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The One and Half Ball Property in Spaces of Vector-Valued Functions

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In this paper we exhibit new classes of Banach spaces that have the strong-$1\frac{1}{2}$-ball property and the $1\frac{1}{2}$-ball property by considering direct-sums of Banach spaces. We introduce the notion of sectional strong-$1\frac{1}{2}$-ball property and show that in c_0-direct sum of reflexive spaces, proximinal and factor reflexive spaces with the sectional strong-$1\frac{i}{2}$-ball property have the strong-$1\frac{1}{2}$-ball property. We give examples of proximinal hyperplanes in c_0 that fail the $1\frac{1}{2}$-ball property and show that this property is in general, not preserved under finite intersections or sums. We show that the range of a bi-contractive projection in ℓ^∞ has the strong-$1\frac{1}{2}$-ball property. For a separable subspace $Y \subset X$ with the strong-$1\frac{1}{2}$-ball property and for any positive, σ-finite, non-atomic measure space $(\Omega, \mathcal{A}, \mu)$, we show that $L^1(\mu, Y)$ has the strong-$1\frac{1}{2}$-ball property in $L^1(\mu, X)$. We show that for any compact set Ω and $Y \subset X$ with the $1\frac{1}{2}$-ball property, $C(\Omega, Y)$ has the $1\frac{1}{2}$-ball property in $C(\Omega, X)$.

Keywords: One and half ball property, spaces of vector-valued functions

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1. Introduction

Let X be a real Banach space. Let $B(x, r)$ denote the closed ball centered at x and radius $r > 0$. When $x = 0$ and $r = 1$, we denote the closed unit ball $B(0, 1)$ by X_1. We say that a finite or infinite collection $\{B(x_i, r_i)\}$ of closed balls intersect in X if there exists a $x \in \bigcap B(x_i, r_i)$. We note that two balls, $B(x_1, r_1)$ and $B(x_2, r_2)$ intersect if and only if $\|x_1 - x_2\| \leq r_1 + r_2$. We say that a collection of balls almost intersect, if for any $\epsilon > 0$, there is a $x_\epsilon \in \bigcap B(x_i, r_i + \epsilon)$. We recall ([12]) that a closed subspace $Y \subset X$ is said to have the $1\frac{1}{2}$-ball (strong-$1\frac{1}{2}$-ball property) if for $x \in X$ and $y \in Y$ and for any two ball $B(x, r_1), B(y, r_2)$ with $\|x - y\| \leq r_1 + r_2$ and $B(x, r_1) \bigcap Y \neq \emptyset$, $B(x, r_1), B(y, r_2)$ almost intersect (intersect) in Y. A simple compactness argument shows that if Y is a reflexive space or more generally a weak*-closed subspace of a dual space, $1\frac{1}{2}$-ball property implies the strong-$1\frac{1}{2}$-ball property. In particular if $Y \subset Z \subset X^*$ and Y is a weak*-closed subspace, then if Y has the $1\frac{1}{2}$-ball property in Z, then it has the strong-$1\frac{1}{2}$-ball property in Z.

Intersection properties of balls play an important role in the study of the geometry

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of Banach spaces. See [6] Chapter I for various forms of intersection properties of balls and their implication to the geometry of Banach spaces. It is known that if Y has the $1\frac{1}{2}$-ball property then it is a proximinal subspace of X, i.e. for any $x \in X$ there exists a $y \in Y$ such that $d(x, Y) = \|x - y\|$. In this case y is said to be a best approximation to x in Y. Let $P(x)$ denote the set of best approximations. In fact $1\frac{1}{2}$-ball property (strong-$1\frac{1}{2}$-ball property) is equivalent to the existence of $y \in P(x)$ such that $\|x\| = \|x - y\| + \inf\{\|z\| : z \in P(x)\}$ ($\|x\| = \|x - y\| + \|y\|$). An advantage of $1\frac{1}{2}$-ball property is that it is a self dual property in the sense that Y has the $1\frac{1}{2}$-ball property if and only if $Y^{\perp}$ has the $1\frac{1}{2}$-ball (and hence the strong-$1\frac{1}{2}$-ball property).

Examples of spaces with the $1\frac{1}{2}$-ball property include, span$\{1\}$, in the space $C(K)$ of continuous functions on a compact set, M-ideals, semi-M-ideals, semi-L-summands. See [6] for these concepts and also page 95 of the monograph for more examples.

In [1] the authors have studied a new class of proximinal subspaces called ball proximinal subspaces. $Y \subset X$ is said to be a ball proximinal subspace, if the closed unit ball Y_1 is a proximinal set in X. It is known that such a Y is a proximinal subspace and this class of spaces is smaller than proximinal subspaces. One of the motivation for the authors of [1], is to exhibit certain properties similar to 'compact sets', of ball proximinal subspaces. These ideas were taken a step forward recently by Indumathi and Lalithambigai ([7]). Combining Theorem 2.4 and Theorem 2.11 of [7], we get the following result.

Theorem 1.1. *Let $Y \subset X$ be a closed subspace. Y has the strong-$1\frac{1}{2}$-ball property if and only if Y has the $1\frac{1}{2}$-property and Y is ball proximinal.*

While dealing these properties for spaces of vector-valued functions one can more often easily show that the $1\frac{1}{2}$-ball property gets preserved and then show that the subspace is ball proximinal and appeal to the above theorem to conclude the strong-$1\frac{1}{2}$-ball property.

In the first section of the paper we prove stability results under c_0, ℓ^1 and ℓ^∞ sums for the $1\frac{1}{2}$-ball property and the strong $1\frac{1}{2}$-ball property. Using the duality of the $1\frac{1}{2}$-ball property, we show that for any family of Banach spaces $\{X_i\}_{i \in I}$ and closed subspaces $Y_i \subset X_i$ with the strong-$1\frac{1}{2}$-ball property for all $i \in I$, the c_0-direct sum, $\bigoplus_{c_0} Y_i$ has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_{c_0} X_i$.

For a reflexive subspace $Y \subset X$ with the $1\frac{1}{2}$-ball property we show that the ℓ^1-direct sum, $\bigoplus_{\ell^1} Y$ has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_{\ell^1} X$.

For notational convenience we have dropped the indexing set while writing the direct sum spaces. Also when all the spaces coincide with a Banach space X, we denote direct sum by $\bigoplus X$. Throughout the paper we assume that all Banach spaces considered are non-trivial.

We give examples to show that unlike proximinality, $1\frac{1}{2}$-ball property is difficult to handle even in finite dimensional spaces. It is in general not transitive and not stable under finite intersections. In order to deal with these stability questions in infinite dimensional, for proximinal and factor reflexive spaces, we introduce the notion of sectional $1\frac{1}{2}$-ball property and show that in c_0-direct sum of reflexive spaces, these have the strong-$1\frac{1}{2}$-ball property.

It well-known that the kernel of a projection of norm one in X is a proximinal subspace. We next study the strong-$1\frac{1}{2}$-ball property for the range of a projection P such that $\|P\| = \|I - P\| = 1$, the so called bi-contractive projections. We show that such subspaces have the strong-$1\frac{1}{2}$-ball property in ℓ^1 and ℓ^∞.

In the third section we study these properties in the space of Bochner integrable functions and vector-valued continuous functions. If $Y \subset X$ is a separable subspace with the strong-$1\frac{1}{2}$-ball property, we show that the space of Bochner integrable functions, $L^1(\mu, Y)$ has the strong-$1\frac{1}{2}$-ball property in $L^1(\mu, X)$ for the Lebesgue measure μ on $[0, 1]$. We also show that the classical proof of proximinality of $L^1(\lambda, Y)$ in $L^1(\lambda, X)$ for a reflexive subspace Y can be used to establish that strong-$1\frac{1}{2}$-ball property gets preserved.

For a finite dimensional space $Y \subset X$ with the $1\frac{1}{2}$-ball property and for any extremally disconnected compact Hausdorff space Ω, we show that the space of Y-valued continuous functions, $C(\Omega, Y)$ has the strong-$1\frac{1}{2}$-ball property in $C(\Omega, X)$.

2. Sequence spaces

Let $\{X_i\}$ be a family of Banach spaces and let $\bigoplus_{\ell^1} X_i$, $\bigoplus_{\ell^\infty} X_i$ and $\bigoplus_{c_0} X_i$ denote the ℓ^1, ℓ^∞ and c_0-direct sum of these spaces with the usual norm. Let $Y_i \subset X_i$ be closed subspaces. It is fairly routine to see that $\bigoplus_{\ell^\infty} Y_i$ has the strong-$1\frac{1}{2}$-ball property or $1\frac{1}{2}$-ball property in $\bigoplus_{\ell^\infty} X_i$ if and only if Y_i' have the same.

Theorem 2.1. *Let $\{X_i\}$ be a family of Banach spaces and let $Y_i \subset X_i$ be closed subspaces. If Y_i's have the $1\frac{1}{2}$-ball property, so does $\bigoplus_{\ell^1} Y_i$ and $\bigoplus_{c_0} Y_i$ in $\bigoplus_{\ell^1} X_i$ and $\bigoplus_{c_0} X_i$ respectively. If Y_i's have the strong-$1\frac{1}{2}$-ball property so does $\bigoplus_{c_0} Y_i$.*

Proof. Since $(\bigoplus_{\ell^1} X_i)^* = \bigoplus_{\ell^\infty} X_i^*$ and since $(\bigoplus_{\ell^1} Y_i)^\perp = \bigoplus_{\ell^\infty} (Y_i)^\perp$, if Y_i's have the $1\frac{1}{2}$-ball property, from our remark above and by duality, we get that $\bigoplus_{\ell^1} Y_i$ has the $1\frac{1}{2}$-ball property.

In a similar way using the identification $(\bigoplus_{c_0} X_i)^* = \bigoplus_{\ell^1} X_i^*$ and the earlier remark, it is easy to deduce that $\bigoplus_{c_0} Y_i$ has the $1\frac{1}{2}$-ball property when ever Y_i's have the $1\frac{1}{2}$-ball property.

Now suppose every $Y_i \subset X_i$ has the strong-$1\frac{1}{2}$-ball property. By Theorem 1.1, we have that Y_i's are ball proximinal. As remarked in [1], $\bigoplus_{c_0} Y_i$ is ball proximinal in $\bigoplus_{c_0} X_i$. Therefore by Theorem 1.1 again we get that $\bigoplus_{c_0} Y_i$ has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_{c_0} X_i$. □

Remark 2.2. We do not know if Y_i's have the strong-$1\frac{1}{2}$-ball property in X_i, then $\bigoplus_{\ell^1} Y_i$ also has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_{\ell^1} X_i$. As remarked in [1] we do not know if ball proximinality is preserved under ℓ^1-direct sums.

Proposition 2.3. *Let $Y_i \subset X_i$ be reflexive subspaces with the $1\frac{1}{2}$-ball property. Then $\bigoplus_{\ell^1} Y_i$ has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_{\ell^1} X_i$.*

Proof. Let X_i be canonically embedded in its bidual X_i^{**}. Since Y_i' are reflexive, there exist closed subspaces $Z_i \subset X_i^*$ such that $Z_i^\perp = Y_i$. It is easy to see that $(\bigoplus_{c_0} Z_i)^\perp = \bigoplus_{\ell^1} Y_i \subset \bigoplus_{\ell^1} X_i \subset \bigoplus_{\ell^1} X^{**} = (\bigoplus_{c_0} X_i^*)^*$ is a weak*-closed subspace with the $1\frac{1}{2}$-ball property. Thus by our earlier remarks, $\bigoplus_{\ell^1} Y_i$ has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_{X_i}$. $\square$

In [12] Payá and Yost gave examples in $\ell^1(3)$ to shows that $1\frac{1}{2}$-ball property is in general not transitive. We next show that transitivity can be achieved if one of the subspaces satisfies better intersection properties of balls. We recall that a closed subspace $Y \subset X$ is said to be a M-ideal if there exists a projection $P : X^* \to X^*$ such that $\mathrm{Ker}(P) = Y^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. Such spaces satisfy the 3-ball intersection property and hence have the strong-$1\frac{1}{2}$-ball property. Also $X^{**} = Y^{\perp\perp} \bigoplus_\infty N$ for some closed subspace $N \subset X^{**}$. See [6] Chapter I. Being a M-ideal is transitive, preserved under finite intersections and sums.

Proposition 2.4. *Let $Z \subset Y \subset X$ be closed subspaces such that Z has the $1\frac{1}{2}$-ball property in Y and Y is a M-ideal. Then Z has the $1\frac{1}{2}$-ball property in X.*

Proof. We use the duality and show that $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in X^{**}. Note that $Z^{\perp\perp} \subset Y^{\perp\perp} \subset X^{**}$. Since $X^{**} = Y^{\perp\perp} \bigoplus_\infty N$ and by duality again $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in $Y^{**} = Y^{\perp\perp}$, by our earlier remarks it follows that $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in X^{**}. Hence Z has the $1\frac{1}{2}$-ball property in X. $\square$

It is well-known and easy to see that for any family of Banach space $\{X_i\}$, $\bigoplus_{c_o} X_i$ is a M-ideal in $\bigoplus_{\ell^\infty} X_i$ (see [6]). We can use the above proposition to easily improve on Theorem 2.1.

Corollary 2.5. *Let $\{X_i\}$ be a family of Banach spaces and let $Y_i \subset X_i$ be closed subspaces. If Y_i's have the $1\frac{1}{2}$-ball property, $\bigoplus_{c_0} Y_i$ has the $1\frac{1}{2}$-ball property in $\bigoplus_{\ell^\infty} X_i$.*

Remark 2.6. Let $Y \subset X$ be a M-ideal and $Z \subset X$ be a closed subspace. Then $Y \cap Z$ has the $1\frac{1}{2}$-ball property in X if and only if it has the $1\frac{1}{2}$-ball property in Y and hence in Z.

Motivated by the above considerations, we now introduce the concept of sectional $1\frac{1}{2}$-ball property for spaces that are rich in M-ideals. We introduce this definition specifically to deal with the case of c_0 and ℓ^∞ direct sums of spaces, where there is a canonical way of generating non-trivial M-ideals. This may be taken as an auxiliary notion.

Definition 2.7. A closed subspace $Y \subset X$ is said to have the sectional $1\frac{1}{2}$-ball property, if for every proper M-ideal $Z \subset X$, $Y \cap Z$ has the $1\frac{1}{2}$ ball property in Z.

We recall that $Z \subset X$ is said to be factor reflexive, if the quotient space X/Z is a reflexive space. For an infinite family $\{X_i\}$ of reflexive spaces, $\bigoplus_\infty X_i$ is the bidual of $\bigoplus_{c_0} X_i$. As noted before, $\bigoplus_{c_0} X_i$, being a M-ideal, has the strong -$1\frac{1}{2}$-ball property in $\bigoplus_\infty X_i$.

Theorem 2.8. *Let $\{X_i\}$ be an infinite family of reflexive Banach spaces. Let $X = \bigoplus_{c_0} X_i$. Let $\mathcal{N}$ denote the class of $Y \subset X$ that are factor reflexive, proximinal subspaces that have the sectional $1\frac{1}{2}$-ball property. Then any $Y \in \mathcal{N}$ has the strong-$1\frac{1}{2}$-ball property in X as well as in X^{**}. For $Y, Z \in \mathcal{N}$, if $Y \cap Z$ has the sectional $1\frac{1}{2}$-ball property, then $Y \cap Z$ is proximinal and hence is in $\mathcal{N}$.*

Proof. For any $f \in Y^\perp = (X/Y)^*$, by reflexivity, f attains its norm and since Y is proximinal, f attains its norm on X. Thus if $NA(X)$ denoted the set of all norm attaining functionals in X^*, then $Y^\perp \subset NA(X)$. Since $X^* = \bigoplus_{\ell^1} X_i^*$, it is easy to see that for any $f \in NA(X)$ there exist a finite index set A such that $f(i) = 0$ for $i \notin A$. As $Y^\perp$ is a Banach space, a simple Baire category argument gives that there is a finite index set A such that $f(i) = 0$ for all $i \notin A$ and for all $f \in Y^\perp$. Thus under the canonical identification, $Y^\perp \subset \bigoplus_{i\in A} X_i^* \subset X^*$. Also note that $X = \bigoplus_{i\in A} X_i \bigoplus_\infty \bigoplus_{i\notin A} X_i$. Here the first summand is a finite ℓ^∞-sum and the last summand is a c_0-direct sum. Thus we have that $Y = Y \cap \bigoplus_{i\in A} X_i \bigoplus_\infty \bigoplus_{i\notin A} X_i$.

Since A is a finite set, $\bigoplus_{i\in A} X_i$ is a reflexive space. It is also a M-summand in X. Thus by hypothesis, $Y \cap \bigoplus_{i\in A} X_i$ has the $1\frac{1}{2}$-ball in X and hence strong-$1\frac{1}{2}$-ball property in X. As $\bigoplus_{i\notin A} X_i$ is common to both Y and X, it is easy to see that Y has the strong-$1\frac{1}{2}$-ball property.

Note that since $X^{**} = \bigoplus_{i\in A} X_i \bigoplus_\infty \bigoplus_{i\notin A} X_i$ (the last summand is the ℓ^∞-direct sum). Similar arguments show that Y has the strong-$1\frac{1}{2}$-ball property in X^{**}.

Let $Y, Z \in \mathcal{N}$. It is easy to see that there is a finite index set A, such that $f(i) = 0$ for all $i \notin A$ and $f \in Y \cap Z$. Thus as before, $Y \cap Z = (Y \cap Z) \cap \bigoplus_{i\in A} X_i \bigoplus_\infty N$ for some closed subspace N. Since $\bigoplus_{i\in A} X_i$ is a reflexive space, $(Y \cap Z) \bigoplus_{i\in A} X_i$ is a proximinal space. Thus as before $Y \cap Z$ is proximinal in X and hence $Y \cap Z \in \mathcal{N}$. Hence $Y \cap N$ has the strong-$1\frac{1}{2}$-ball property. □

The following corollaries are easy to prove using the arguments given during the proof of the above theorem.

Corollary 2.9. *Let Γ be an infinite discrete set. Any proximinal subspace $Y \subset c_0(\Gamma)$ of finite co-dimension that has the sectional $1\frac{1}{2}$-ball property has the strong-$1\frac{1}{2}$-ball property in $\ell^\infty(\Gamma)$.*

Corollary 2.10. *Let $\{X_i\}$ be an infinite family of reflexive Banach spaces. Let $X = \bigoplus_\infty X_i$. Let $Y \subset X$ be a factor reflexive, subspace that any $f \in Y^\perp$ has at most finitely many non-zero coordinates and has the sectional $1\frac{1}{2}$-ball property. Then Y has the strong-$1\frac{1}{2}$-ball property in X.*

It is known that proximinality notions behave well in the class of subspaces of finite co-dimension of c_0, see [5]. In particular for this class, proximinality is transitive and intersection of two spaces in this class is again in this class. We next examine this stability for the $1\frac{1}{2}$-ball property. From our analysis above it is clear that it is enough to consider these questions for subspaces of $\ell^\infty(k)$. By duality this also leads to answers in $\ell^1(k)$. Also when $k = 2$, since $(x, y) \to (\frac{x+y}{2}, \frac{x-y}{2})$ is an isometry between $\ell^\infty(2)$ and $\ell^1(2)$, the answers are identical. As noted before the diagonal $(\alpha, \alpha, ..., \alpha)$ has the $1\frac{1}{2}$-ball property in any $\ell^\infty(k)$. Since permutations and multiplication of coordinates by ± 1 are isometries of ℓ^∞ more examples can be generated from this example. Also for any subset A of the coordinate basis, span$\{A\}$ has the $1\frac{1}{2}$-ball property.

We first note that in the case of $\ell^\infty(2)$, the above described examples are the only ones with the $1\frac{1}{2}$-ball property.

Example 2.11. $Y = \text{span}\{(1, 2)\}$ fails the $1\frac{1}{2}$-ball property in $\ell^\infty(2)$. Let $a = (2, 4)$ and $b = (-1, 2)$. Consider the balls $B(a, \frac{5}{3})$ and $B(b, \frac{4}{3})$. $\|a - b\| = 3$ and for $y = (\frac{1}{3}, \frac{2}{3}) \in Y$, $\|b - y\| = \frac{4}{3}$. It is easy to see that $Y \cap B(a, \frac{5}{3}) \cap B(b, \frac{4}{3}) = \emptyset$. Similar arguments show that span$\{(\alpha, \beta)\}$ fails the $1\frac{1}{2}$-ball property, unless α *or* $\beta = 0$, $|\alpha| = 1 = |\beta|$.

By using the canonical embedding into c_0, we thus have examples of proximinal hyperplanes in c_0 that fail the $1\frac{1}{2}$-ball property. We next recall an example (Example 6) from [12] to show that intersection of two spaces with the $1\frac{1}{2}$-ball property can fail to have the $1\frac{1}{2}$-ball property.

Example 2.12. Let $Y = \{(x, y, z) : x = y\}$. Since the diagonal has the $1\frac{1}{2}$-ball property in $\ell^1(2)$, we see that Y has the $1\frac{1}{2}$-ball property in $\ell^1(3)$. Let $Z = \{(x, y, z) : z = x + y\}$, since the annihilator, span$\{(1, 1, -1)\}$ has the $1\frac{1}{2}$-ball property in $\ell^\infty(3)$, we get that Z has the $1\frac{1}{2}$-ball property in $\ell^1(3)$. Note that $Y \cap Z = \{(\alpha, \alpha, 2\alpha)\}$ fails the $1\frac{1}{2}$-ball property in $\ell^1(3)$, as noted in [12]. Thus the intersection of two spaces with the strong-$1\frac{1}{2}$-ball property can fail to have the strong-$1\frac{1}{2}$-ball property. Since $(Y \cap Z)^\perp = Y^\perp + Z^\perp$, by duality we see that the sum of two spaces (closed, since the spaces are finite dimensional) with the strong-$1\frac{1}{2}$-ball property can fail to have the strong-$1\frac{1}{2}$-ball property.

Remark 2.13. We do not know a complete description of subspaces of $\ell^\infty(k)$ $(k > 2)$ that have the $1\frac{1}{2}$-ball property. We also do not have an example of a subspace with sectional $1\frac{1}{2}$-ball property but not the $1\frac{1}{2}$-ball property.

We next consider the strong-$1\frac{1}{2}$-ball property for ranges of projections P with $\|P\| = 1$ and $\|I - P\| = 1$, the so called bi-contractive projections. It is easy to see that the range of such a projection is a proximinal subspace. We are interested in knowing when such subspaces have the strong-$1\frac{1}{2}$-ball property. Note that if P is a bi-contractive projection then so is P^*, thus duality can be used again to deduce the strong-$1\frac{1}{2}$-ball property.

Theorem 2.14. *The range of bi-contractive projection in ℓ^∞ and ℓ^1 have the strong-$1\frac{1}{2}$-ball property.*

Proof. Let P be a bi-contractive projection in ℓ^∞. We use the results from [4] that describe such projections in terms isometries involving involutions. There exists a permutation ϕ of natural numbers that is an involution such that $P(x) = \frac{x\circ\phi - x}{2}$ for all $x \in \ell^\infty$. Since ϕ is an involution, it has fixed points and transpositions (permutations of two elements). Let A denote the set of fixed points of ϕ. Clearly $P(x)(k) = 0$ for all $k \in A$. For $k, l \notin A$ since $\phi(k) = l$ and $\phi(l) = k$, we have that $P(x)(k) = \frac{x(l)-x(k)}{2} = -P(x)(l)$. Thus in the part corresponding to a transposition we have that some α and $-\alpha$ appear. We can decompose A^c into orbits of ϕ. We can treat ℓ^∞ as the ℓ^∞-direct sum of $\ell^\infty(k)$ according to this decomposition.

Let Y be the range of P. Let $x \in \ell^\infty$, $y, y' \in Y$ be such that $\|x - y\| \leq r_1 + r_2$, $\|x - y'\| \leq r_1$. We recall that permutations and multiplication by signs are isometries. Since the one-dimensional span of any of $\{(\pm 1, \pm 1)\}$ has the $1\frac{1}{2}$-ball property, it follows that span$\{y, y'\} \subset Y$ has the strong-$1\frac{1}{2}$-ball property in ℓ^∞. Thus there is a $y_0 \in Y$ such that $\|x - y_0\| \leq r_1$ and $\|y - y_0\| \leq r_2$. Hence Y has the strong-$1\frac{1}{2}$-ball property.

Now let $M \subset \ell^1$ be the range of a bi-contractive projection. Since P^* is a bi-contractive projection in ℓ^∞ we get that $M^\perp$ has the strong-$1\frac{1}{2}$-ball property and thus M has the $1\frac{1}{2}$-ball property. It is known that the range of a projection of norm one in ℓ^1 is weak*-closed w.r.t. the duality, $c_0^* = \ell^1$ (one way of deducing this in the current situation is to consider the concrete representation of P^* given above and note that this is the bi-transpose of a projection in c_0). Therefore M has the strong-$1\frac{1}{2}$-ball property. □

In the next proposition we use the specific description of hyperplanes in c_0 that are ranges of bi-contractive projections due to Baronti and Papini [3]. As before these subspaces have the strong-$1\frac{1}{2}$-ball property also in ℓ^∞ as c_0 is a M-ideal in ℓ^∞.

Proposition 2.15. *Let $Y \subset c_0$ be a subspace of finite co-dimension. Suppose Y is an intersection of hyperplanes that are ranges of bi-contractive projections. Then Y has the strong-$1\frac{1}{2}$-ball property.*

Proof. Suppose $Y = \cap_1^k \ker(f_i)$, where $f_i \in \ell^1$ and each $\ker(f_i)$ is the range of a bi-contractive projection.

For any i, as $\ker(f_i)$ is the range of a bi-contractive projection it follows from [3] that either f_i has only one non-zero coordinate or exactly two non-zero components which have the same absolute value. Thus we may assume w.l.o.g. that $f_i = e_i$ or $f_i = e_i - e_j$ for a $j \neq i$. In either case we have that span$\{f_i\}$ has the $1\frac{1}{2}$-ball property so that $\ker(f_i)$ has the $1\frac{1}{2}$-ball property. Since permutations and multiplication by signs are isometries the case of finite intersections can be similarly handled. □

3. Function spaces

In this section we study the $1\frac{1}{2}$-ball property and the strong-$1\frac{1}{2}$-ball property for spaces of Bochner integrable functions and for spaces of vector-valued continuous functions.

Theorem 3.1. *Let X be a Banach space and let $Y \subset X$ be a separable subspace with the strong $1\frac{1}{2}$-ball property. Let μ denote the Lebesgue measure on $[0,1]$. $L^1(\mu, Y)$ has the strong -$1\frac{1}{2}$-ball property in $L^1(\mu, X)$.*

Proof. Since Y has the strong-$1\frac{1}{2}$-ball property it is a proximinal subspace. As Y is separable by a result of Mendoza, [11] we get that $L^1(\mu, Y)$ is a proximinal subspace of $L^1(\mu, X)$. Let $f \in L^1(\mu, X)$. We only need the show that a best approximation $g \in L^1(\mu, Y)$ satisfies, $\|f - g\| + \|g\| \leq \|f\|$. Since g is a best approximation, it is well-known, (see [9] Corollary 2.11) that, $\|g(t) - f(t)\| = d(f(t), Y)$ a.e. Also this equation defines a measurable function of t. Since Y is separable and, f is a.e. separably-valued, by discarding a null set we may assume that, the functions are every where defined.

For $t \in [0,1]$, let $\mathcal{N}(f(t)) = \{y \in Y : d(f(t), Y) = \|f(t) - y\|\}$. Define a set-valued function, $\mathcal{H} : [0,1] \to 2^Y$ by

$$\mathcal{H}(t) = \{y \in \mathcal{N}(f(t)) : \|f(t)\| = \|f(t) - y\| + \|y\|\}.$$

Since Y has the strong-$1\frac{1}{2}$-ball property, $\mathcal{H}(t) \neq \emptyset$ for all t. In order to apply von-Neumann measurable selection theorem, we need to show that the graph of $\mathcal{H}$, $\{(t, y) : y \in \mathcal{H}(t)\}$ is a measurable set in the product space. But this is same as $\{(t, y) : \|f(t)\| = \|f(t) - y\| (= d(f(t), Y)) + \|y\|\}$. As all the functions involved are measurable, the set under consideration is measurable. Therefore the graph is a measurable set. Thus by Von-Neumann's theorem ([13] Corollary 5.5.8) there is a measurable function $H : [0,1] \to Y$ such that $H(t) \in \mathcal{H}(t)$ for all t. Since H is separably valued, it is strongly measurable. Further as $H(t)$ is a point-wise best approximation in Y, for $f(t)$, we get that $H \in L^1(\mu, Y)$ and $d(f, L^1(\mu, Y)) = \|f - H\|$.

Thus we only need to show that $\|f - H\| + \|H\| \leq \|f\|$. Again since $H(t) \in \mathcal{H}(t)$, we have $\|f(t) - H(t)\| + \|H(t)\| \leq \|f(t)\|$ a.e. Therefore by integrating on both sides we get that $\|f\| = \|f - H\| + \|H\|$ and $d(f, Y) = \|f - H\|$. □

Remark 3.2. Since Bochner integrable functions have σ-finite support and are a.e. separably-valued, standard application of the Borel isomorphism theorem will allow us to prove the above theorem for general non-atomic, σ-finite countably generated measure spaces.

Remark 3.3. Let $Y \subset X$ be a separable space with the strong-$1\frac{1}{2}$-ball property. We now get from Theorem 1.1 that, $L^1(\mu, Y)$ is ball proximinal in $L^1(\mu, X)$. Under the assumption that Y is a separable ball proximinal space, we could show in [1] that $L^1(\mu, Y_1)$ is a proximinal set. Whether $L^1(\mu, Y)$ is ball proximinal or not is still an open problem. We still do not know how to deal with $1\frac{1}{2}$-ball property in this situation.

We next prove a measurable version of Proposition 2.3 in Section 2. Our proof relies on the proof of the proximinality of $L^1(\lambda, Y) \subset L^1(\lambda, X)$ for reflexive subspace $Y \subset X$. See Theorem 2.13 of [9].

Theorem 3.4. *Let $(\Omega, \mathcal{A}, \lambda)$ be a positive measure space. Let $Y \subset X$ be a reflexive Banach space with the $1\frac{1}{2}$-ball property. Then $L^1(\lambda, Y)$ has the strong-$1\frac{1}{2}$-ball property in $L^1(\lambda, X)$.*

Proof. Since we have already covered the discrete version in Proposition 2.3, we assume w.l.o.g. that the measure space is non-atomic. Let $f \in L^1(\lambda, X)$. Since $d(f, L^1(\lambda, Y)) = \lim \|f - h_n\|$ for a sequence $\{h_n\} \subset L^1(\lambda, Y)$ and as integrable functions have σ-finite support, we may assume w.l.o.g. that λ is a σ-finite measure.

As before we need to exhibit a point-wise best approximation g that satisfies, $\|f - g\| + \|g\| \leq \|f\|$. As in the proof of the classical case, let $\{s_n\} \subset L^1(\lambda, X)$ be a sequence of simple functions such that $s_n \to f$. Again using result for the discrete case, and since Y has the strong-$1\frac{1}{2}$-ball property, we have a sequence $\{s'_n\} \subset L^1(\lambda, Y)$ of simple functions such that $\|s_n(t) - s'_n(t)\| + \|s'_n(t)\| = \|s_n(t)\|$ a.e. and $d(s_n(t), Y) = \|s_n(t) - s'_n(t)\|$ a.e. Thus $\|s_n - s'_n\| + \|s'_n\| = \|s_n\|$.

Now using the reflexivity of Y and the Dunford and Pettis theorem, as in the classical case, there exists a subsequence, still denoted by $\{s'_n\}$ and a $g \in L^1(\lambda, Y)$ such that, $d(f, L^1(\lambda, Y)) = \|f - g\|$, $s'_n \to g$ in the weak topology. Note that $s_n - s'_n \to f - g$ in the weak topology. By the lower semi-continuity of the norm in the weak-topology, $\|f-g\|+\|g\| \leq \liminf \|s_n - s'_n\| + \|s'_n\|$. Now since $\|s_n\| \to \|f\|$, we get that $\|f - g\| + \|g\| \leq \|f\|$. □

Remark 3.5. Arguments given during the proof of the above theorem can be used to give a direct proof of the result that for a reflexive subspace Y, $L^1([0,1], \mu, Y)$ is ball proximinal in $L^1([0,1], \mu, X)$. This was proved by different methods in [1].

We next consider these questions in the space $C(\Omega, X)$, the space of X-valued continuous functions defined on extremally disconnected Hausdorff spaces Ω, equipped with the supremum norm. See Chapter 3, Section 11 of [8] for properties of $C(\Omega)$ for extremally disconnected compact spaces Ω.

Theorem 3.6. *Let Ω be an extremally disconnected compact Hausdorff space. Let $Y \subset X$ be a finite dimensional space having the $1\frac{1}{2}$-ball property. Then $C(\Omega, Y)$ has the strong-$1\frac{1}{2}$-ball property in $C(\Omega, X)$.*

Proof. Let S be a discrete set. We will first prove the theorem for the Stone-Čech compactification, $\Omega = \beta(S)$. Since Y is finite dimensional, it is easy to see that $C(\beta(S), Y)$ is isometric to $\bigoplus_\infty Y$ where the sum is taken over the cardinal $|S|$. Similarly one can show that $C(\beta(S), X)$ is isometric to a subspace of $\bigoplus_\infty X$. By our earlier remarks, we have that $C(\beta(S), Y)$ has the strong-$1\frac{1}{2}$-ball property in $\bigoplus_\infty X$ and hence in $C(\beta(S), X)$.

Since Ω is extremally disconnected, there exists a discrete set S such that Ω is homeomorphic to a subset of $\beta(S)$ and there exists a retract $\phi : \beta(S) \to \Omega$ (see [8]). Now considering the balls in $C(\Omega, X)$ with one center from $C(\Omega, Y)$ and composing with ϕ, one can use the result already established for $\beta(S)$. Now restricting the functions to Ω, as ϕ is a retract, gives that $C(\Omega, Y)$ has the strong-$1\frac{1}{2}$-ball property in $C(\Omega, X)$. □

We next adapt a proof from [10] for proving intersection properties of balls in $C(\Omega, X)$, to deal with the $1\frac{1}{2}$-ball property for general compact spaces.

Theorem 3.7. *Let Ω be a compact Hausdorff space and let $Y \subset X$ have the $1\frac{1}{2}$-ball property. $C(\Omega, Y)$ has the $1\frac{1}{2}$-ball property in $C(\Omega, X)$.*

Proof. Let $f \in C(\Omega, X)$ and $g, g' \in C(\Omega, Y)$ such that $\|f - g\| \leq r_1 + r_2$ and $\|f - g'\| \leq r_1$. Let $\epsilon > 0$. As noted in [10] Corollary 4.2, there exists subspaces $Z_1 \subset C(\Omega, X)$ and $Z_2 \subset C(\Omega, Y)$ isometric to finite ℓ^∞-direct sums of X and Y respectively (we can take the same finite index set by repeating the spaces if necessary) such that $d(f, Z_1), d(g, Z_2), d(g', Z_2)$ are all smaller than ϵ. Now using the observation that $1\frac{1}{2}$-ball property is preserved under finite ℓ^∞-sums, it can be seen that $C(\Omega, Y)$ has the $1\frac{1}{2}$-ball property in $C(\Omega, X)$. $\square$

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