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Weak Convexity of the Distance Function

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We investigate the sets which have neighborhood where the distance function from a point to the set is weakly convex. It is shown that such sets are proximally smooth. We obtain precise estimates for parameters of weak convexity via the size of the neighborhood and the constant of proximal smoothness of the set in the Hilbert space.

Keywords: Distance function, uniform convexity, uniform smoothness, weak convexity, modulus of weak convexity, proximal smoothness, P-supporting condition

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1. Introduction and main notations

We denote by E a Banach space, and by $\mathcal{H}$ a Hilbert space, over $\mathbb{R}$. For a point p from the topological dual space E^* and for $x \in E$ we define by $\langle p, x\rangle$ the value of the functional p at the point x. Also $\langle p, x\rangle$ will be the scalar product for vectors p and x from $\mathcal{H}$. Let $B_r(a) = \{x \in E \mid \|x - a\| \le r\}$. For a subset $A \subset E$ we denote by ∂A, $\operatorname{int} A$, $\operatorname{cl} A$ the boundary, the interior and the closure of the set A, respectively. The diameter of a set A is defined by $\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$.

Let a subset $A \subset E$ be fixed. Then the *distance function* from the point $x \in E$ to the set A is given by the formula

$$\varrho(x) = \varrho(x, A) = \inf_{a\in A} \|x - a\|.$$

For points $x_0, x_1 \in E$, a number $\lambda \in [0, 1]$ and a function $f : E \to \mathbb{R}$ we put

$$x_\lambda = (1-\lambda)x_0+\lambda x_1, \qquad [x_0, x_1] = \{x_\lambda \mid \lambda \in [0, 1]\}, \qquad f_\lambda = (1-\lambda)f(x_0)+\lambda f(x_1).$$

For a subset $A \subset E$ and a number $\delta > 0$ we define

$$U(\delta, A) = \{x \in E \mid 0 < \varrho(x) < \delta\}.$$

Definition 1.1. Let $U \subset \mathcal{H}$ be an open subset. A function $f : U \to \mathbb{R}$ is called *weakly convex with the constant* $C > 0$ on the set U, if it is continuous on the set U and the function $f(x) + \frac{C}{2}\|x\|^2$ is convex on each convex subset $V \subset U$.

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It's easy to see that the condition of convexity in Definition 1.1 is equivalent to the next inequality: for each pair of points $x_0, x_1 \in U$, such that $[x_0, x_1] \subset U$, and for any number $\lambda \in [0,1]$, we have

$$f(x_\lambda) \le f_\lambda + \frac{C}{2}\lambda(1-\lambda)\|x_0 - x_1\|^2. \tag{1}$$

So, we shall use Definition 1.1 as well as Formula (1) when speaking about weakly convex function with the constant $C > 0$ in the space $\mathcal{H}$.

Definition 1.1 and properties of weakly convex functions have been considered in [19], [20], [11] (see also the bibliographies in these papers). In particular, very close relationship between the class of $C^{1,1}$ functions and the class of weakly convex functions was established in [20] and especially in [11].

Definition 1.2. Let $U \subset E$ be an open subset. A function $f : U \to \mathbb{R}$ is called *weakly convex with the modulus* $\mu : [0, \operatorname{diam} U) \to [0, +\infty)$, ($\mu(t)$ increasing, $\mu(0) = 0$) on the set U, if it is continuous on the set U and for any pair of points $x_0, x_1 \in U$, such that $[x_0, x_1] \subset U$, and for any number $\lambda \in [0,1]$, we have

$$f(x_\lambda) \le f_\lambda + \lambda(1-\lambda)\mu(\|x_0 - x_1\|). \tag{2}$$

To our knowledge Definition 1.2 appeared in the papers of S. Rolewicz [16, 17] as *strong* $\alpha(\cdot)$ *paraconvexity.* See the details in [17].

Under the name *semiconvex functions* the concept can be found in [6].

It's easy to see that Definition 1.1 is some special case of Definition 1.2 for the modulus $\mu(t) = \frac{C}{2}t^2$ in the case $E = \mathcal{H}$.

Definition 1.2 is very similar to the definition of strongly convex function [14, 21], but instead of sign "minus" before the modulus of convexity we have sign "plus".

Each function which is pointwisely not less than

$$\mu_0(t) = \sup_{x_0, x_1 \in U:\ \|x_0 - x_1\| = t,\ \lambda \in (0,1)} \frac{f(x_\lambda) - f_\lambda}{\lambda(1-\lambda)}$$

can be taken in Definition 1.2 as the modulus of convexity μ.

We shall call the function μ_0 *the precise modulus of weak convexity.*

We recall that for all x_0, x_1 in the space E

$$|\varrho(x_0, A) - \varrho(x_1, A)| \le \|x_0 - x_1\|. \tag{3}$$

Lemma 1.3. *Let $A \subset E$ be a closed subset. Then the distance function $\varrho(x) = \varrho(x, A)$ satisfies the next condition*

$$\varrho(x_\lambda) \le \varrho_\lambda + 2\lambda(1-\lambda)\|x_0 - x_1\|. \tag{4}$$

Proof. By the formulae $x_\lambda - x_0 = \lambda(x_1 - x_0)$, $x_\lambda - x_1 = (1-\lambda)(x_0 - x_1)$ and Formula (3) we get $|\varrho(x_\lambda) - \varrho_\lambda| \leq$

$$\begin{aligned} &\leq (1-\lambda)|\varrho(x_\lambda) - \varrho_0| + \lambda|\varrho(x_\lambda) - \varrho_1| \\ &\leq (1-\lambda)\|x_\lambda - x_0\| + \lambda\|x_\lambda - x_1\| = 2\lambda(1-\lambda)\|x_0 - x_1\|. \qquad \square \end{aligned}$$

Thus, the distance function satisfies Definition 1.2 with the modulus $\mu(t) = 2t$ for any closed subset. It is also worth recalling that for a closed subset $A \subset E$ the distance function $\varrho(x, A)$ is convex on the space E if and only if the set A is convex.

Note that if the precise modulus of weak convexity equals zero (or nonpositive) for a continuous function, then such function is convex.

For nonconvex function the precise modulus of nonconvexity can not be an arbitrary function. Using ideas from [18, Lemma 1, §7], [21] we shall prove the next result.

Lemma 1.4. *Let $U \subset E$ be an open set and $f : U \to \mathbb{R}$ be a weakly convex function with the precise modulus of weak convexity μ_0. Then for any number $c > 1$ we have inequality*

$$\mu_0(ct) \leq c^2 \mu_0(t). \tag{5}$$

In particular, if the function f is not convex, then

$$t^2 = O(\mu_0(t)), \quad \text{as } t \to +0. \tag{6}$$

Proof. Let us fix $t > 0$, $\varepsilon > 0$ and $c \in (1, 2)$. Choose points $x_0, x_1 \in U$ and a number $\lambda \in (0,1)$ such that $[x_0, x_1] \subset U$, $\|x_0 - x_1\| = ct$ and

$$-\varepsilon + \mu_0(ct) \leq \frac{f(x_\lambda) - f_\lambda}{\lambda(1-\lambda)} \leq \mu_0(ct).$$

From the previous formula we have

$$f(x_\lambda) - f_\lambda \geq \lambda(1-\lambda)\mu_0(ct) - \lambda(1-\lambda)\varepsilon. \tag{7}$$

Suppose without loss of generality that $\lambda \in (0, \frac{1}{2}]$. Then $0 < \lambda \leq \lambda c < 1$. Consider the point

$$x_2 = \left(1 - \frac{1}{c}\right) x_0 + \frac{1}{c} x_1$$

and observe that $\|x_0 - x_2\| = \frac{\|x_0 - x_1\|}{c} = t$ and

$$x_\lambda = (1 - \lambda c)x_0 + \lambda c x_2.$$

From the definitions of the point x_2 and the precise modulus of weak convexity we obtain, that

$$f(x_2) \leq \left(1 - \frac{1}{c}\right) f(x_0) + \frac{1}{c} f(x_1) + \frac{1}{c}\left(1 - \frac{1}{c}\right) \mu_0(ct), \tag{8}$$

and also

$$f(x_\lambda) \le (1-\lambda c)\, f(x_0) + \lambda c f(x_2) + \lambda c\,(1-\lambda c)\,\mu_0(t). \tag{9}$$

Now we multiply (8) by λc and rewrite Formulae (8) and (9) as

$$\lambda\left(1-\frac{1}{c}\right)\mu_0(ct) \ge \lambda c f(x_2) - (\lambda c - \lambda) f(x_0) - \lambda f(x_1), \tag{10}$$

$$\lambda c(1-\lambda c)\mu_0(t) \ge f(x_\lambda) - \lambda c f(x_2) - (1-\lambda c) f(x_0). \tag{11}$$

By Formulae (10) and (11), taking into account (7), we get

$$\begin{aligned}
&\lambda\left(1-\frac{1}{c}\right)\mu_0(ct) + \lambda c(1-\lambda c)\mu_0(t)\\
&\ge f(x_\lambda) - (1-\lambda c) f(x_0) - (\lambda c-\lambda) f(x_0) - \lambda f(x_1)\\
&= f(x_\lambda) - (1-\lambda) f(x_0) - \lambda f(x_1) = f(x_\lambda) - f_\lambda\\
&\ge \lambda(1-\lambda)\mu_0(ct) - \lambda(1-\lambda)\varepsilon.
\end{aligned}$$

Divided on λ, we obtain inequality (note that $\lambda \in (0, \frac{1}{2}]$, $c \in (1,2)$)

$$c^2\mu_0(t) - \mu_0(ct) \ge -\frac{\varepsilon c}{1-\lambda c} \ge -\varepsilon\frac{c}{1-\frac{c}{2}}.$$

The number $\varepsilon > 0$ is arbitrary in the last formula, hence $c^2\mu_0(t) - \mu_0(ct) \ge 0$. Formula (5) is proved when $c \in (1,2)$.

In the case $c \ge 2$, we can choose such number $b \in (1,2)$ and natural n, that $c = b^n$. Changing c on b, we can repeat the previous proof and obtain that $\mu_0(ct) = \mu_0(b^n t) \le b^2\mu_0(b^{n-1}t) \le \cdots \le b^{2n}\mu_0(t) = c^2\mu_0(t)$. Thus Formula (5) is proved for any $c > 1$.

If the function f is not convex, then we can find $t > 0$ with $\mu_0(t) > 0$. We can rewrite Formula (5) for λt and $c = \frac{1}{\lambda}$ as

$$\frac{\mu_0(t)}{t^2} = \frac{\mu_0\left(\frac{1}{\lambda}\lambda t\right)}{t^2} \le \frac{\mu_0(\lambda t)}{\lambda^2 t^2}, \quad \forall \lambda \in (0,1).$$

Taking $C = \frac{t^2}{\mu_0(t)} > 0$, we have $s^2 \le C\mu_0(s)$, $\forall s \in (0,t)$. The last condition is equivalently to Formula (6). □

The purpose of the present paper is to characterize (nonconvex) closed subsets $A \subset E$ with weakly convex distance function on some neighborhood $A \cup U(\delta, A)$ of the set A, where $\delta > 0$.

As we can see from Lemma 1.3, the modulus of weak convexity of the first order takes place for any set. Thus, taking into account Lemma 1.4, we shall consider the situation when the modulus from Definition 1.2 has order $t^2 = O(\mu(t))$, $t \to +0$ and $\mu(t) = o(t)$, $t \to +0$. Herewith the class of proximally smooth sets with constant $R > 0$ from the space E shall be very important for us.

Modulus of convexity of a Banach space E is the function $\delta_E : (0; 2] \to \mathbb{R}$, which is defined by the formula

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| \mid \ x, y \in E,\ \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\}.$$

A Banach space E is called *uniformly convex*, if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0; 2]$.

Modulus of smoothness of a Banach space E is the function $\varrho_E : [0; +\infty) \to \mathbb{R}$, which is defined by the formula

$$\varrho_E(\tau) = \sup\left\{\left\|\frac{x+y}{2}\right\| + \left\|\frac{x-y}{2}\right\| - 1 \mid \ \|x\| = 1,\ \|y\| = \tau\right\}.$$

A Banach space E is called *uniformly smooth*, if

$$\lim_{\tau \to +0} \frac{\varrho_E(\tau)}{\tau} = 0.$$

Remark 1.5. Any uniformly convex or uniformly smooth Banach space, as well as it dual Banach space, is reflexive [9, Chapter 2, §4, Theorem 2 and Corollary 2].

For the subset $A \subset E$ and the point $x \in E$ define $\pi(x, A) = \{a \in A \mid \|x - a\| = \varrho(x, A)\}$.

Definition 1.6 ([8]). A subset $A \subset E$ is called *proximally smooth* with the constant $R > 0$, if the distance function $x \to \varrho(x, A)$ is continuously differentiable on the set $U(R, A)$.

Definition 1.7. We say that a subset $A \subset E$ satisfies *P-supporting condition* with the constant $R > 0$, if for any such point $x \in U(R, A)$, that $a \in \pi(x, A)$, we have inequality

$$\varrho\left(a + \frac{R}{\|x-a\|}(x-a), A\right) \ge R. \tag{12}$$

Proposition 1.8. *Let E be a uniformly convex and uniformly smooth Banach space. Let $A \subset E$ be a closed subset, $R > 0$. The next conditions are equivalent:*

(*1*) *The set A satisfies to the P-supporting condition with the constant R.*
(*2*) *The set A is proximally smooth with the constant R.*
(*3*) *The projection mapping $x \to \pi(x, A)$ is singleton and continuous on the set $U(R, A)$.*

Equivalences in Proposition 1.8 were established in the papers [10, 15, 19] in finite dimension case. In the case of the real Hilbert space Proposition 1.8 was proved in the papers [8, 13]. In arbitrary uniformly convex and uniformly smooth Banach space Proposition 1.8 was established in [3].

2. Weak convexity of the distance function in a Hilbert space

Definition 2.1. Let $x_0, x_1 \in E$, $\|x_1 - x_0\| \le 2R$. The set

$$D_R(x_0, x_1) = \bigcap_{a \in E:\ \{x_0, x_1\} \subset B_R(a)} B_R(a)$$

is called *strongly convex segment*, and the set

$$D_R^0(x_0, x_1) = D_R(x_0, x_1) \setminus \{x_0, x_1\}.$$

is called *strongly convex segment without endpoints.*

Definition 2.2 ([19]). A subset $A \subset E$ is called *weakly convex* with the constant $R > 0$, if for any pair of points $x_0, x_1 \in A$ such that $0 < \|x_1 - x_0\| < 2R$, the next condition is valid

$$A \bigcap D_R^0(x_0, x_1) \neq \emptyset.$$

By [1, Theorem 2.2] the class of proximally smooth sets with the constant $R > 0$ from $\mathcal{H}$ coincides with the class of weaklly convex sets (in the sense of J.-Ph. Vial, Definition 2.2) with the same constant $R > 0$.

Proposition 2.3 ([11, Theorem 1.16.1]). *Let a subset $A \subset \mathcal{H}$ be closed, connected and locally weakly convex with the constant $R > 0$, i.e., for any point $a \in A$ there exists some positive number $\delta(a) > 0$ such that the set $A \cap B_{\delta(a)}(a)$ is weakly convex with the constant R. Let* $\operatorname{diam} A < 2R$. *Then the set A is weakly convex with the constant R.*

Theorem 2.4. *Let a subset $A \subset \mathcal{H}$ be closed, connected and $\delta > 0$. Let the distance function $\varrho(x) = \varrho(x, A)$ be weakly convex with the constant $C > 0$ on the set $A \cup U(\delta, A)$ and* $\operatorname{diam} A < \frac{2}{C}$. *Then the set A is proximally smooth with the constant $R = \frac{1}{C}$.*

Proof. Let $x_0, x_1 \in A$, $0 < \|x_0 - x_1\| < \beta = \min\{\frac{2}{C}, \delta\}$. Then $[x_0, x_1] \subset A \cup U(\delta, A)$ and $\varrho(x_{\frac{1}{2}}) \le \frac{C}{8}\|x_0 - x_1\|^2$ (Formula (1) with $f = \varrho$ and $\lambda = \frac{1}{2}$). Taking $\varepsilon = \|x_0 - x_1\|$, we have

$$A \cap B_{\frac{C\varepsilon^2}{8}}\left(x_{\frac{1}{2}}\right) \neq \emptyset. \tag{13}$$

Put $R = \frac{1}{C}$. Using inequality $\frac{\varepsilon^2}{8R} \le R - \sqrt{R^2 - \frac{\varepsilon^2}{4}}$, by Formula (13) we obtain that

$$A \cap B_{R - \sqrt{R^2 - \frac{\varepsilon^2}{4}}}\left(x_{\frac{1}{2}}\right) \neq \emptyset. \tag{14}$$

By Pythagoras theorem we easily conclude that (in the case $\varepsilon < \beta$)

$$B_{R - \sqrt{R^2 - \frac{\varepsilon^2}{4}}}\left(x_{\frac{1}{2}}\right) \subset D_R^0(x_0, x_1).$$

By the last inclusion and by Formula (14) we get

$$A \cap D^0_R(x_0, x_1) \neq \emptyset, \quad \forall x_0, x_1 \in A: \ 0 < \|x_0 - x_1\| < \beta.$$

Choose a point $a \in A$ and arbitrary points $x_0, x_1 \in A \cap B_{\frac{\beta}{3}}(a)$. Note that $\|x_0 - x_1\| < \beta$ and $D^0_R(x_0, x_1) \subset B_{\frac{\beta}{3}}(a)$ (from inequality $\frac{\beta}{3} < R$). We have

$$\left(A \bigcap B_{\frac{\beta}{3}}(a)\right) \bigcap D^0_R(x_0, x_1) = A \bigcap D^0_R(x_0, x_1) \neq \emptyset.$$

Thus for any point $a \in A$ we can find the number $\delta(a) = \frac{1}{3}\beta > 0$ (the same for all points) such that the set $A \cap B_{\frac{1}{3}\beta}(a)$ is weakly convex with the constant R. By Proposition 2.3 the set A is weakly convex with the constant R, and hence the set A is also proximally smooth with the constant R. □

Corollary 2.5. *Suppose that the conditions of Theorem* 2.4 *hold true. Then the distance function $\varrho(x)$ is continuously differentiable on the set $U(\frac{1}{C}, A)$.*

Proof. Follows from Theorem 2.4 and Definition 1.6. □

Lemma 2.6. *Let a space E be uniformly convex and uniformly smooth. Let a closed subset $A \subset E$ satisfies the P-supporting principle with the constant $R > 0$ and $\delta \in (0, R)$. Define $A_0 = \mathrm{cl}\,(A + \mathrm{int}\, B_\delta(0))$. Then the set A_0 satisfies P-supporting condition with the constant $r = R - \delta$.*

Proof. Fix a point $x \in U(r, A_0) \subset U(R, A)$. By Proposition 1.8 $\pi(x, A) = \{a\}$ and for the point $z = a + \frac{x-a}{\|x-a\|}R$ we have $A \cap \mathrm{int}\, B_R(z) = \emptyset$. By the last equality we have $\mathrm{cl}(A + \mathrm{int}\, B_\delta(0)) \cap \mathrm{int}\, B_{R-\delta}(z) = \emptyset$ and $y = a + \frac{x-a}{\|x-a\|}\delta \in \partial A_0$, $\|x - y\| = \varrho(x, A) - \delta = \varrho(x, A_0)$. Because of arbitrariness of the point $x \in U(r, A_0)$ by Proposition 1.8 we have P-supporting condition for the set A_0 with the constant $r = R - \delta$ (and also the condition of proximal smoothness with the constant r). □

Theorem 2.7. *Let a subset $A \subset \mathcal{H}$ be closed and proximally smooth with the constant $R > 0$. Let $\delta \in (0, R)$. Then the distance function $\varrho(x) = \varrho(x, A)$ is weakly convex on the set $A \cup U(\delta, A)$ with the constant of weak convexity*

$$C = \frac{1}{R - \delta}.$$

Proof. Choose $x_0, x_1 \in A \cup U(\delta, A)$, $[x_0, x_1] \subset A \cup U(\delta, A)$. Without loss of generality we may assume that $0 < \varrho(x_0) \leq \varrho(x_1) < \delta$. Define the number $r = R - \delta > 0$.

Let us define the set $A_0 = \mathrm{cl}\left(A + \mathrm{int}\, B_{\varrho(x_0)}(0)\right)$. By Lemma 2.6 (see also [11, Theorem 1.12.4]) the set A_0 is proximally smooth with the constant $R - \varrho(x_0)$, i.e. with the constant r too. By [3, Theorem 6.10] (see also Lemma 2.6) $x_0 \in \partial A_0$. Let $a_i = \pi(x_i, A)$, $i = 0, 1$.

Define the point $y \in [a_1, x_1] \cap \partial A_0$, $\|a_1 - y\| = \varrho(x_0)$. Let $\varrho = \|y - x_1\| = \varrho(x_1) - \varrho(x_0) \geq 0$, $w = \frac{1}{2}(y + x_0)$, $z = \frac{1}{2}(y + x_1)$, $p = (y - x_1)/\|y - x_1\|$. Define halfspaces $H^+ = \{x \mid \langle p, x\rangle \geq \langle p, z\rangle\}$, $H^- = \{x \mid \langle p, x\rangle < \langle p, z\rangle\}$. Let

$$\varphi(t) = t + \frac{t^2}{r}.$$

Using the above definitions we have

$$\begin{aligned}\varrho(x_{\frac{1}{2}}) &\leq \|x_{\frac{1}{2}} - w\| + \varrho(w, A_0) + \varrho(x_0)\\ &= \frac{1}{2}\|y - x_1\| + \varrho(w, A_0) + \varrho(x_0) = \varrho_{\frac{1}{2}} + \varrho(w, A_0).\end{aligned} \tag{15}$$

For any natural k define $\tilde{\varepsilon}_k = 2^{-k}$ and

$$r_k = \begin{cases} \frac{1}{2}r, & r - 2^{-k} \leq 0,\\ r - 2^{-k}, & r - 2^{-k} > 0.\end{cases}$$

By the asymptotic equality $r - \sqrt{r^2 - \frac{t^2}{4}} \sim \frac{t^2}{8r}$, $t \to 0$, we obtain that for every sufficiently large natural k we can find the number $\varepsilon_k \in (0, \tilde{\varepsilon}_k)$ such that for all $t \in (0, \varepsilon_k)$ the next inequality is valid

$$r - \sqrt{r^2 - \frac{t^2}{4}} \leq \frac{t^2}{8r_k} \leq \frac{\varphi^2(t)}{8r_k}. \tag{16}$$

We shall choose the points x_0 and x_1 with one more additional property $\varepsilon = \|x_0 - x_1\| < \varepsilon_k$.

If $x_0 \in H^+$, then $\|y - x_0\| \leq \|x_0 - x_1\| = \varepsilon < \varepsilon_k$. The set A_0 satisfies P-supporting condition in $\mathcal{H}$ with the constant r, hence it is weakly convex and proximally smooth with the constant r. Thus

$$D_r^0(x_0, y) \cap A_0 \neq \emptyset,$$

and, moreover, by [11, Theorem 1.4.1] the points x_0 and y can be connected by some continuous curve Γ from the set $A_0 \cap D_r^0(x_0, y)$. By Pythagoras theorem

$$\Gamma \cap B_{r - \sqrt{r^2 - \frac{\|y - x_0\|^2}{4}}}(w) \neq \emptyset,$$

and by Formula (16) we obtain that

$$\varrho(w, A_0) \leq \frac{\|y - x_0\|^2}{8r_k} \leq \frac{\varepsilon^2}{8r_k}. \tag{17}$$

Let $x_0 \in H^-$. By P-supporting condition we have $x_0 \notin B_r(y - rp)$, and by Pythagoras theorem (in the plane yx_1x_0) we obtain that

$$\varepsilon^2 \geq \left[r^2 - \left(r - \frac{\varrho}{2}\right)^2\right] + \frac{\varrho^2}{4} = r\varrho,$$

hence $\varrho \le \frac{\varepsilon^2}{r}$. By the triangle inequality

$$\|y - x_0\| \le \|x_0 - x_1\| + \|y - x_1\| \le \varepsilon + \frac{\varepsilon^2}{r} = \varphi(\varepsilon).$$

Repeating the previous case $x_0 \in H^+$, we have

$$\varrho(w, A_0) \le \frac{\|y - x_0\|^2}{8r_k} \le \frac{\varphi^2(\varepsilon)}{8r_k}. \tag{18}$$

Taking together the results of Formula (17) (in the case $x_0 \in H^+$) and Formula (18) (in the case $x_0 \in H^-$) we deduce by Formula (15), that in any case under the condition $\|x_0 - x_1\| = \varepsilon \in (0, \varepsilon_k)$ we have the estimate

$$\varrho(x_{\frac{1}{2}}) \le \varrho_{\frac{1}{2}} + \varrho(w, A_0) \le \varrho_{\frac{1}{2}} + \frac{\varphi^2(\varepsilon)}{8r_k}. \tag{19}$$

Put $l_k = \left(1 + \frac{\tilde{\varepsilon}_k}{r}\right)^{-2} r_k$. By the condition $\|x_0 - x_1\| = \varepsilon \in (0, \varepsilon_k)$ and by the definition of the function φ we obtain that

$$\frac{\varphi^2(\varepsilon)}{8r_k} \le \frac{\varepsilon^2}{8l_k},$$

and using Formula (19)

$$\varrho(x_{\frac{1}{2}}) \le \varrho_{\frac{1}{2}} + \frac{\varepsilon^2}{8l_k}. \tag{20}$$

Thus the function $g_k(x) = \varrho(x) + \frac{1}{2l_k}\|x\|^2$ satisfies the inequality of convexity with $\lambda = \frac{1}{2}$ in the neighborhood of radius $\frac{1}{2}\varepsilon_k$ of any point from the set $A \cup U(\delta, A)$. Using arguments from [2, Lemma 3.2, page 757] and continuity of the function $g_k(x)$ we obtain that the function $g_k(x)$ is convex on each convex subset $V \subset A \cup U(\delta, A)$. Hence the limit

$$\lim_{k\to\infty} g_k(x) = \varrho(x) + \frac{1}{2r}\|x\|^2$$

also has the same property. By Definition 1.1 the function $\varrho(x)$ is weakly convex with the constant $C = \frac{1}{r} = \frac{1}{R-\delta}$. $\square$

Example 2.8. Consider in 2-dimensional Euclidean space $\mathbb{R}^2$ the set

$$A = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0,\ x_2 \ge 0,\ x_1^2 + x_2^2 = R^2\right\}.$$

The set A is the arc of the circle of radius $R > 0$ from the first quarter of Cartesian system of coordinates. It is easy to see that the set A is proximally smooth with the constant R.

Put

$$U_0 = \left\{x_1 \ge 0,\ x_2 \ge 0,\ x_1^2 + x_2^2 \le R^2\right\}, \qquad U_1 = \{x_2 \le 0,\ x_1 \ge x_2\},$$
$$U_2 = \{x_1 \le 0,\ x_2 \ge x_1\}, \qquad U_3 = \left\{x_1 \ge 0,\ x_2 \ge 0,\ x_1^2 + x_2^2 \ge R^2\right\}.$$

It's easy to see that the distance function from the point $x = (x_1, x_2) \in \mathbb{R}^2$ to the set A is given by the formula

$$\varrho(x) = \varrho(x, A) = \begin{cases} R - \|x\|, & x \in U_0, \\ \sqrt{(x_1 - R)^2 + x_2^2}, & x \in U_1, \\ \sqrt{x_1^2 + (x_2 - R)^2}, & x \in U_2, \\ \|x\| - R, & x \in U_3. \end{cases}$$

Note that the function ϱ is convex on the sets U_1, U_2, U_3, and concave on the set U_0.

Let $C > 1$. Define $g_C(x) = R - \|x\| + \frac{C}{2}\|x\|^2$, $g_C(x) = \varrho(x) + \frac{C}{2}\|x\|^2$ for $x \in U_0$.

Convexity of the function g_C can be checked by checking the positive definiteness of Hessian

$$g_C''(x) = \begin{pmatrix} C - \dfrac{1}{\|x\|} + \dfrac{x_1^2}{\|x\|^3} & \dfrac{x_1 x_2}{\|x\|^3} \\ \dfrac{x_1 x_2}{\|x\|^3} & C - \dfrac{1}{\|x\|} + \dfrac{x_2^2}{\|x\|^3} \end{pmatrix}.$$

By Sylvester's criteria the positive definiteness of the matrix g_C'' is equivalent to the conditions $(g_C''(x))_{11} = C - \frac{1}{\|x\|} + \frac{x_1^2}{\|x\|^3} > 0$ and $\det g_C''(x) > 0$. The condition $\det g_C''(x) > 0$ is equivalent to the condition $C - \frac{1}{\|x\|} > 0$, which entails also the inequality $(g_C''(x))_{11} > 0$. Thus, the function g_C is convex on each convex subset of the set $U_0 \cap \{\|x\| > \frac{1}{C}\}$. From this fact we obtain that the function ϱ is weakly convex with any constant $C \geq \frac{1}{R-\delta}$ on the set $A \cup U(\delta, A)$, $\delta \in (0, R)$. Thus the estimate of Theorem 2.7 is exact. □

3. Weak convexity of the distance function in uniformly convex and uniformly smooth Banach spaces

Recall that *Clarke's subdifferential* of a locally Lipschitz function $f : E \to \mathbb{R}$ at the point x_0 is nonempty bound closed convex subset $\partial_{Cl} f(x_0) \subset E^*$, which can be given via it's supporting function by the formula (see [7]): for any $y \in E$

$$\limsup_{\lambda \to +0,\ x \to x_0} \frac{f(x + \lambda y) - f(x)}{\lambda} = s(y, \partial_{Cl} f(x_0)) = \sup_{p \in \partial_{Cl} f(x_0)} \langle p, y \rangle. \qquad (21)$$

Lemma 3.1. *Let a Banach space E be uniformly convex and uniformly smooth. Let a subset $A \subset E$ be a closed cavern, i.e. A is closed and* $\mathrm{cl}(E \backslash A)$ *is a convex set with nonempty interior. Suppose that there exists $\delta > 0$ such that the distance function $\varrho(x) = \varrho(x, A)$ is weakly convex on the set $U(\delta, A)$ with the modulus $\mu(t)$, $\mu(t) = o(t)$, $t \to +0$. Then the set A is proximally smooth with some constant $R \geq \delta$.*

Proof. Fix $x_0 \in U(\delta, A)$, also consider such point x_1, that $[x_0, x_1] \subset U(\delta, A)$.

It's known [4, Lemma 3.1], that for any cavern A the distance function is concave on the set $E\backslash A$, i.e. it is concave on each convex subset $V \subset U(\delta, A)$. Consider both conditions of concavity and weak convexity together: for any $\lambda \in [0,1]$

$$\varrho_\lambda \leq \varrho(x_\lambda) \leq \varrho_\lambda + \lambda(1-\lambda)\mu(\|x_0 - x_1\|).$$

We obtain, that (denoting $x_1 = x$)

$$\varrho(x) - \varrho(x_0) \leq \frac{\varrho(x_0 + \lambda(x - x_0)) - \varrho(x_0)}{\lambda} \leq \varrho(x) - \varrho(x_0) + (1-\lambda)\mu(\|x_0 - x\|),$$

The function ϱ is concave, $x_0 \in U(\delta, A)$, and hence superdifferential

$$\partial^+\varrho(x_0) = \left\{ p \in E^* \mid \langle p, y\rangle \geq \lim_{\lambda \to +0} \frac{\varrho(x_0 + \lambda y) - \varrho(x_0)}{\lambda},\ \forall y \in E \right\}$$

is nonempty set and it coincides (for the same reason as for the convex functions) with $\partial_{Cl}\varrho(x_0)$.

Take a point $p \in \partial^+\varrho(x_0) = \partial_{Cl}\varrho(x_0)$. We have

$$0 \leq \varrho(x_0) + \langle p, x - x_0\rangle - \varrho(x) \leq \mu(2\|x_0 - x\|), \quad \text{for all } x_0, x :\ [x_0, x] \subset U(\delta, A).$$

The last formula gives the continuous differentiability of the function ϱ on the set $U(\delta, A)$, $\varrho'(x_0) = p$. $\square$

The result of Lemma 3.1 holds true for an arbitrary closed set. We use the results from [3] to prove this.

Theorem 3.2. *Let a Banach space E be uniformly convex and uniformly smooth with moduli of convexity and smoothness of power type. Let $\delta > 0$ and $A \subset E$ be a closed set such that the distance function $\varrho(x) = \varrho(x, A)$ is weakly convex in the sense of Definition* 1.2 *on the set $U(\delta, A)$ with the modulus $\mu(t) = o(t)$, $t \to +0$. Then the set A is weakly convex with some constant $R \geq \delta$.*

Proof. Choose $x_0, x_1 \in U(\delta, A)$, $[x_0, x_1] \subset U(\delta, A)$. By Formula (2) we obtain that for any $\lambda \in [0,1]$

$$\varrho(x_1) \geq \varrho(x_0) + \frac{\varrho(x_0 + \lambda(x_1 - x_0)) - \varrho(x_0)}{\lambda} - (1-\lambda)\mu(\|x_0 - x_1\|).$$

Taking Clarke's derivative at the point x_0, we have

$$\varrho(x_1) \geq \varrho(x_0) + s(x_1 - x_0, \partial_{Cl} f(x_0)) - \mu(2\|x_0 - x_1\|).$$

From the nonemptiness of Clarke's subdifferential (21) we obtain that there exists at least one point $p \in \partial_{Cl} f(x_0)$ such that $\langle x_1 - x_0, p\rangle \leq s(x_1 - x_0, \partial_{Cl} f(x_0))$ for all x_1. Taking $x_1 = x$, we have, that for any number $\alpha \geq \varrho(x)$

$$\alpha - \varrho(x_0) + \mu(2\|x - x_0\| + 2|\alpha - \varrho(x_0)|) \geq \langle p, x - x_0\rangle.$$

In the space $E \times \mathbb{R}$ the above inequality is equivalent to the next one

$$(p,-1)\cdot(x-x_0,\alpha-\varrho(x_0)) \le \mu(2\|x-x_0\|+2|\alpha-\varrho(x_0)|), \ \text{where}\ \mu(t)=o(t),\ t\to+0.$$

The last inequality shows that the vector p is a Frechet normal functional of the function ϱ at the point x_0 [5] in the space $E\times\mathbb{R}$ with the norm $\|\cdot\|+|\cdot|$.

So, the set of Frechet normal functionals $\partial_F\varrho(x_0)$ at any point $x_0\in U(\delta,A)$ is not empty. If $x_0\in A$, then $0\in\partial_F\varrho(x_0)$. By [3, Theorem 6.2 parts (b) and (f)], the set A is proximally smooth with the constant δ. $\square$

Remark 3.3. The power type for the moduli of convexity and smoothness of the space E is not necessary. We use this condition because we refer on the results from [3]. Using the results from [1, 3, 5], one can prove Theorem 3.2 in an arbitrary uniformly convex and smooth Banach space. We don't do it because of bulky calculations.

Proposition 3.4 ([12, Lemma 4.1]). *Let E be a uniformly convex and uniformly smooth Banach space with the modulus of uniform smoothness ϱ_E. Suppose that a subset $A\subset E$ satisfies P-supporting condition with the constant R. Let $x_0,x_1\in A$ and $\lambda\in[0,1]$. Then*

$$\varrho(x_\lambda,A)\le 8R\lambda(1-\lambda)\varrho_E\left(\frac{\|x_0-x_1\|}{R}\right).$$

Theorem 3.5. *Let E be a uniformly convex and uniformly smooth Banach space with modulus of smoothness ϱ_E. Suppose that a set $A\subset E$ is closed and proximally smooth with the constant $R>0$. Let $\delta\in(0,R)$. Then the distance function $\varrho(x)=\varrho(x,A)$ is weakly convex on the set $A\cup U(\delta,A)$ in the sense of Definition* 1.2 *with the modulus*

$$\mu(\varepsilon)=8(R-\delta)\varrho_E\left(\frac{2\varepsilon}{R-\delta}\right).$$

Proof. Fix $x_0,x_1\in A\cup U(\delta,A)$, $[x_0,x_1]\subset A\cup U(\delta,A)$. Without loss of generality we may assume that $0<\varrho(x_0)\le\varrho(x_1)<\delta$. Define $r=R-\delta>0$.

Let $A_0=\operatorname{cl}\left(A+\operatorname{int}B_{\varrho(x_0)}(0)\right)$. By Lemma 2.6 the set A_0 satisfies P-supporting condition with the constant $R-\varrho(x_0)$, i.e. with the constant r too. Also by Lemma 2.6 $x_0\in\partial A_0$. Let $a_i=\pi(x_i,A)$, $i=0,1$.

Fix $\lambda\in(0,1)$.

Define the point $y\in[a_1,x_1]\cap\partial A_0$, $\|a_1-y\|=\varrho(x_0)$. Let $\varrho=\|y-x_1\|=\varrho(x_1)-\varrho(x_0)\ge0$, $w=\lambda y+(1-\lambda)x_0$.

We have the next estimate

$$\begin{aligned}\varrho(x_\lambda)&\le\|x_\lambda-w\|+\varrho(w,A_0)+\varrho(x_0)\\&=\lambda\|y-x_1\|+\varrho(w,A_0)+\varrho(x_0)=\varrho_\lambda+\varrho(w,A_0).\end{aligned}\tag{22}$$

By Proposition 3.4 we have

$$\varrho(w, A_0) \leq 8r\lambda(1-\lambda)\varrho_E\left(\frac{\|y - x_0\|}{r}\right). \tag{23}$$

By Formula (3) $\|y - x_1\| = \varrho(x_1) - \varrho(x_0) \leq \|x_0 - x_1\|$ and

$$\|y - x_0\| \leq \|y - x_1\| + \|x_0 - x_1\| \leq 2\|x_0 - x_1\|.$$

By Formula (23) we obtain that

$$\varrho(w, A_0) \leq 8r\lambda(1-\lambda)\varrho_E\left(\frac{2\|x_0 - x_1\|}{r}\right).$$

Substituting the last inequality in the estimate (22), we have

$$\varrho(x_\lambda) \leq \varrho_\lambda + 8(R-\delta)\lambda(1-\lambda)\varrho_E\left(\frac{2\|x_0 - x_1\|}{R-\delta}\right). \qquad \square$$

Thus, for any uniformly convex and uniformly smooth Banach space any nonconvex proximally smooth set has weakly convex distance function in some neighborhood with modulus μ of the type $t^2 = O(\mu(t))$, $t \to +0$ and $\mu(t) = o(t)$, $t \to +0$. The converse statement also holds true.

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