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Smooth Selections of Convex-Valued Multifunctions

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We establish a class of multifunctions having smooth (C^∞) selections and formulate assumptions on a multifunction F under which for any continuous selection f of F there is a sequence of smooth selections of F converging uniformly to f. Moreover, we obtain a Castaing type representation of multifunctions by a sequence of smooth selections, i.e. we construct a sequence $\{f_k\}$ of smooth selections of F satisfying the condition $F(x) = \overline{\cup_{k\geq 1}\ f_k(x)}$ for all $x \in X$.

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1. Introduction

Let F be a multifunction from an open set $X \subset R^m$ to nonempty convex subsets of R^n. A function $f : X \mapsto R^n$ is a selection of F if $f(x) \in F(x)$ for all $x \in X$. Due to the axiom of choice at least one such mapping exists if all $F(x)$ are nonempty.

This paper is concerned with the problem of regularity of selections. We find conditions on F under which:

(a) There exists a smooth ($C^\infty(X)$) selection of F.
(b) For given isolated points $(x_i, y_i) \in \text{graph}(F)$ there is a smooth selection φ of F such that $\varphi(x_i) = y_i$ for every i.
(c) For every continuous selection f of F there is a sequence of smooth selections of F converging uniformly to f.
(d) There exists a sequence $\{f_k\}$ of smooth selections of F satisfying the density condition that $F(x) = \overline{\cup_{k\geq 1}\ \{f_k(x)\}}$ for all $x \in X$.
(e) The space $CS(F)$ of continuous selections of F equipped with the supremum norm contains a countable dense subset (of smooth selections of F).

We use the technique of convolutions. Without going into details, to obtain a smooth selection we prove that a convolution of a suitably chosen smooth function and a continuous selection of F is a smooth selection of F.

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It is well known that the convolution-type operator is a tool to obtain smooth mappings. For instance, in [7] such operator is considered and smoothing effects are proved. Our aim is to show that a convolution-type operator is a proper tool to get smooth selections.

In order to get a smooth selection φ passing through given points (x_i, y_i) we need the existence of a continuous selection g such that $g(x_i) = y_i$. A suitably chosen convolution-type operator transforms a continuous selection g into a smooth selection φ.

To solve the problem (c) we find a sequence of convolution type operators. We construct a sequence of smooth mappings $\{\eta_k(\cdot)\}$ and an auxiliary continuous selection $a(\cdot)$ such that the convolutions of $a(\cdot)$ and $\eta_k(\cdot)$ provide a sequence $\{f_k\}$ of smooth selections convergent uniformly to the given continuous selection f.

The solution to the problem (d) is based on the construction of a sequence $\{F_k\}$ of multifunctions whose graphs are contained in the graph of F. This sequence is created in such a way that each F_k has a smooth selection f_k. The condition $F(x) = \overline{\cup_{k\geq 1} \{f_k(x)\}}$ for all $x \in X$ follows from the construction of $\{F_k\}$ and from earlier considerations.

The result on the existence of a countable dense subset of the space of continuous selections of F equipped with the supremum norm is related to the problem (e). The relationship between the existence of such a countable dense subset and the existence of a smooth Castaing type representation is discussed. A few examples showing the necessity of some assumptions are given.

The regularity of selections was studied by many authors. After several papers on continuous selections by Michael [12, 13], various types of regularity of selections have been considered. It was proved that a sufficiently regular multifunction has a measurable, a Lipschitz continuous or a differentiable selection (see for instance [10, 6, 14], respectively). There are also results on the existence of smooth selections ([6, 5]).

There are many different ways to prove the existence of smooth selections. Some recent papers establish only the existence of smooth selections while other also give its explicit formula. A candidate for a formula for a differentiable selection is the Steiner point of $F(x)$ (in [8]) or the generalized Steiner point of $F(x)$ (in [3], Thm. 3.11).

Other results on the existence of smooth selections are [3], Thm. 4.3, [6], Prop. 5 and [5], Thm. 3.7. In [3], Thm. 4.3, F is a continuous multifunction defined on a linear normed space with convex closed images in R^n. The existence of a directionally differentiable selection of F is obtained via metric projections.

In [6], Prop. 5, F is acting from a closed subset X of R^m to nonempty convex subsets of R^n. Assuming that the preimages $F^{-1}(y)$ are open in X for all $y \in R^n$ and employing a smooth (C^∞) partition of unity on R^m, the existence of a smooth selection of F is proved. This result is generalized in [5], Thm. 3.7. Here X is an infinite dimensional Banach space, Y is a linear normed space and the preimages of the multifunction $F : X \mapsto 2^Y$ are open in X for all $y \in F(X)$. Then, under suitable assumptions on F and X, if X admits a C^k-smooth bump function, it is shown that F admits a C^k-smooth selection.

In comparison to these results, in this paper a class of multifunctions defined on a finite dimensional space is considered. In order to get a smooth selection we construct a convolution kernel for a local smoothing of a continuous selection.

A suitable convolution-type formula describes a smooth selection. This is an explicit-type formula, however, it is only local, and, in order to get a smooth selection defined on the whole domain of the multifunction F, it is necessary to use a smooth partition of unity on R^n. Under suitable assumptions, locally we have a formula for a smooth selection of F also if the Steiner selection of F is not continuous. The convolution-type formula can be modified in such a way that it is possible to construct a smooth selection passing through some given points.

In [2] Castaing provided a characterization of measurable multifunctions known as the Castaing Representation Theorem. Assuming that (X, M_X, μ_X) is a complete σ-finite measure space and Y is a complete separable metric space, Castaing proved that a measurable multifunction $F : X \mapsto 2^Y$ with nonempty closed images has a sequence $\{f_k\}$ of measurable selections satisfying the following condition

$$\forall_{x\in X} \quad F(x) = \overline{\cup_{k\geq 1} \, \{f_k(x)\}}. \tag{1}$$

A sequence $\{f_k\}$ satisfying (1) has been called the Castaing representation of F.

Dentcheva [3, 4] studied the existence of a Castaing representation which inherits the regularity properties of the multifunction and constructed many different types of such representations including measurable, continuous, resp. Hölder-continuous or directionally differentiable, if the multifunction had corresponding properties. The key ingredient of the proof was the notion of generalized Steiner point.

The starting point of our considerations is to provide a characterization of set-valued mappings similar to the Castaing result. In comparison to [2, 3, 4], we would like to find assumptions on F under which it has a sequence of smooth (C^∞) selections dense in F in the Castaing sense, i.e. satisfying the condition (1). For this purpose we use the technique of convolution that was mentioned at the beginning of this paragraph.

The method of convolution provides also another result. We find a family of smooth selections of F dense in the family of all continuous selections of F in the sense of the topology of the uniform convergence. This result leads us to the problem on the separability of the space of continuous selections of F equipped with the supremum norm. We notice that a suitable convolution type formula creates a sequence of smooth selections dense in the space of continuous selections of F.

We recall the problem of the existence of a selection f of F inheriting the regularity from F such that the graph of f contains given points. For instance, [6], (Prop. 3) states that a Lipschitz continuous multifunction has a Lipschitz continuous selection passing through given points. Another result [3], (Thm. 4.3) states that if a continuous multifunction F is semidifferentiable at some points

$(x_i, y_i) \in \mathrm{graph}(F)$, then F has a continuous selection φ such that $\varphi(x_i) = y_i$ and φ is directionally differentiable at these points. In comparison to these results, we notice that the convolution type operator enables us to increase the regularity of selections passing through given points.

As a conclusion arising from our considerations, in the last paragraph we obtain a solution to a problem on the existence of smooth selections formulated by Ioffe in [9].

Here are some basic notations. The supremum norm of a function $f : X \mapsto R^n$ is denoted as $\|f\|_{L_\infty(X)}$; the graph of $F : X \mapsto 2^Y$ is the set $\{(x, y) \in X \times Y \ |y \in F(x)\}$; $B(x, r)$ is the Euclidean open ball in R^n with center x and radius r; $\|\cdot\|$ is the Euclidean norm; $d(y, A)$ is the distance of a point y to $A \subset R^n$. For $X \subset R^m$ we denote the closure of X as $\overline{X}$; the interior of X as $\mathrm{Int}(X)$; the Minkowski sum of X and $B(0, r)$ as $B(X, r)$.

Recall that a multifunction $F : X \mapsto 2^{R^n}$ is lower semicontinuous (l.s.c.) at $x_0 \in X$ if for any open subset $V \subset R^n$ such that $V \cap F(x_0) \neq \emptyset$,

$$\exists \sigma > 0 \ \ \forall x \in X : \ \ \|x - x_0\| \leq \sigma \ \Rightarrow F(x) \cap V \neq \emptyset. \tag{2}$$

and F is l.s.c. if it is l.s.c. at each point of the domain (see [1], p. 39).

2. Main results

In this paragraph we formulate the main results of our paper and discuss a relationship between them. The proofs are given in the next section. First we deal with the problem (a) of the existence of a smooth selection and the problem (c) whether every continuous selection f of F can be approximated by a sequence of smooth selections of F in the sense of the topology of the uniform convergence. The solutions to problems (b), (d), (e) follow from the first result.

Theorem 2.1. *Let $X \subset R^m$ be open. Assume that $F : X \mapsto 2^{R^n}$ is a lower semicontinuous, convex-valued multifunction such that* $\mathrm{Int}\, F(x) \neq \emptyset$ *for all $x \in X$. Then for every continuous selection f of F and every $\varepsilon > 0$ there exists a function $f^\varepsilon : X \mapsto R^n$ such that f^ε is a selection of F, $\|f - f^\varepsilon\|_{L_\infty(X)} < \varepsilon$ and $f^\varepsilon \in C^\infty(X)$.*

See Paragraph 3.1 for the proof.

Modifying the convolution-type formula used in the proof of the first result we get a smooth selection passing through some given points.

Theorem 2.2. *Let $X \subset R^m$ be open and let $F : X \mapsto 2^{R^n}$ be a lower semicontinuous, convex-valued multifunction such that* $\mathrm{Int}\, F(x) \neq \emptyset$ *for all $x \in X$. Let $K = \{(x_i, y_i) : x_i \in X, \ y_i \in \mathrm{Int}\, F(x_i)\}$ be a family of isolated points such that* $\inf\{\|x_i - x_j\| : (x_i, y_i), (x_j, y_j) \in K, \ i \neq j\} > 0$. *Then there is a smooth selection φ of F such that $\varphi(x_k) = y_k$ for every k.*

The proof is given in Paragraph 3.2.

Next we construct a smooth Castaing-type representation. First, we remark that Theorem 2.1 implies the existence of a family $\{f_i\}_{i\in I}$ of smooth selections of F dense in the family of all continuous selections of F in the sense of uniform convergence topology. In Theorem 2.3 below we obtain that a multifunction F having such family $\{f_i\}_{i\in I}$ has also a smooth Castaing type representation – there is a sequence $\{f_k\}$ of smooth selections belonging to the above family $\{f_i\}_{i\in I}$ and satisfying $\overline{\cup_{k\geq 1}\ \{f_k(x)\}} = \overline{F(x)}$ for all $x \in X$.

Theorem 2.3. *Let $X \subset R^m$ be open and $F : X \mapsto 2^{R^n}$ be lower semicontinuous. Assume that* $\operatorname{Int} F(x) \neq \emptyset$ *and $F(x)$ is convex for all $x \in X$. Then there is a sequence $\{f_k\}$ of smooth selections of F such that $\overline{F(x)} = \overline{\cup_{k\geq 1}\ \{f_k(x)\}}$ for all $x \in X$.*

The proof is given in Paragraph 3.3.

If in addition we assume F to be closed valued, then in Theorem 2.3 we get a sequence $\{f_k\}$ satisfying the Castaing condition (1) and we have a solution to the problem (d).

Let $CS(F)$ be the the space of continuous selections of a multifunction F equipped with the supremum norm. We consider the problem (e) of the separability of the space $CS(F)$. In general the existence of a smooth Castaing type representation of F does not imply that $CS(F)$ is separable (see the last paragraph). Therefore to prove the separability of the space $CS(F)$ we need stronger assumptions on F than in Theorem 2.3. Additional assumptions concern the diameter of $F(x)$. Let $\operatorname{diam} F(x)$ be the diameter of $F(x)$, i.e. let $\operatorname{diam} F(x) = \sup\{\|y_1 - y_2\| \ :\ y_1, y_2 \in F(x)\}$.

Now we establish a class of multifunctions F such that the space $CS(F)$ is separable and have a smooth Castaing-type representation.

Theorem 2.4. *Let $X \subset R^m$ be open. Let $F : X \mapsto 2^{R^n}$ be a convex-valued, lower semicontinuous multifunction such that* $\operatorname{Int} F(x) \neq \emptyset$ *for all $x \in X$. Assume that for any $x_0 \in \overline{X} \setminus X$ and for any sequence $\{x_k\}$ in X convergent to x_0 we have* $\operatorname{diam} F(x_k) \to 0$. *We also assume that* $\operatorname{diam} F(x_k) \to 0$ *for any $\{x_k\}$ in X such that $\|x_k\| \to +\infty$. Then the space $CS(F)$ is separable – there is a sequence of smooth selections of F dense in $CS(F)$.*

The proof is given in Paragraph 3.4. In the last paragraph we describe the relationship between the existence of a smooth Castaing-type representation of F and the separability of the space $CS(F)$. As a conclusion arising from our considerations we get a solution to a problem formulated by Ioffe in [9].

3. Proofs

3.1. Proof of Theorem 2.1

A sketch of the proof. Let $X \subset R^m$ be the domain of the multifunction F. First we need an auxiliary assumption that X is compact and $\overline{\operatorname{Int}(X)} = X$. Then we obtain that for every $\varepsilon > 0$ and every continuous selection f there are a continuous map $a : X \mapsto R^n$ and a number $\varepsilon_1 \in (0, \varepsilon)$ such that $B(a(x), \varepsilon_1) \subset \operatorname{Int} F(x) \cap B(f(x), \varepsilon)$ for all $x \in X$.

We introduce a smooth function $\eta_\delta : R^m \mapsto R^1$ depending on some $\delta \in (0, \varepsilon)$. Next, for a fixed δ we define a map $a^\delta : X \mapsto R^n$ by the formula

$$a^\delta(x) = \left(\int_{B(0,\delta)} \eta_\delta(y) \cdot \tilde{a}_1(x - y)\, dy, \ldots, \int_{B(0,\delta)} \eta_\delta(y) \cdot \tilde{a}_n(x - y)\, dy \right),$$

where $B(0, \delta) \subset R^m$ and $\tilde{a} \equiv (\tilde{a}_1, \ldots, \tilde{a}_n) : \overline{B(X, \varepsilon)} \mapsto R^n$ is a continuous extension of the function $a : X \mapsto R^n$. We prove there is a sufficiently small positive δ_0 depending on $(\varepsilon_1, \varepsilon)$ such that $a^{\delta_0}(x) \in B(a(x), \varepsilon_1) \subset \operatorname{Int} F(x) \cap B(f(x), \varepsilon)$ for all $x \in X$. Hence, by the arbitrariness of ε, any continuous selection f on a compact domain can be approximated by smooth selections.

In order to get a smooth selection f^ε approximating f on an open domain we introduce a locally finite open cover $\{\Omega_k\}$ of X such that $\overline{\Omega_k} \subset X$ is compact for every k. Employing earlier considerations we obtain that for every Ω_k there is a smooth selection $a_k^\delta : \overline{\Omega_k} \mapsto R^n$ of $F|_{\overline{\Omega_k}}$ approximating f on a compact domain $\overline{\Omega_k}$. Applying a smooth partition of unity subordinate to the cover $\{\Omega_k\}$ we get the smooth selection f^ε of F.

3.1.1. Auxiliary results

Lemma 3.1. *Let $X \subset R^m$ and $G : X \mapsto 2^{R^n}$ be lower semicontinuous. Suppose $G(x) \neq \emptyset$ is open and convex for all $x \in X$. Then* $\operatorname{graph}(G)$ *is open in $X \times R^n$.*

Proof. By contradiction. Suppose that $\operatorname{graph}(G)$ is not open. Then there are a point $(x_0, y_0) \in \operatorname{graph}(G)$ and a sequence $\{(x_k, y_k)\}$ such that

$$\forall_{k \in N} \quad (x_k, y_k) \notin \operatorname{graph}(G) \quad \text{and} \quad \|(x_k, y_k) - (x_0, y_0)\| < 2^{-k}.$$

Let $\sigma(K, \cdot)$ be the support function of a convex set K. Then, since $y_k \notin G(x_k)$ and due to the convexity of $G(x_k)$, there is $v_k \in R^n$ such that $\|v_k\| = 1$ and $\langle y_k, v_k \rangle \geq \sigma(G(x_k), v_k)$. The sequence $\{v_k\}$ has a subsequence $\{v_{k_j}\}$ convergent to some $v_0 \in R^n$.

Let $r > 0$ be such that $B(y_0, r) \subset G(x_0)$. Then, employing our earlier considerations, we obtain $B(y_0, r) \subset \{y : \langle y, v_0 \rangle \leq \langle y_0, v_0 \rangle + r\}$. Similarly we estimate

$$\begin{aligned} G(x_{k_j}) &\subset \{y \in R^n : \langle y, v_{k_j} \rangle \leq \langle y_{k_j}, v_{k_j} \rangle\} \\ &= \{y : \langle y, v_0 \rangle \leq \langle y_0, v_0 \rangle + (\langle y_{k_j}, v_{k_j} \rangle - \langle y_0, v_0 \rangle) + (\langle y, v_0 \rangle - \langle y, v_{k_j} \rangle)\} \\ &\subset \{y : \langle y, v_0 \rangle \leq \langle y_0, v_0 \rangle + r/2\}, \end{aligned}$$

and the last inclusion holds for k_j sufficiently large. Let $\tilde{k}$ be the smallest positive integer number such that for every $k_j > \tilde{k}$ the above inclusion holds. Then for $k_j > \tilde{k}$ we get

$$B(y_0, r) \cap G(x_{k_j}) \subset \{y : \langle y, v_0 \rangle \leq \langle y_0, v_0 \rangle + r/2\}. \tag{3}$$

Let us define $Z = B(y_0, r) \cap \{y \in R^n : \langle y, v_0\rangle > \langle y_0, v_0\rangle + \frac{r}{2}\}$. Then Z is open and $Z \cap G(x_0) \neq \emptyset$. However, $Z \cap G(x_{k_j}) = \emptyset$ for all $k_j > \tilde{k}$ due to (3). This contradicts the lower semicontinuity of $G(\cdot)$ at x_0. □

Lemma 3.2. *Let $X \subset R^m$ be compact and $G : X \mapsto 2^{R^n}$ be a lower semicontinuous convex-valued multifunction such that $\operatorname{Int} G(x) \neq \emptyset$ for all $x \in X$. Let $G_1 : X \mapsto 2^{R^n}$ be a multifunction defined by the formula $G_1(x) = \operatorname{Int} G(x)$. Then $G_1(\cdot)$ is lower semicontinuous and has a continuous selection.*

Proof. The lower semicontinuity of $G_1(\cdot)$ follows straightforward from the lower semicontinuity of $G(\cdot)$. The existence of a continuous selection of $G_1(\cdot)$ follows from

> **Lemma 2.1 ([6], Prop. 4).** *Let T be a compact metric space, $\Gamma : T \mapsto 2^{R^n}$ be a multifunction with nonempty convex image sets. Let the preimage $\Gamma^{-1}(y)$ be open in T for each $y \in R^n$. Then Γ has a Lipschitz continuous selection.*

The multifunction $G_1(\cdot) : X \to 2^{R^n}$ satisfies the assumptions of Lemma 2.1. Indeed, let us prove that the preimage $(G_1(y))^{-1}$ is open in X for each $y \in R^n$. Suppose there is $y_0 \in R^n$ such that $(G_1(y_0))^{-1}$ is not open in X. Then there are $x_0 \in (G_1(y_0))^{-1}$ and a sequence $\{x_k\}$ in $X \setminus (G_1(y_0))^{-1}$ convergent to x_0. The sequence $\{(x_k, y_0)\}$ does not belong to the graph of $G_1(\cdot)$ and converges to (x_0, y_0). Hence, for every open neighbourhood V of (x_0, y_0) open in $X \times R^n$, we have $V \not\subset \operatorname{graph}(G_1(\cdot))$. However, since $(x_0, y_0) \in \operatorname{graph}(G_1(\cdot))$, then by Lemma 3.1 there is a neighbourhood U of (x_0, y_0) open in $X \times R^n$ such that $U \subset \operatorname{graph}(G_1(\cdot))$. A contradiction. □

Lemma 3.3. *Let $X \subset R^m$ be compact; $G : X \mapsto 2^{R^n}$ be lower semicontinuous; $G(x)$ be open, convex, nonempty and $G(x) \neq R^n$ for all x. Let $g : X \mapsto R^n$ be a continuous selection of G. Then*

$$\exists \varepsilon_1 > 0 \quad \forall x \in X \quad B(g(x), \varepsilon_1) \subset G(x).$$

Proof. Let $\|\cdot\|$ be the Euclidean norm in R^n and $d_1(x, y) = \|x - y\|$. For $x \in R^n$ and a nonempty $A \subset R^n$ we define $d(x, A) = \inf\{d_1(x, a) : a \in A\}$.

Let $\varepsilon_0(x) = d(g(x),\ R^n \setminus G(x))$. Since $R^n \setminus G(x)$ is nonempty and closed, then we have $\varepsilon_0(x) = \min\{d_1(g(x), z(x)) : z(x) \in R^n \setminus G(x)\}$. Hence, the multifunction $Z : X \mapsto 2^{R^n}$ defined as

$$Z(x) = \{y \in R^n \setminus G(x) \ : \ \|y - g(x)\| = \varepsilon_0(x)\}. \tag{4}$$

is nonempty-valued for all $x \in X$.

Let $\varepsilon_1 = \inf\{\varepsilon_0(x) \mid x \in X\}$. Our aim is to prove that $\varepsilon_1 > 0$. Indeed, suppose that $\varepsilon_1 = 0$. Then there is a sequence $\{x_k\}$ in X such that $\varepsilon_0(x_k) \to 0$. Due to the compactness of X there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ convergent to some $x_0 \in X$.

Let $z : X \mapsto R^n$ be a selection of Z. The sequence $(x_{k_j}, z(x_{k_j}))$ does not belong to the graph of $G(\cdot)$ and, by the continuity of $g(\cdot)$ and due to (4), converges to $(x_0, g(x_0))$.

Applying Lemma 3.1 to $G(\cdot)$ we obtain there is a neighbourhood V of $(x_0, g(x_0))$ open in $X \times R^n$ such that $V \subset \mathrm{graph}(G(\cdot))$. Combining this fact with the fact that $(x_{k_j}, z(x_{k_j})) \to (x_0, g(x_0))$ we obtain that $(x_{k_j}, z(x_{k_j})) \in V$ for k_j sufficiently large. Hence, $z(x_{k_j}) \in G(x_{k_j})$ for k_j large enough.

On the other hand, we also have $z(x_{k_j}) \notin G(x_{k_j})$ due to (4). A contradiction. □

Lemma 3.4. *Let $X \subset R^m$ be compact and $F : X \to 2^{R^n}$ satisfy the assumptions of Theorem 2.1. Suppose $f : X \mapsto R^n$ is a continuous selection of F. Let $A : X \mapsto 2^{R^n}$ be defined as $A(x) = \mathrm{Int}\, F(x) \cap B(f(x), \varepsilon)$ for some $\varepsilon > 0$ fixed. Then:*

(*i*) *The multifunction $A(\cdot)$ is lower semicontinuous and has a continuous selection.*

(*ii*) *For any continuous selection $a(\cdot)$ of $A(\cdot)$ we have*

$$\exists \varepsilon_1 \in (0, \varepsilon) \quad \forall x \in X \quad B(a(x), \varepsilon_1) \subset A(x). \tag{5}$$

Proof. (*i*) The lower semicontinuity of $A(\cdot)$ is a consequence of the following

> **Lemma 4.1. (see [11], p. 121, Theorem B).** *Let:*
> *1) X be a topological space and $F_i : X \mapsto 2^{R^n}$, i=1,2, be closed valued,*
> *2) V be a neighbourhood of $x_0 \in X$ such that $F_i|_V$ are convex valued,*
> *3) F_i be lower semicontinuous at x_0; $\mathrm{Int}\, F_1(x_0) \cap \mathrm{Int}\, F_2(x_0) \neq \emptyset$,*
> *4) $G : X \mapsto 2^{R^n}$ definied as $G(x) = F_1(x) \cap F_2(x)$ be nonempty-valued.*
> *Then the multifunction $G(\cdot)$ is lower semicontinuous at x_0.*

Indeed, let us define $F_1(\cdot) = \overline{B(f(\cdot), \varepsilon)}$ and $F_2(\cdot) = \overline{F(\cdot)}$. Then, by Lemma 4.1, the map $G(x) \equiv F_1(x) \cap F_2(x)$ is l.s.c. at any $x_0 \in X$. Therefore $G(\cdot)$ fulfills the assumptions of Lemma 3.2, whence we obtain that the multifunction $G_1(x) = \mathrm{Int}\, G(x)$ is l.s.c. and has a continuous selection. Since $A(\cdot) \equiv G_1(\cdot)$, then Lemma 3.4(*i*) follows.

(*ii*) Since we have proved that the multifunction $A(\cdot)$ satisfies the assumptions of Lemma 3.3, then a positive number $\varepsilon_1 \in (0, \varepsilon)$ satisfying (5) exists. □

3.1.2. Proof of Theorem 2.1

Step 1. Let us recall from ([7], pp. 629–631) a result about convolution and smoothing. Let $\eta : R^n \mapsto R$ be defined as follows

$$\eta(x) = \begin{cases} c \cdot \exp(\frac{1}{\|x\|^2 - 1}), & \|x\| < 1 \\ 0, & \|x\| \geq 1, \end{cases}$$

and the constant $c > 0$ be such that $\int_{R^n} \eta(x) dx = 1$. For $\varepsilon > 0$ we define $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(\frac{x}{\varepsilon})$. Let $U \subset R^n$ be open and let $U_\varepsilon = \{x \in U \ : \ d(x, \partial U) > \varepsilon\}$. For a locally integrable function $f : U \mapsto R$ we define $f_\varepsilon : U_\varepsilon \mapsto R$ as follows $f_\varepsilon(x) = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y) dy$. Then $f_\varepsilon \in C^\infty(U_\varepsilon)$ (see [7], p. 630, Thm. 6).

Step 2. Find a candidate for a smooth selection. Let $X \subset R^m$ be compact, let $X = \overline{\mathrm{Int}(X)}$ and let f be a continuous selection of F. Then, employing Lemmas 3.1–3.4, we obtain that for every $\varepsilon > 0$ there are a continuous map $a : X \mapsto R^n$ and a number $\varepsilon_1 \in (0, \varepsilon)$ such that $B(a(x), \varepsilon_1) \subset \mathrm{Int}\, F(x) \cap B(f(x), \varepsilon)$ for all $x \in X$.

Tietze extension theorem implies there is a continuous extension $\tilde{a} : \overline{B(X, \varepsilon)} \mapsto R^n$ of the selection $a : X \mapsto R^n$. By the Heine-Cantor theorem $\tilde{a}$ is uniformly continuous since its domain is a compact subset of R^m.

Let $\varepsilon_2 \equiv \frac{\varepsilon_1}{n}$ with $\varepsilon_1 > 0$ introduced in (5). The uniform continuity of $\tilde{a}$ implies that

$$\exists\, \sigma = \sigma(\varepsilon_2) > 0 \ \ \forall\, x, y \in \overline{B(X, \varepsilon)} \ \ \|x - y\| < \sigma \quad \Rightarrow \quad \|\tilde{a}(x) - \tilde{a}(y)\| < \varepsilon_2.$$

We define $\delta_0 = \min\{\sigma, \varepsilon_2\}$. Then $\delta_0 \in (0, \varepsilon)$ and we notice that

$$\forall\, x, y \in \overline{B(X, \varepsilon)} \ \ \|x - y\| < \delta_0 \quad \Rightarrow \quad \|\tilde{a}(x) - \tilde{a}(y)\| < \varepsilon_2.$$

Let δ_0 be as above. We define the function $\eta_{\delta_0} : R^m \mapsto R^1$ as follows

$$\eta_{\delta_0}(x) = \begin{cases} \delta_0^{-m} c \cdot e^{(\|x/\delta_0\|^2 - 1)^{-1}}, & \|x\| < \delta_0 \\ 0, & \|x\| \geq \delta_0, \end{cases} \tag{6}$$

where $c > 0$ is a constant chosen such that $\int_{R^m} \eta_{\delta_0}(x)dx = 1$. It depends only on m.

Let $\tilde{a} : \overline{B(X, \varepsilon)} \mapsto R^n$ be the continuous extension of the map $a : X \mapsto R^n$ introduced above. We define a map $a^{\delta_0} : X \mapsto R^n$ as follows:

$$a^{\delta_0}(x) = \left(\int_{B(0,\delta_0)} \eta_{\delta_0}(y) \cdot \tilde{a_1}(x - y)dy, \ldots, \int_{B(0,\delta_0)} \eta_{\delta_0}(y) \cdot \tilde{a_n}(x - y)dy \right).$$

The mapping $a^{\delta_0}(\cdot)$ is well defined on X. Indeed, for every $x \in X$, $y \in B(0, \delta_0)$ we have $x - y \in B(X, \delta_0) \subset \overline{B(X, \varepsilon)}$ and the domain of $\tilde{a}(\cdot)$ is $\overline{B(X, \varepsilon)}$. Since $\tilde{a_k}(\cdot)$ is continuous, then $\tilde{a_k}(\cdot)$ is locally integrable by the compactness of its domain. Therefore the components of $a^{\delta_0}(\cdot)$ fulfil the assumptions of Thm. 6 in [7], p. 630, whence we obtain that $a^{\delta_0}(\cdot) \in C^\infty(\mathrm{Int}(X))$.

The candidate for a smooth selection f^ε approximating f on a compact domain is the mapping $a^{\delta_0}(\cdot)$. Let $f^\varepsilon \equiv a^{\delta_0}$.

Step 3. The function f^ε is a smooth selection approximating f on a compact domain in the L_∞ norm. Indeed, since $\int_{B(0,\delta)} \eta_\delta(y)dy = 1$, we fix an arbitrary $x \in X$ and estimate

$$\|f^\varepsilon(x) - \tilde{a}(x)\| \leq n \cdot \max_{i \in \{1,..,n\}} \left| \int_{B(0,\delta)} \eta_\delta(y) \cdot \tilde{a}_i(x - y)dy - \tilde{a}_i(x) \right|$$

$$= n \cdot \max_{i \in \{1,..,n\}} \left| \int_{B(0,\delta)} \eta_\delta(y) \cdot (\tilde{a}_i(x - y) - \tilde{a}_i(x))\, dy \right|$$

$$\leq n \cdot \int_{B(0,\delta)} \eta_\delta(y)dy \cdot \max_i \max_{y \in B(0,\delta)} |\tilde{a}_i(x - y) - \tilde{a}_i(x)| < n \cdot 1 \cdot \varepsilon_2 \equiv n\frac{\varepsilon_1}{n} = \varepsilon_1.$$

The last inequality above follows from the uniform continuity of $\tilde{a}$. Since $\tilde{a}|_X \equiv a$, then by the above estimate we obtain that $f^\varepsilon(x) \in B(a(x), \varepsilon_1)$ for every $x \in X$. Combining this estimate with our choice of $\varepsilon_1 > 0$ in (5) we have

$$\forall x \in X \quad f^\varepsilon(x) \in B(a(x), \varepsilon_1) \subset \operatorname{Int} F(x) \cap B(f(x), \varepsilon) \subset F(x).$$

Step 4. Derive a smooth selection f^ε approximating f defined on an open domain. Now we assume that F is a multifunction defined on an open domain $X \subset R^m$ and prove the existence of a smooth selection f^ε approximating a given continuous selection f.

There is an open locally finite cover $\{\Omega_k\}$ of X such that $\overline{\Omega_j} \subset X$ is compact for every j. According to [15], there is a smooth partition of unity subordinate to this cover. There are smooth mappings $\zeta_k : X \mapsto [0, 1]$ such that $\sum_k \zeta_k(x) = 1$ for all $x \in X$ and ζ_k vanish outside Ω_k.

Let f be a continuous selection of F. Then, due to our earlier considerations, for every Ω_j and every $\varepsilon > 0$ there is a smooth function $f_j^\varepsilon : \overline{\Omega_j} \mapsto R^n$ such that f_j^ε is a selection of $F|_{\overline{\Omega_j}}$ and $\|f - f_j^\varepsilon\|_{L_\infty(\Omega_j)} < \varepsilon$. We define a function $f^\varepsilon : X \mapsto R^n$ as

$$f^\varepsilon(x) = \sum_{j \in J(x)} \zeta_j(x) \cdot f_j^\varepsilon(x) \quad \text{where } J(x) = \{j \; : \zeta_j(x) > 0\}.$$

The map f^ε is a selection of F due to the convexity of $F(x)$. The smoothness of f^ε and the fact that $\|f - f^\varepsilon\|_{L_\infty(X)} < \varepsilon$ follow from earlier considerations. □

3.2. Proof of Theorem 2.2

A smooth selection passing through given points is constructed. The proof is similar to the proof of Theorem 2.1, but we need some restrictions on the continuous selection appearing in the formula of the convolution and some adjustments concerning the parameter δ of the smooth map $\eta_\delta(\cdot)$.

Lemma 3.5. *Let $F : X \mapsto 2^{R^n}$ fulfil the assumptions of Theorem 2.2. Let K be a family of isolated points (x_i, y_i) satisfying the assumptions of Theorem 2.2. Let $Z \subset X$ be compact such that $Z = \overline{\operatorname{Int}(Z)}$ and let $K_Z = \{(x_i, y_i) \in K : x_i \in \operatorname{Int}(Z)\}$. Let $F_1 : Z \mapsto 2^{R^n}$ be defined as $F_1(z) = \operatorname{Int} F(z)$. Then there is a continuous selection $g : Z \mapsto R^n$ of F_1 satisfying $g(x_i) = y_i$ for every point $(x_i, y_i) \in K_Z$ and*

$$\exists \gamma > 0 \;\; \forall (x_i, y_i) \in K_Z \quad g|_{B(x_i, \gamma)} \equiv y_i. \tag{7}$$

Proof. Since we assume that $inf\{\|x_i - x_j\| : (x_i, y_i), (x_j, y_j) \in K, \; i \neq j\} > 0$, then the number of points in K_Z is finite by the compactness of Z. We have $K_Z = \{(x_i, y_i)\}_{i=1}^l$ for some positive integer l. Each point $(x_i, y_i) \in K_Z$ has an open neighbourhood in graph(F_1) due to Lemma 3.1. Then there is $r > 0$ such that $B((x_i, y_i), r) \subset \text{graph}(F_1)$ for all $(x_i, y_i) \in K_Z$ and $B((x_i, y_i), r) \cap B((x_i, y_i), r) = \emptyset$ for all $(x_i, y_i), (x_j, y_j) \in K_Z, \; i \neq j$.

Let $\{U_j\}_{j=0}^l$ be an open cover of Z of the form

$$U_0 = Z \setminus \cup_{j=1}^l \overline{B(x_j, r/2)}\,, \qquad U_j = B(x_j, r); \quad j = 1, \ldots, l. \tag{8}$$

There is a partition of unity on Z subordinate to the cover $\{U_j\}$. There are continuous functions $\zeta_j : Z \mapsto [0,1]$ such that $\sum_j \zeta_j(x) = 1$ for every $x \in Z$ and $\zeta_j \equiv 0$ outside U_j. Let $h_i : Z \mapsto R^n; i = 1,..,l;$ be defined as $h_i(x) = y_i$. Let $h_0 : \overline{U_0} \mapsto R^n$ be a continuous selection of $F_1(\cdot)|_{\overline{U_0}}$. It exists in view of Lemma 3.2. We define $g : Z \mapsto R^n$ as

$$g(x) = \sum\nolimits_{j \in J(x)} \zeta_j(x) \cdot h_j(x), \quad \text{where } J(x) = \{j \in \{1, \ldots, l\} \ : \zeta_j(x) > 0\}.$$

The map g is what we are looking for. Indeed, the fact that g is a selection of $F_1(\cdot)$ follows from the convexity of $F(x)$. The condition (7) holds for any positive $\gamma < \frac{r}{2}$ due to our choice of the cover $\{U_j\}$. Inserting for instance $\gamma = \frac{r}{3}$ ends the proof. □

Lemma 3.6. *Let the assumptions of Lemma* 3.5 *hold. Then there is a function* $\varphi : Z \mapsto R^n$ *such that* $\varphi \in C^\infty(Z)$*,* φ *is a selection of* $F|_Z$ *and* $\varphi(x_k) = y_k$ *for every* k*.*

Proof. *Step 1. Derive a candidate for a smooth selection* φ. Let $\{(x_i, y_i)\}$ and $g : Z \to R^n$ be the points and the continuous selection from Lemma 3.5. Lemma 3.3 implies that

$$\exists \varepsilon_1 > 0 \ \ \forall x \in Z \ \ B(g(x), \varepsilon_1) \subset \operatorname{Int} F(x). \tag{9}$$

By the Tietze extension theorem there is a continuous extension $\bar{g} : \overline{B(Z, \varepsilon_1)} \mapsto R^n$ of g. By the Heine-Cantor theorem $\bar{g}$ is uniformly continuous. Hence, for $\varepsilon_2 := \frac{\varepsilon_1}{n}$, we have

$$\exists \sigma = \sigma(\varepsilon_2) > 0 \ \ \forall x, y \in \overline{B(Z, \varepsilon_1)} \ \ \|x - y\| < \sigma \quad \Rightarrow \quad \|\bar{g}(x) - \bar{g}(y)\| < \varepsilon_2.$$

Let $\delta = \min\{\sigma, \varepsilon_2, \frac{\gamma}{2}\}$ with γ introduced in Lemma 3.5. Then

$$\forall x, y \in \overline{B(Z, \varepsilon_1)} \ \ \|x - y\| < \delta \quad \Rightarrow \quad \|\bar{g}(x) - \bar{g}(y)\| < \varepsilon_2. \tag{10}$$

We introduce a map $\varphi : Z \mapsto R^n$ as follows

$$\varphi(x) = \left(\int_{B(0,\delta)} \eta_\delta(y) \cdot \bar{g_1}(x - y) dy, \ldots, \int_{B(0,\delta)} \eta_\delta(y) \cdot \bar{g_n}(x - y) dy \right).$$

Step 2. The function φ *is a smooth selection of* $F|_Z$ *passing through given points.* Indeed, the fact that $\varphi \in C^\infty(\operatorname{Int}(Z))$ follows from the proof of Theorem 2.1. To prove that φ is a selection of $F|_Z$ we fix an arbitrary $x \in Z$ and we estimate

$$\begin{aligned} \|\varphi(x) - g(x)\| \equiv \|\varphi(x) - \bar{g}(x)\| &\leq n \cdot \max_j \left| \int_{B(0,\delta)} \eta_\delta(y) \cdot (\bar{g_j}(x - y) - \bar{g_j}(x)) \, dy \right| \\ &\leq n \cdot \int_{B(0,\delta)} \eta_\delta(y) dy \cdot \max_{y \in \overline{B(0,\delta)}} \|\bar{g}(x - y) - \bar{g}(x)\| \leq n \cdot 1 \cdot \varepsilon_2 \equiv \varepsilon_1, \end{aligned}$$

and the last inequality follows from (10). Combining (9) with the above estimate we obtain $\varphi(x) \in B(g(x), \varepsilon_1) \subset \operatorname{Int} F(x)$ for all $x \in Z$.

Now we prove that $\varphi(x_k) = y_k$ for each given point (x_k, y_k). Indeed, fixing an arbitrary point $(x_i, y_i) \in \{(x_k, y_k)\}$ and performing calculations we obtain that

$$\begin{aligned}\|\varphi(x_i) - y_i\| = \|\varphi(x_i) - g(x_i)\| &\le n \cdot \max_{j\in\{1,\dots,n\}} |\varphi_j(x_i) - g_j(x_i)| \\ &= n \cdot \left| \int_{B(0,\delta)} \eta_\delta(y)\bar{g}_l(x_i - y)dy - g_l(x_i)\right| \\ &= n \cdot \left| \int_{B(0,\delta)} \eta_\delta(y)\big(\bar{g}_l(x_i - y) - g_l(x_i)\big)dy\right| \\ &\le n \cdot \int_{B(0,\delta)} \eta_\delta(y)dy \cdot |\bar{g}_l(z) - g_l(x_i)| \equiv n \cdot 1 \cdot |g_l(z) - g_l(x_i)| = I_1\end{aligned}$$

for some $z \in \overline{B(x_i, \delta)} \subset B(x_i, \gamma) \subset \operatorname{Int}(Z)$. The last identity follows from the fact that $z \in Z$ and $\bar{g}|_Z \equiv g$. Since $g|_{B(x_i,\gamma)} \equiv y_i$, then $g(z) = g(x_i)$ and in particular $I_1 = 0$. □

Lemma 3.7. *Under the assumptions of Lemma* 3.5*, the assertion of Theorem* 2.2 *holds.*

Proof. Let $\{(x_i, y_i)\} \in \operatorname{graph} F(\cdot)$ be given isolated points such that $x_i \neq x_j$ for $i \neq j$, $x_k \in X$ and $y_k \in \operatorname{Int} F(x_k)$. Let $\{\Omega_j\}$ be a locally finite open cover of X such that $\overline{\Omega_j}$ is compact for every j. Then, according to Lemma 3.6, the multifunction $F(\cdot)|_{\overline{\Omega_j}}$ has a smooth selection $\varphi_j : \overline{\Omega_j} \mapsto R^n$ such that $\varphi_j(x_i) = y_i$ for each given point (x_i, y_i) belonging to the graph of $\operatorname{Int} F(\cdot)|_{\operatorname{Int}(\Omega_j)}$.

There is a smooth partition of unity on X subordinate to the above cover $\{\Omega_j\}$ (see [15] for instance). There are smooth functions $\zeta_j : X \to [0, 1]$ such that $\zeta_j \equiv 0$ outside Ω_j and $\sum_j \zeta_j(x) = 1$ for every $x \in X$.

Let $\varphi : X \to R^n$ be defined by the formula

$$\varphi(x) = \sum\nolimits_{j\in J(x)} \zeta_j(x) \cdot \varphi_j(x), \quad \text{where } J(x) = \{j \in \{1, 2, \dots\} : \zeta_j(x) > 0\}.$$

Then φ is a selection of F due to the convexity of $F(x)$. Moreover, $\varphi(x_k) = y_k$ for each given point (x_k, y_k), because the mappings $\varphi_j(\cdot)$ are constructed in Lemma 3.6 in such a way that $\varphi_j(x_k) = y_k$ for every $j \in J(x_k)$. □

3.3. Proof of Theorem 2.3

Let $B_X(x_0, r) = B(x_0, r) \cap X$, let (k, l, α) be positive integer numbers and let $\{A_{k,l}\}$, $\{X_{k,l,\alpha}\}$, $\{Z_{k,l}\}$ be families of subsets of X defined as follows

$$\begin{aligned} A_{k,l} &= \{x \in X : B(y_k, \tfrac{1}{l}) \subset F(x)\}, \\ X_{k,l,\alpha} &= \begin{cases} X, & \text{if } A_{k,l} = \emptyset, \\ \emptyset, & \text{if } A_{k,l} = X, \\ B_X(X \setminus A_{k,l}, \tfrac{1}{\alpha}), & \text{otherwise,} \end{cases} \qquad (11) \\ Z_{k,l,\alpha} &= X \setminus X_{k,l,\alpha}. \end{aligned}$$

We also define a family of set-valued mappings $F_{k,l,\alpha} : X \mapsto 2^{R^n}$ by the formula

$$F_{k,l,\alpha}(x) = \begin{cases} F(x), & x \in X_{k,l,\alpha}, \\ B(y_k, \frac{1}{l}), & x \in Z_{k,l,\alpha}. \end{cases} \tag{12}$$

Our aim is to obtain a sequence $\{f_{k,l,\alpha}\}$ of selections of F such that $f_{k,l,\alpha}$ is a selection of $F_{k,l,\alpha}$ and satisfying $\overline{F(x)} = \overline{\cup_{k,l,\alpha} \{f_{k,l,\alpha}(x)\}}$ for all $x \in X$.

Lemma 3.8. *Let $X \subset R^m$ be open. Let $F : X \mapsto 2^{R^n}$ be a lower semicontinuous, convex-valued multifunction such that* $\operatorname{Int} F(x) \neq \emptyset$ *for all $x \in X$. Let (k, l, α) be fixed and let $F_{k,l,\alpha} : X \mapsto 2^{R^n}$ be the multifunction defined in* (12)*. Then $F_{k,l,\alpha}$ is lower semicontinuous, convex-valued and* $\operatorname{Int}(F_{k,l,\alpha}(x)) \neq \emptyset$ *for all $x \in X$.*

Proof. The convexity of $F_{k,l,\alpha}(x)$ and the fact that $\operatorname{Int}(F_{k,l,\alpha}(x)) \neq \emptyset$ for all $x \in X$ are clear. Only the lower semicontinuity of $F_{k,l,\alpha}$ at any $x_0 \in X$ should be justified.

If (k, l, α) are such that $X_{k,l,\alpha} = \emptyset$ or $X_{k,l,\alpha} = X$, then we have either $F_{k,l,\alpha}(x) \equiv B(y_k, \frac{1}{l})$ or $F_{k,l,\alpha}(x) \equiv F(x)$, respectively, and the lower semicontinuity of $F_{k,l,\alpha}$ at any $x_0 \in X$ is clear.

If (k, l, α) are such that $X_{k,l,\alpha} \neq \emptyset$ and $X_{k,l,\alpha} \neq X$, then for every $x_0 \in X$ we have either $x_0 \in X_{k,l,\alpha}$ or $x_0 \in Z_{k,l,\alpha}$. Hence, to prove that the multifunction $F_{k,l,\alpha}$ is l.s.c. we have to show that $F_{k,l,\alpha}$ is l.s.c. at every $x_0 \in Z_{k,l,\alpha}$ and every $x_0 \in X_{k,l,\alpha}$. We may prove that for any open set $U \subset R^n$ such that $U \cap F_{k,l,\alpha}(x_0) \neq \emptyset$ we have

$$\exists \delta > 0 \ \ \forall x \in B_X(x_0, \delta), \quad U \cap F_{k,l,\alpha}(x) \neq \emptyset. \tag{13}$$

Let us prove that $F_{k,l,\alpha}$ is l.s.c. at any $x_0 \in Z_{k,l,\alpha}$. First, in order to prove the lower semicontinuity of $F_{k,l,\alpha}$ at $x_0 \in Z_{k,l,\alpha}$, we notice that $B_X(x_0, \frac{1}{\alpha}) \subset A_{k,l}$ for every $x_0 \in Z_{k,l,\alpha}$. If it was not true, then there would be $x_0 \in Z_{k,l,\alpha}$ and $y_0 \in B_X(x_0, \frac{1}{\alpha}) \setminus A_{k,l}$. Then we would have $y_0 \in X \setminus A_{k,l}$ and $x_0 \in B_X(y_0, \frac{1}{\alpha}) \subset B_X(X \setminus A_{k,l}, \frac{1}{\alpha}) \equiv X_{k,l,\alpha} = X \setminus Z_{k,l,\alpha}$, a contradiction with our assumption that $x_0 \in Z_{k,l,\alpha}$.

Since $x_0 \in Z_{k,l,\alpha}$, $B_X(x_0, \frac{1}{\alpha}) \subset A_{k,l}$ and $F_{k,l,\alpha}(x_0) = B(y_k, \frac{1}{l})$, then we obtain

$$F_{k,l,\alpha}(x_0) = B(y_k, l^{-1}) \subset F(x) \ \text{ for all } x \in B_X(x_0, \alpha^{-1}). \tag{14}$$

Let an open set $U \subset R^n$ be such that $U \cap F_{k,l,\alpha}(x_0) \neq \emptyset$. Then $U \cap F_{k,l,\alpha}(x) \neq \emptyset$ for all $x \in B_X(x_0, \frac{1}{\alpha})$ due to (14). Hence, (13) holds with a positive constant $\delta = \frac{1}{\alpha}$.

Now we prove that $F_{k,l,\alpha}$ is l.s.c. at any $x_0 \in X_{k,l,\alpha}$. By the definition of the multifunction $F_{k,l,\alpha}$ we have the identity

$$F_{k,l,\alpha}|_{X_{k,l,\alpha}} \equiv F|_{X_{k,l,\alpha}}. \tag{15}$$

Since $X_{k,l,\alpha}$ is open in X and since F is lower semicontinuous, then employing (15) we obtain that $F_{k,l,\alpha}$ is l.s.c. at any $x_0 \in X_{k,l,\alpha}$. Hence, the assertion of Lemma 3.8 follows. □

Lemma 3.9. *Let $F : X \mapsto 2^{R^n}$ satisfy the assumptions of Theorem 2.3 and $\{F_{k,l,\alpha}\}$ be the family of multifunctions defined in (12). Then for every (k, l, α) there is a smooth selection $f_{k,l,\alpha}$ of $F_{k,l,\alpha}$. Moreover,*

$$\forall\ (x_0 \in X,\ \xi \in F(x_0),\ \varepsilon > 0)\ \ \exists\ f_{k,l,\alpha}\ \ \|\xi - f_{k,l,\alpha}(x_0)\| < \varepsilon. \tag{16}$$

Proof. The existence of a smooth selection $f_{k,l,\alpha}$ of $F_{k,l,\alpha}$ follows from Lemma 3.8 and Theorem 2.2. Only (16) should be justified.

To start with, let the triple $x_0 \in X$, $\xi \in F(x_0)$, $\varepsilon > 0$ be fixed and let $\{y_i\}_{i=1}^{\infty}$ be dense in R^n. There is y_k such that $y_k \in \operatorname{Int} F(x_0)$ and $\|\xi - y_k\| < \frac{\varepsilon}{2}$. Applying Lemma 3.1 to the multifunction $\operatorname{Int} F(\cdot)$ and its point (x_0, y_k) we obtain

$$\exists r > 0 \ \ B_{X\times R^n}((x_0, y_k), r) \subset \operatorname{graph}(\operatorname{Int}(F(\cdot))). \tag{17}$$

Let r be as above, let l be the smallest positive integer number satisfying

$$l^{-1} < \min\left\{\frac{\varepsilon}{2}, \frac{r}{2}\right\} \tag{18}$$

and let $\alpha = l$. In order to get (16) we need to justify that for the above triple (k, l, α) we have $B(y_k, \frac{1}{l}) \subset F(x)$ for all $x \in B_X(x_0, \frac{1}{\alpha})$.

Let an arbitrary $(x, y) \in B_X(x_0, \frac{1}{\alpha}) \times B(y_k, \frac{1}{l})$ be fixed. Then, employing (18) and (17), we get

$$(x, y) \in B_{X\times R^n}((x_0, y_k), r) \subset \operatorname{graph}(\operatorname{Int}(F(\cdot))).$$

This implies that $B(y_k, \frac{1}{l}) \subset F(x)$ for all $x \in B_X(x_0, \frac{1}{\alpha})$ and therefore $B_X(x_0, \frac{1}{\alpha}) \subset A_{k,l}$. Since $B_X(x_0, \frac{1}{\alpha}) \subset A_{k,l}$, then $X_{k,l,\alpha} \subset X \setminus x_0$ by the definition of $X_{k,l,\alpha}$ in (11). Hence $Z_{k,l,\alpha} \ni x_0$ and therefore $F_{k,l,\alpha}(x_0) = B(y_k, \frac{1}{l})$.

Now we prove (16). Indeed, for a fixed triple $x_0 \in X$, $\xi \in F(x_0)$, $\varepsilon > 0$ there are positive integer numbers (k, l, α) such that $F_{k,l,\alpha}(x_0) = B(y_k, \frac{1}{l})$. Since $F_{k,l,\alpha}$ has a smooth selection $f_{k,l,\alpha}$, we estimate

$$\|\xi - f_{k,l,\alpha}(x_0)\| \le \|\xi - y_k\| + \|y_k - f_{k,l,\alpha}(x_0)\| \le \frac{\varepsilon}{2} + l^{-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

3.4. Proof of Theorem 2.4

Lemma 3.10. *Let $X \subset R^m$ be open, $Z \subset X$ be compact, $F : X \mapsto 2^{R^n}$ be a lower semicontinuous multifunction and $F(x)$ be closed, convex. Then the space $CS(F|_Z)$ is separable.*

Proof. The existence of at least one continuous selection of F follows from the Michael theorem [12]. Next, the space $CS(F|_Z)$ is separable, since it is a subset of a metric separable space $C(Z, R^m)$ of continuous functions from a compact $Z \subset R^m$ to R^n. $\square$

Lemma 3.11. *Let $X \subset R^m$ be open, $Z \subset X$ be closed, $F : X \mapsto 2^{R^n}$ be a lower semicontinuous multifunction and $F(x)$ be closed, convex. Let $\varphi : Z \mapsto R^n$ be a continuous selection of $F|_Z$. Then there is a continuous selection $f : X \mapsto R^n$ of F such that $f|_Z \equiv \varphi$.*

Proof. Let $F_1 : X \mapsto 2^{R^n}$ be defined as

$$F_1(x) = \begin{cases} \varphi(x), & x \in Z \\ F(x), & x \in X \setminus Z. \end{cases}$$

Then by the Michael theorem ([12]) a continuous selection f of F_1 exists. $\square$

Lemma 3.12. *Let $F : X \mapsto 2^{R^n}$ satisfy the assumptions of Theorem 2.4. Let $\{P_k\}$ be a sequence of compact subsets of X such that* $\operatorname{Int} P_k \neq \emptyset$*,* $P_k \subset \operatorname{Int} P_{k+1}$*, and* $\cup_{k\geq 1} P_k = X$*. Then*

$$\lim_{k\to+\infty} \sup_{x\in X\setminus P_k} \operatorname{diam} F(x) = 0.$$

Proof. Suppose that lemma is not true. Then there is a sequence $\{P_k\}$ of compact subsets of X such that $\sup_{x\in X\setminus P_k} \operatorname{diam} F(x) \not\to 0$.Hence, there is a sequence $\{x_k\}$ such that $x_k \in X \setminus P_k$ and $\operatorname{diam} F(x_k) \not\to 0$. If there was a subsequence $\{x_{k_j}\}$ such that $\|x_{k_j}\| \to +\infty$, then we would get a contradiction to our assumptions on F. Therefore $\{x_k\}$ is bounded and has a subsequence $\{x_{k_j}\}$ convergent to some x_0. If $x_0 \in \overline{X} \setminus X$, then we have a contradiction to our assumptions on F. Hence $x_0 \in X$. Therefore $x_0 \in P_\lambda$ for some positive integer λ and $x_{k_j} \in \operatorname{Int} P_{\lambda+1}$ for k_j sufficiently large. On the other hand, for k_j large enough we have $\operatorname{Int} P_{\lambda+1} \subset P_{k_j}$. This implies that $x_{k_j} \in X \setminus P_{k_j} \subset X \setminus \operatorname{Int} P_{\lambda+1}$. This contradiction ends the proof. $\square$

Lemma 3.13. *Let $F : X \mapsto 2^{R^n}$ fulfill the assumptions of Theorem 2.4. Then for every $\varepsilon > 0$ there is a compact $Z \subset X$ such that for all selections f, g of F satisfying $\|f - g\|_{L_\infty(Z)} < \varepsilon$ we have $\|f - g\|_{L_\infty(X\setminus Z)} < \varepsilon$.*

Proof. Suppose the lemma is not true. Then there is $\varepsilon_0 > 0$ such that for every compact $Z \subset X$ there are selections f, g of F such that $\|f - g\|_{L_\infty(Z)} < \varepsilon$ and $\|f - g\|_{L_\infty(X\setminus Z)} \geq \varepsilon$.

Let $\{P_k\}$ be the family of compact subsets of X introduced in Lemma 3.12. There is a sequence $\{f_k, g_k\}$ of selections of F such that $\|f_k - g_k\|_{L_\infty(P_k)} < \varepsilon_0$ and

$$\|f_k - g_k\|_{L_\infty(X\setminus P_k)} \geq \varepsilon_0.$$

Hence, there is $\{x_k\}$ such that $x_k \in X \setminus P_k$ and $\|f(x_k) - g(x_k)\| \not\to 0$. Since $\operatorname{diam} F(x_k) \geq \|f(x_k) - g(x_k)\|$, then $\operatorname{diam} F(x_k) \not\to 0$. On the other hand, since $\{x_k\}$ is a sequence such that $x_k \in X\setminus P_k$, then $\operatorname{diam} F(x_k) \to 0$ due to Lemma 3.12, a contradiction. $\square$

Lemma 3.14. *Let $F : X \mapsto 2^{R^n}$ fulfill the assumptions of Theorem 2.4 and let $F(x)$ be closed for all $x \in X$. Assume that $\{Z_k\}_{k=1}^\infty$ is a family of compact subsets of X such that for all continuous selections f, g of F satisfying $\|f - g\|_{L_\infty(Z_k)} < \frac{1}{k}$ we have $\|f - g\|_{L_\infty(X\setminus Z_k)} < \frac{1}{k}$. Let $\{\varphi_{k,l}\}_{l=1}^\infty$ be a sequence of continuous selections of $F|_{Z_k}$ dense in $CS(F|_{Z_k})$. Let $\{f_{k,l}\}_{k,l=1}^\infty$ be a sequence of continuous selections of F such that $f_{k,l}|_{Z_k} \equiv \varphi_{k,l}$. Then the sequence $\{f_{k,l}\}_{k,l=1}^\infty$ is dense in $CS(F)$.*

Proof. The sequences $\{Z_k\}$, $\{\varphi_{k,l}\}$ and $\{f_{k,l}\}$ exist due to Lemmas 3.13, 3.10, 3.11, respectively. Let us prove that $\{f_{k,l}\}_{k,l=1}^{\infty}$ is dense in $CS(F)$. Let be given an arbitrary continuous selection f of F and an arbitrary $\varepsilon > 0$. Let k be the smallest positive integer number such that $\frac{1}{k} < \varepsilon$. Then there are a compact $Z_k \subset X$, a sequence $\{\varphi_{k,l}\}_{l=1}^{\infty}$ of continuous selections of $F|_{Z_k}$ dense in $CS(F|_{Z_k})$ and there is $\varphi_{k,l_0} \in \{\varphi_{k,l}\}_{l=1}^{\infty}$ such that

$$\|f - \varphi_{k,l_0}\|_{L_\infty(Z_k)} < k^{-1}.$$

Let $f_{k,l_0} \in \{f_{k,l}\}_{k,l=1}^{\infty}$ be a continuous selection of F and an extension of φ_{k,l_0}. Then, employing our considerations, we have

$$\|f - f_{k,l_0}\|_{L_\infty(X)} \le \max\{\|f - f_{k,l_0}\|_{L_\infty(Z_k)},\ \|f - f_{k,l_0}\|_{L_\infty(X\setminus Z_k)}\} < k^{-1} < \varepsilon\ . \quad \square$$

Lemma 3.15. *Let $F : X \mapsto 2^{R^n}$ be as in Theorem 2.4. Then the space $CS(F)$ is separable. Moreover, there is a sequence of smooth ($C^\infty(X)$) selections of F dense in $CS(F)$.*

Proof. Let $G : X \mapsto 2^{R^n}$ be defined as $G(x) = \overline{F(x)}$. Then the space $CS(G)$ is separable due to Lemma 3.14. Therefore the space $CS(F)$ is separable, since it is a subset of a metric separable space $CS(G)$. Hence, there is a sequence $\{f_k\}$ of continuous selections of F dense in $CS(F)$.

In order to get a sequence of smooth selections of F dense in $CS(F)$ we apply Theorem 2.1 to the sequence $\{f_k\}$. Theorem 2.1 implies that there is a sequence $\{g_{kl}\}_{k,l=1}^{\infty}$ of smooth selections of F such that $\|f_k - g_{kl}\|_{L_\infty(X)} < \frac{1}{l}$ for all k. Clearly $\{g_{kl}\}_{k,l=1}^{\infty}$ is dense in $CS(F)$. This ends the proofs of Lemma 3.15 and Theorem 2.4.
$\square$

4. Final remarks.

Remark 4.1. Let us describe the relationship between the separability of the space $CS(F)$ and the existence of a smooth Castaing-type representation of F.

The existence of a sequence $\{f_k\}$ of smooth selections of F dense in $CS(F)$ was shown in Theorem 2.4. Under the assumptions of Theorem 2.4 such sequence is a Castaing type representation of F – we have $\overline{F(x)} = \overline{\cup_{k\ge1}\ f_k(x)}$ for all $x \in X$. On the other hand, due to Theorem 2.3 there is another sequence $\{\varphi_k\}$ of smooth selections of F satisfying $\overline{F(x)} = \overline{\cup_{k\ge1}\ \varphi_k(x)}$ for all $x \in X$. However, $\{\varphi_k\}$ is not dense in $CS(F)$ in general. In fact the assumptions of Theorem 2.3 on F do not imply the space $CS(F)$ is separable.

Indeed, let $F : R^1 \mapsto 2^{R^1}$ be a multifunction defined as $F(x) = [-1, 1]$. Then F has a smooth Castaing-type representation, but the space $CS(F)$ is not separable, since $\{\sin(\alpha x)\}_{\alpha\in R_+}$ is an uncountable family of continuous selections of F such that

$$\inf_{a>b>0}\ \sup_{x\in R} |\sin(ax) - \sin(bx)| \ge 1/2. \quad \square$$

If we omit the assumption of convexity of $F(x)$, then an opposite situation appears – the separability of the space $CS(F)$ does not imply the existence of a smooth

Castaing-type representation of F. For instance, let $F : [0,2] \mapsto 2^{R^1}$ be defined as follows

$$F(x) = \begin{cases} [0,1], & 0 \leq x \leq 1 \\ [0,1] \cup \{2\}, & 1 < x \leq 2. \end{cases}$$

Then every continuous selection f of F is also a continuous selection of a multifunction $F_1(x) = [0,1]$, $x \in [0,2]$. The space $CS(F)$ is separable, because $CS(F) = CS(F_1)$ and, according to Lemma 3.10, the space $CS(F_1)$ is separable.

However, F does not have a smooth Castaing-type representation. If there was such representation, then there would be a sequence $\{f_i\}$ of smooth selections of F such that $\overline{\cup_{i=1}^{\infty} \{f_i(\frac{3}{2})\}} = F(\frac{3}{2})$. However, the point $(\frac{3}{2}, 2)$ belongs to the graph of F and, for every continuous selection f of F we have $\|f(\frac{3}{2}) - 2\| \geq 1$. A contradiction. $\square$

Remark 4.2. Let us mention the problem of the existence of smooth selections formulated by Ioffe in [9], p. 648, Remark(g). (See also [9], Thm. 3, p. 646, and the Corollary on p. 648 for the reason for introducing this problem).

Ioffe considered a time-dependent multifunction $F : T \times X \mapsto 2^Y$, defined another multifunction $F_L : T \times X \mapsto 2^Y$ as

$$\begin{aligned} F_L(t,x) = \{y \in F(t,x) :\ & \text{there is a selection } f \text{ of } F \text{ measurable in } t, \\ & C^r \text{ in } x \text{ such that } f(t,x) = y\} \end{aligned}$$

and suggested the problem what are the assumptions on F under which: (a) $F_L(t,x)$ is nonempty and (b) $F_L(t,x)$ is dense in $F(t,x)$ for all (t,x).

We have found a solution to this problem in a particular case for a time-independent multifunction F from an open $X \subset R^m$ to nonempty convex subsets of R^n. In Theorem 2.3 we have constructed a smooth Castaing-type representation and therefore we obtained an answer to the Ioffe's problem for $r = +\infty$.

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