction $f : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is said cyclically pseudomonotone if for all $x_1, x_2, \ldots, x_n \in \mathbb{X}$ the following implication holds: if there exists $i \in \{1,2,\ldots,n\}$ such that $f(x_i, x_{i+1}) > 0$, then there exists $j \in \{1,2,\ldots,n\}$ such that $f(x_j, x_{j+1}) < 0$, where $x_{n+1} = x_1$.

As for the multivalued operator, the cyclical pseudomonotonicity of f may be equivalently reformulated in the following way: $f(x_i, x_{i+1}) \geq 0$ for all $i = 1,2,\ldots,n-1$

implies $f(x_n, x_1) \leq 0$. Finally we observe that the cyclical pseudomonotonicities are stronger than the pseudomonotonicities (both for functions and operators). In fact in case $n = 2$ the above definitions turn out to be actually pseudomonotonicities.

Theorem 3.10. *If f is a cyclically pseudomonotone equilibrium bifunction, then T_f is a cyclically pseudomonotone operator.*

Proof. Fix $x_1, x_2, \ldots, x_n \in \mathbb{X}$ and $x_i^* \in T_f(x_i)$ with $i = 1, 2, \ldots, n-1$ such that $\langle x_i^*, x_{i+1} - x_i \rangle \geq 0$. Hence, from the pseudoconvexity of $f(x_i, \cdot)$ we deduce

$$f(x_i, x_{i+1}) \geq f(x_i, x_i) = 0, \quad \forall i = 1, 2, \ldots, n-1.$$

Since f is cyclically pseudomonotone then $f(x_n, x_1) \leq 0$. Applying the Lebourg's Mean Value Theorem to the function $f(x_n, \cdot)$ on the segment $[x_1, x_n]$, we deduce the existence of $z = tx_1 + (1-t)x_n$, with $t \in (0,1)$, and $z^* \in \partial^\circ f(x_n, \cdot)(z)$ such that

$$0 \leq f(x_n, x_n) - f(x_n, x_1) = \langle z^*, x_n - x_1 \rangle = t^{-1} \langle z^*, x_n - z \rangle.$$

Since the operator $\partial^\circ f(x_n, \cdot)$ is pseudomonotone [23], then

$$0 \geq \langle x_n^*, z - x_n \rangle = t \langle x_n^*, x_1 - x_n \rangle, \quad \forall x_n^* \in T_f(x_n)$$

which completes the proof. □

The function of Example 2.5 shows that the converse of Theorem 3.10 does not hold. Indeed, since f is not pseudomonotone, then it is not cyclically pseudomonotone. Nevertheless $T_f(x) = \{Ax\}$ is cyclically monotone (it coincides with the gradient of a quadratic convex function) and hence it is cyclically pseudomonotone. Instead the function of Example 2.6 is trivially cyclically pseudomonotone but T_f is not even pseudomonotone.

4. Characterization of the solution set of equilibrium problems

Pseudomonotone$_*$ single–valued operators were firstly introduced in [10]. Subsequently this concept was extended for multivalued operators in [16]. The importance of pseudomonotonicity$_*$ is double. First of all it is, in some way, a minimal condition ensuring that the cutting plane property for variational inequalities holds independently of the set C. Moreover it is possible to prove that the Clarke subdifferential of a locally Lipschitz ∂°–pseudoconvex function is pseudomonotone$_*$ [16]. This latter result is fundamental for characterizing the solution set of a pseudoconvex minimization problem [8].

Definition 4.1. The multivalued operator $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is pseudomonotone$_*$ if

(i) it is pseudomonotone,

(ii) for every $x, y \in D(T)$ and $x^* \in T(x)$, $y^* \in T(y)$ such that $\langle x^*, y - x \rangle = 0$ and $\langle y^*, x - y \rangle = 0$ then $x^* \in \widehat{T}(y)$ and $y^* \in \widehat{T}(x)$.

D–maximal and cyclic pseudomonotonicity are sufficient conditions for the pseudomonotonicity$_*$ of a multivalued operator [16, Proposition 3.9]. Taking into account Corollary 3.7 and Theorem 3.10, we immediately achieve the pseudomonotonicity$_*$ of the diagonal subdifferential operator T_f.

Corollary 4.2. *Assume that the equilibrium bifunction f is*

(i) *upper sign continuous,*
(ii) *cyclically pseudomonotone.*

Then T_f is a pseudomonotone$_*$ operator.

Now we are up to establishing a characterization of the set of solutions of the equilibrium problem EP(f, C), which will be denoted by $S(f, C)$.

Theorem 4.3. *Assume that the equilibrium bifunction f is*

(i) *upper sign continuous,*
(ii) *cyclically pseudomonotone.*

Suppose that $\bar{x} \in S(f, C)$. Then $S(f, C) = \overline{S}$, where

$$\overline{S} = \{x \in C \ : \ \widehat{T}_f(x) \cap -N(x, C) = \widehat{T}_f(\bar{x}) \cap -N(\bar{x}, C)\}.$$

Proof. We first prove that $S(f, C) \subseteq \overline{S}$. Fix $x \in S(f, C)$. Since x and $\bar{x}$ solve the minimization problems

$$\min\{f(x, y) \ : \ y \in C\} \quad \text{and} \quad \min\{f(\bar{x}, y) \ : \ y \in C\}$$

respectively, we deduce the following necessary optimality conditions:

$$T_f(x) \cap -N(x, C) \neq \emptyset \quad \text{and} \quad T_f(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset.$$

Since $\widehat{T}_f$ is a maximal element with respect to graph inclusion, we have

$$\widehat{T}_f(x) \cap -N(x, C) \neq \emptyset \quad \text{and} \quad \widehat{T}_f(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset.$$

Let $x^* \in \widehat{T}_f(x) \cap -N(x, C)$ and $\bar{x}^* \in \widehat{T}_f(\bar{x}) \cap -N(\bar{x}, C)$ be two fixed elements. Since $-x^*$ and $-\bar{x}^*$ belong to the normal cones to C at x and $\bar{x}$ respectively, we have

$$\langle x^*, \bar{x} - x\rangle \geq 0 \quad \text{and} \quad \langle \bar{x}^*, x - \bar{x}\rangle \geq 0. \tag{1}$$

Moreover, the pseudomonotonicity of $\widehat{T}_f$ implies

$$\langle \bar{x}^*, x - \bar{x}\rangle \leq 0 \quad \text{and} \quad \langle x^*, \bar{x} - x\rangle \leq 0, \tag{2}$$

and, comparing inequalities (1) and (2), we achieve the following equalities:

$$\langle x^*, \bar{x} - x\rangle = 0 = \langle \bar{x}^*, x - \bar{x}\rangle. \tag{3}$$

Corollary 4.2 guarantees that the operator T_f is pseudomonotone$_*$. It has been proved in [16] that if T_1 and T_2 are two equivalent pseudomonotone operators and

T_1 is pseudomonotone$_*$, then T_2 is pseudomonotone$_*$ too. In particular we deduce that $\widehat{T}_f$ is pseudomonotone$_*$. Hence $\bar{x}^* \in \widehat{T}_f(x)$ and $x^* \in \widehat{T}_f(\bar{x})$. Furthermore for any $y \in C$

$$\langle \bar{x}^*, y - x\rangle = \langle \bar{x}^*, y - \bar{x}\rangle + \langle \bar{x}^*, \bar{x} - x\rangle = \langle \bar{x}^*, y - \bar{x}\rangle \geq 0,$$

where the second equality descends from (3), while the inequality is due to the fact that $\bar{x}^* \in -N(\bar{x}, C)$. Thus $\bar{x}^* \in -N(x, C)$. Analogously $x^* \in -N(\bar{x}, C)$ which proves the required inclusion.

Now we focus on the reverse inclusion $\overline{S} \subseteq S(f, C)$ and fix $x \in \overline{S}$. If $x \in Z(T_f)$, then it solves the variational inequality problem $\mathrm{VIP}(T_f, C)$ and, from Theorem 2.2, we deduce that $x \in S(f, C)$. Otherwise, suppose $x \in \overline{S} \setminus Z(T_f)$. Since $\bar{x} \in S(f, C)$, we have just seen that $\widehat{T}_f(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset$. Therefore there exists $\hat{x}^* \in \widehat{T}_f(x) \cap -N(x, C)$. This implies there exist $k > 0$ and $x^* \in T_f(x)$ such that $\hat{x}^* = kx^*$. Since $-N(x, C)$ is a cone, $x^* = k^{-1}\hat{x}^* \in -N(x, C)$. Then $T_f(x) \cap -N(x, C) \neq \emptyset$ or, equivalently, x solves $\mathrm{VIP}(T_f, C)$. Invoking Theorem 2.2 again, the result follows. $\square$

This result extends a previously one stated for a minimization problem. More precisely, consider the minimization problem

$$\min\{\varphi(x) \;:\; x \in C\}, \tag{4}$$

where $\varphi : \mathbb{X} \to \mathbb{R}$ and $C \subseteq \mathbb{X}$. Problem (4) is a particular equilibrium problem setting $f(x, y) = \varphi(y) - \varphi(x)$. Let us observe that f is cyclically pseudomonotone. Applying Theorem 4.3 to this particular equilibrium bifunction, we achieve the following characterization [8].

Corollary 4.4. *Consider the minimization problem* (4)*. Suppose φ is a locally Lipschitz and ∂°–pseudoconvex function, C is a closed and convex set, and $\bar{x} \in S(\varphi, C)$ is a global solution. Define*

$$\overline{S} = \{x \in C \;:\; \widehat{\partial^\circ\varphi}(x) \cap -N(x, C) = \widehat{\partial^\circ\varphi}(\bar{x}) \cap -N(\bar{x}, C)\};$$

then $S(\varphi, C) = \overline{S}$.

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