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Inequalities for Polynomials on the Unit Square via the Krein-Milman Theorem

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In this paper we provide sharp Bernstein and Markov inequalities for 2-homogeneous polynomials on the square $\square \subset \mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. If $\mathcal{P}(^2\square)$ is the space of such polynomials, we also find the polarization constant of $\mathcal{P}(^2\square)$ and the unconditional constant for the canonical basis of $\mathcal{P}(^2\square)$. All the results are obtained by means of the Krein-Milman Theorem, using a characterization of the extreme 2-homogeneous polynomials on $\square$ which is also given in the paper.

Keywords: Convexity, extreme points, polynomial norms, Bernstein and Markov inequalities, polarization constants

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1. Preliminaries

According to the Krein-Milman Theorem, every nonempty compact convex set in a Banach space is fully described by the set of its extreme points. We recall that if C is a convex set in a Banach space, a point $e \in C$ is said to be *extreme* if $x, y \in C$

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and $\lambda x + (1-\lambda)y = e$, for some $0 < \lambda < 1$, entails $x = y = e$. Equivalently, $e \in C$ is extreme if and only if $C \setminus \{e\}$ is convex.

In this paper we are interested in the study of the extreme points in the unit ball of finite dimensional spaces of polynomials. This question has been addressed by many authors in the past. Just to mention a few, we can say that Konheim and Rivlin [19] found a characterization of the extreme polynomials of the unit ball of the space of polynomials in one real variable endowed with the sup norm on $[-1, 1]$. Konheim and Rivlin's characterization does not include an explicit description of the extreme polynomials. In the authors' humble opinion, providing such an explicit description might be a too ambitious task, however, it is possible to solve the problem partially for smaller sets of polynomials. For instance, in [28] the authors describe, explicitly, the extreme polynomials in the unit ball of the space of trinomials with independent term on the real line with the sup norm on symmetric intervals. This contribution completes a previous work by Aron and Klimek [1]. In a similar direction, Choi and Kim [5, 6, 7] considered the same problem for scalar-valued 2-homogeneous polynomials on the real spaces ℓ_1^2, ℓ_2^2 and ℓ_∞^2 whereas Grecu [14] treated the case of scalar-valued 2-homogeneous polynomials on the real spaces ℓ_p^2 with $1 < p < \infty$. See also [12, 13, 14, 15, 16, 17] for related questions concerning real or complex homogeneous polynomials of degree 2 or 3. Trigonometric trinomials (real or complex) have also been studied by Aron and Klimek [1], Neuwirth [31] and Révész [32].

The geometry of finite dimensional spaces of polynomials on non-symmetric convex bodies has also been studied. Recall that a convex body is a closed bounded convex set with nonempty interior and therefore, a convex body in a finite dimensional space is a compact convex set with nonempty interior. For instance, in [29] it can be found a full description of the extreme points of a space of 2-homogeneous polynomials defined on the simplex (the triangle of vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$). Using this characterization the authors obtained sharp Bernstein and Markov type inequalities and the polarization constants for such polynomials. As a continuation of this work, in this paper we address the same questions for another space of polynomials on a non-symmetric convex body, namely the space of 2-homogeneous polynomials defined on the square of vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Further studies in the geometry of the space of polynomials of degree at most 2 on the simplex are due to [22].

In the following, $\square$ will denote the square of vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Notice that $\square$ has central symmetry but it is not balanced. Hence the study of polynomials on $\square$ cannot be included in the category of polynomials on a Banach space with the usual sup norm over the unit ball.

If P is a 2-homogeneous polynomial on $\mathbb{R}^2$, we define its norm by

$$\|P\|_\square := \sup\{|P(\mathbf{x})| : \mathbf{x} \in \square\}.$$

Similarly, if L is a bilinear form on $\mathbb{R}^2$, we also define

$$\|L\|_\square := \sup\{|L(\mathbf{x}, \mathbf{y})| : \mathbf{x}, \mathbf{y} \in \square\}.$$

The spaces of 2-homogeneous polynomials, bilinear forms and symmetric bilinear forms on $\mathbb{R}^2$ endowed with the above norms will be represented, respectively, by $\mathcal{P}(^2\square)$, $\mathcal{L}(^2\square)$ and $\mathcal{L}^s(^2\square)$.

The mapping that assigns to every $(a,b,c) \in \mathbb{R}^3$ the polynomial $P(x,y) = ax^2 + bxy + cy^2$ allows us to identify $\mathcal{P}(^2\square)$ with $(\mathbb{R}^3, \|\cdot\|_\square)$, where $\|(a,b,c)\|_\square = \|P\|_\square$. This latter representation together with the usual polynomial representation of $\mathcal{P}(^2\square)$ will be interchanged throughout the paper.

According to a well-known fact, a convex function (like a polynomial norm, for instance) defined on a convex body (like the unit ball of a finite dimensional polynomial space) attains its maximum at one extreme point of the convex body. From now on we will refer to this method as the *Krein-Milman Approach*. In this paper we apply this method in order to obtain a number of polynomial inequalities involving polynomial norms.

Bernstein and Markov type inequalities for polynomials on arbitrary normed spaces are considered in many papers, see, e.g., [18, 26, 33, 34] and the references therein. These results, however, make essential use of the fact that the unit ball of a normed space is a *balanced* convex body, which is not the case for $\square$. Bernstein and Markov type inequalities for polynomials on a non necessarily balanced convex body were studied first by Wilhelmsen in 1974 [36]. Many other publications on this topic have appeared ever since (see, e.g., [2, 20, 21, 29, 30]).

In Section 3 we use the Krein-Milman approach to give sharp estimates on the Euclidean length of the gradient of a polynomial in $\mathcal{P}(^2\square)$. To the author's knowledge, the best known general estimate on the length of the gradient of a multivariate polynomials on a convex body can be found in [20, Theorem 3]. From that result we deduce that

$$\sup_{(x,y)\in\square} \|\nabla P(x,y)\|_2 \leq 12 \cdot \|P\|_\square,$$

for all $P \in \mathcal{P}(^2\square)$. However we can prove the optimal inequality

$$\sup_{(x,y)\in\square} \|\nabla P(x,y)\|_2 \leq \sqrt{13} \cdot \|P\|_\square,$$

for all $P \in \mathcal{P}(^2\square)$ together with an improvement of the previous estimate in the interior points of $\square$.

Using the Krein-Milman Approach, sharp Bernstein and Markov type inequalities for trinomials on the real line have been obtained in [27]. Other sharp Bernstein and Markov type inequalities for trinomials can be seen in [24, 25]. Actually the Krein-Milman approach is a classical method for many purposes, used in its origin by Konheim and Rivlin [19], and much earlier, by Voronovskaya [35].

In Section 4 we will see how a sharp Markov type inequality for $\mathcal{P}(^2\square)$ provides the polarization constant for polynomials in $\mathcal{P}(^2\square)$. In order to discuss polarization constants, let us introduce some general definitions and basic results. If E is a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with closed unit ball B and $n \in \mathbb{N}$, then we will denote by $\mathcal{P}(^nE)$ and $\mathcal{L}^s(^nE)$ the space of all scalar-valued continuous

n-homogeneous polynomials on E and the space of all continuous symmetric n-linear forms on E^n endowed respectively with the norms given by

$$\begin{aligned}\|P\|_{\mathsf{B}} &= \sup\{|P(x)| : \ x \in \mathsf{B}\},\\ \|L\|_{\mathsf{B}} &= \sup\{|L(x_1, \ldots, x_n)| : \ x_1, \ldots, x_n \in \mathsf{B}\}.\end{aligned}$$

For simplicity, $\|P\|$ and $\|L\|$ are used sometimes instead of $\|P\|_{\mathsf{B}}$ and $\|L\|_{\mathsf{B}}$ respectively.

A well-known algebraic result states that for every $P \in \mathcal{P}(^nE)$ there exists a unique $L \in \mathcal{L}^s(^nE)$ such that $P(x) = L(x, \overset{n}{\ldots}, x)$, for every $x \in E$. In this case it is convenient to write $P = \widehat{L}$. Actually, the mapping $\mathcal{L}^s(^nE) \ni L \mapsto \widehat{L} \in \mathcal{P}(^nE)$ is an isomorphism satisfying

$$\|\widehat{L}\| \leq \|L\| \leq \frac{n^n}{n!}\|\widehat{L}\|, \tag{1}$$

for every $L \in \mathcal{L}^s(^nE)$. The first inequality in (1) is trivial whereas the second one was proved by Martin in his Ph.D. dissertation [23]. It can be shown that the constant $\frac{n^n}{n!}$ cannot be improved in general since equality is achieved for ℓ_1^n and the polynomial defined by $\Phi(x_1, \ldots, x_n) = x_1 \cdots x_n$ for every $(x_1, \ldots, x_n) \in \mathbb{K}^n$. However, for a specific space E the constant $\frac{n^n}{n!}$ in (1) can be improved. The best constant $\mathbb{K}(n; E)$ in the inequality

$$\|L\| \leq M\|\widehat{L}\|,$$

for every $L \in \mathcal{L}^s(^nE)$ is called the nth polarization constant of E.

Interestingly, it was proved in [29] that (1) does not hold for polynomials on a non-symmetric convex body, in particular for 2-homogeneous polynomials on the simplex (the triangle of vertices $(0,0)$, $(0,1)$ and $(1,0)$). In Section 4 we give a version of (1) for polynomials on $\square$, showing that

$$\|L\|_{\square} \leq \frac{3}{2}\|\widehat{L}\|_{\square},$$

for every $L \in \mathcal{L}^s(^2\square)$ and that $\frac{3}{2}$ is optimal. Observe that the polarization constant of ℓ_∞^2 is 2.

A third issue addressed in this paper has to do with unconditional constants in polynomial spaces. Let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_k \in \mathbb{N} \cup \{0\}$, $1 \leq k \leq m$. If $P(\mathbf{x}) = \sum_{|\alpha| \leq n} a_\alpha \mathbf{x}^\alpha$ is a polynomial of degree n on $\mathbb{R}^m$, we define its modulus $|P|$ by $|P|(\mathbf{x}) = \sum_{|\alpha| \leq n} |a_\alpha| \mathbf{x}^\alpha$. If $\mathsf{B} \subset \mathbb{R}^m$ is a convex body, we let $\mathcal{P}_n(\mathsf{B})$ and $\mathcal{P}(^n\mathsf{B})$ represent, respectively, the space of polynomials of degree at most n and the space of n-homogeneous polynomials on $\mathbb{R}^m$ endowed with the norm

$$\|P\|_{\mathsf{B}} = \sup\{|P(x)| : \ x \in \mathsf{B}\}.$$

Similarly, $\mathcal{L}_n(\mathsf{B})$ and $\mathcal{L}^s(^n\mathsf{B})$ represent, respectively, the space of n-linear forms and the space of symmetric n-linear forms on $\mathbb{R}^m$ endowed with the norm

$$\|L\|_{\mathsf{B}} = \sup\{|L(x_1, \dots, x_n)| : \ x_1, \dots, x_n \in \mathsf{B}\}.$$

Now let $\mathcal{B}_n = \{\mathbf{x}^\alpha : |\alpha| \leq n\}$ be the canonical basis of $\mathcal{P}_n(\mathsf{B})$ or $\mathcal{P}(^n\mathsf{B})$ and $\mathcal{S} \subset \mathcal{B}_n$. Then it is easy to see that the unconditional constant of $\mathcal{S}$ coincides with the best possible constant $C_{\mathsf{B},\mathcal{S}}$ in the inequality

$$|||P|||_{\mathsf{B}} \leq C_{\mathsf{B},\mathcal{S}}\|P\|_{\mathsf{B}}, \tag{2}$$

for every P in the space generated by $\mathcal{S}$. This type of inequalities have been studied since a long time ago. Of special importance are the results obtained by Bohr in 1914 for infinite complex power series [4] since they are the starting point of a prolific research line that continues to be productive nowadays (see for instance [3, 8, 9, 10, 11]). It is interesting to note that the relationship between unconditional constants in polynomial spaces and inequalities of the type (2) was already noticed in [9].

From now on, $\mathsf{S}_\square$ and $\mathsf{B}_\square$ will denote, respectively, the unit sphere and unit ball of the space $\mathcal{P}(^2\square)$. If C is a convex body, $\mathrm{ext}(C)$ will denote the set of extreme points of C. Also, π_{ac} will denote the linear projection given by $\pi_{ac}(a, b, c) = (a, c)$, for every $(a, b, c) \in \mathbb{R}^3$.

The plots appearing in this paper were rendered using *Metapost*, *Maple* and *Matlab*.

2. The geometry of $\mathcal{P}(^2\square)$

First of all we obtain a formula for $\|\cdot\|_\square$:

Theorem 2.1. *If $P(x, y) = ax^2 + bxy + cy^2$, then*

$$\|P\|_\square = \begin{cases} \max\left\{|a|, |c|, |a+b+c|, \frac{b^2-4ac}{4|c|}\right\} & \text{if } b^2 - 4ac > 0,\ c \neq 0 \text{ and } -\frac{b}{2c} \in (0,1). \\ \max\left\{|a|, |c|, |a+b+c|, \frac{b^2-4ac}{4|a|}\right\} & \text{if } b^2 - 4ac > 0,\ a \neq 0 \text{ and } -\frac{b}{2a} \in (0,1). \\ \max\{|a|, |c|, |a+b+c|\} & \text{otherwise.} \end{cases}$$

Proof. Notice that if $(x_0, y_0) \in \overset{\circ}{\square}$ (the interior of $\square$) then there exists a $\lambda > 1$ such that $\lambda(x_0, y_0) \in L = \{(x, 1) : x \in [0, 1]\} \cup \{(1, y) : y \in [0, 1]\}$. Then $|P(x_0, y_0)| < \lambda^2|P(x_0, y_0)| = |P(\lambda x_0, \lambda y_0)|$, which shows that the norm of P is attained on the set L. Hence, $\|P\|_\square = \max\{\|ax^2 + bx + c\|_\square, \|a + by + cy^2\|_\square\}$. Considering the

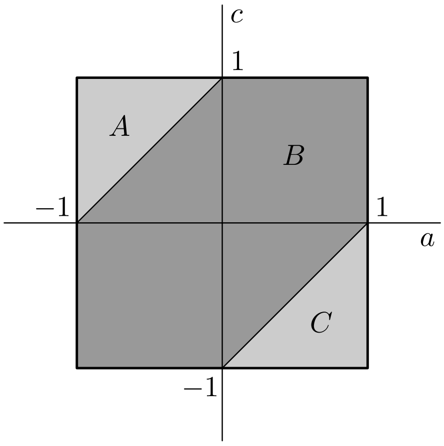

Figure 2.1: $\pi_{ac}(\mathsf{B}_\square)$.

zeros of the derivatives of these polynomials ($-\frac{b}{2a}$ and $-\frac{b}{2c}$ respectively), we obtain

$$\|P\|_\square = \begin{cases} \max\left\{|a|,|c|,|a+b+c|,\left|\frac{4ac-b^2}{4a}\right|,\left|\frac{4ac-b^2}{4c}\right|\right\} & \text{if } -\frac{b}{2a},-\frac{b}{2c}\in(0,1), \\ \max\left\{|a|,|c|,|a+b+c|,\left|\frac{4ac-b^2}{4a}\right|\right\} & \text{if } -\frac{b}{2a}\in(0,1) \text{ and } -\frac{b}{2c}\notin(0,1), \\ \max\left\{|a|,|c|,|a+b+c|,\left|\frac{4ac-b^2}{4c}\right|\right\} & \text{if } -\frac{b}{2c}\in(0,1) \text{ and } -\frac{b}{2a}\notin(0,1), \\ \max\{|a|,|c|,|a+b+c|\} & \text{otherwise.} \end{cases}$$

If $b^2-4ac>0$ and $-\frac{b}{2a}\in(0,1)$, then $-\frac{b}{2c}\notin(0,1)$. If $b^2-4ac<0$, then

$$\frac{4ac-b^2}{4|a|}\le|a+b+c| \quad\text{and}\quad \frac{4ac-b^2}{4|c|}\le|a+b+c|.$$

Therefore, we obtain the desired formula. □

In order to sketch $\mathsf{S}_\square$ and obtain the extreme points of $\mathsf{B}_\square$, it is important to have a parametrization of $\mathsf{S}_\square$. This parametrization can be constructed by projecting $\mathsf{B}_\square$ onto the ac-plane. For this matter we will use the subsets A, B and C of $[-1,1]^2$ (see Figure 2.1):

$$\begin{aligned} A &:= \{(a,c)\in[-1,1]^2:\ -1\le a\le 0 \text{ and } a+1\le c\le 1\}, \\ B &:= \left\{(a,c)\in[-1,1]^2:\ -1\le a\le 1 \text{ and } \max\{-1,a-1\}\le c\le\min\{1,a+1\}\right\}, \\ C &:= \{(a,c)\in[-1,1]^2:\ 0\le a\le 1 \text{ and } -1\le c\le a-1\}. \end{aligned}$$

Theorem 2.2. *The projection of* $\mathsf{S}_\square$ *onto the ac-plane is* $[-1,1]^2$.

Proof. If $(a,b,c)\in\mathsf{S}_\square$, then it is straightforward from Theorem 2.1 that $\max\{|a|,|c|\}\le 1$, or in other words $\pi_{ac}(a,b,c)=(a,c)\in[-1,1]^2$. This shows that $\pi_{ac}(\mathsf{S}_\square)\subset[-1,1]^2$. Now let $(a,c)\in[-1,1]^2$ then one of the following statements holds:

1. $(a,c) \in A$. In this case using Theorem 2.1 we have $(a,b,c) \in \mathsf{B}_\square$ for all b such that
$$-2\sqrt{ac+|c|} \le b \le 2\sqrt{ac+|a|}.$$
If $a=-1$ or $c=1$, then all those points are in $\mathsf{S}_\square$. However, if $a>-1$ and $c<1$ only $(a,-2\sqrt{ac+|c|},c)$ and $(a,2\sqrt{ac+|a|},c)$ lay in $\mathsf{S}_\square$. All this can be easily checked by applying Theorem 2.1. In any case we deduce that $(a,c) \in \pi_{ac}(\mathsf{S}_\square)$.
2. $(a,c) \in B$. In this case using Theorem 2.1 we see that $(a,b,c) \in \mathsf{B}_\square$ for all b such that
$$-1-a-c \le b \le 1-a-c.$$
In fact, if $|a|=1$ or $|c|=1$ then all those points are in $\mathsf{S}_\square$. However, if $|a|<1$ and $|c|<1$ then only $(a,-1-a-b,c)$ and $(a,1-a-b,c)$ lay in $\mathsf{S}_\square$. Again, all this can be easily proved using Theorem 2.1 once more. Therefore we deduce that $(a,c) \in \pi_{ac}(\mathsf{S}_\square)$.
3. $(a,c) \in C$. Taking into consideration that $(-a,-c) \in A$, it follows from case 1 that $(a,b,c) \in \mathsf{B}_\square$ for all b such that
$$-2\sqrt{ac+|a|} \le b \le 2\sqrt{ac+|c|}.$$
If $a=1$ or $c=-1$, we have in fact that those points are in $\mathsf{S}_\square$. Otherwise, only the points $(a,2\sqrt{ac+|c|},c)$ and $(a,-2\sqrt{ac+|a|},c)$ lay in $\mathsf{S}_\square$. In any case we have proved that $(a,c) \in \pi_{ac}(\mathsf{S}_\square)$ in this case too. $\square$

Theorem 2.3. *If for every $(a,c) \in [-1,1]^2$ we define the mappings*
$$F(a,c) = \begin{cases} 2\sqrt{ac+|a|} & \text{if } (a,c) \in A, \\ 2\sqrt{ac+|c|} & \text{if } (a,c) \in C, \\ 1-a-c & \text{if } (a,c) \in B, \end{cases}$$
$$G(a,c) = -F(-a,-c),$$
where A, B and C are as in Figure 2.1 and the set
$$H = \{(a,b,c) \in \mathbb{R}^3 : (a,c) \in \partial[-1,1]^2 \text{ and } G(a,c) \le b \le F(a,c)\},$$
then

(a) $\mathsf{S}_\square = \operatorname{graph}(F) \cup \operatorname{graph}(G) \cup H$.
(b) *The extreme points of $\mathsf{B}_\square$ have the form*
$$\pm(t,2\sqrt{1-t},-1) \quad \text{and} \quad \pm(-1,2\sqrt{1-t},t) \quad \text{with } t \in [0,1]$$
or
$$\pm(1,-1,1),\ \pm(1-3,1),\ \pm(1,0,0),\ \pm(0,0,1).$$

Proof. The proof of (a) is outlined in the proof of Theorem 2.2. As for the proof of part (b), notice that $\operatorname{graph}(F|_A)$, $\operatorname{graph}(F|_B)$, $\operatorname{graph}(F|_C)$, $\operatorname{graph}(G|_A)$, graph($G|_B$), graph($G|_C$) and H are all ruled surfaces, therefore extreme point can only happen at their intersections. By symmetry we just need to focus on the intersections of the surfaces graph($F|_A$), graph($F|_B$), graph($F|_C$) and H.

First we investigate the intersections with H. The surfaces graph($F|_A$) and H meet along the curve $\gamma_1 = \{(-1, 2\sqrt{1-c}, c) : 0 \le c \le 1\}$ and the segment joining the points $(-1,0,1)$ and $(0,0,1)$. Notice that $(-1,0,1) \in \gamma_1$, so the only candidates to be extreme points in this intersection are the points in γ_1 and $(0,0,1)$. Similarly, the surfaces graph($F|_C$) and H meet along the curve $\gamma_2 = \{(a, 2\sqrt{1-a}, -1) : 0 \le a \le 1\}$ and the segment joining the points $(1,0,-1)$ and $(1,0,0)$. Notice that $(1,0,-1) \in \gamma_2$, so the only candidates to be extreme points in this intersection are the points in γ_2 and $(1,0,0)$. Now the intersection of graph($F|_B$) with H are the segments joining the point $(1,-1,1)$ with the points $(0,0,1)$ and $(1,0,0)$ on the one hand, and the point $(-1,3,-1)$ with the points $(-1,2,0)$ and $(0,2,-1)$. Of course, only the end points of these segments can be extreme.

As for the other intersections, the surfaces graph($F|_A$) and graph($F|_B$) meet along the segment joining the points $(-1,2,0)$ and $(0,0,1)$ whereas the surfaces graph($F|_C$) and graph($F|_B$) meet along the segment joining the points $(0,2,-1)$ and $(1,0,0)$. Notice that the end points of these segments have already appeared above.

Therefore, by symmetry, the only candidates to be extreme points are the points in the curves $\pm\gamma_k$ with $k = 1, 2$, together with the points $\pm(1,-1,1)$, $\pm(1,-3,1)$, $\pm(1,0,0)$ and $\pm(0,0,1)$. Since the curves $\pm\gamma_k$ with $k = 1, 2$ are nowhere affine, it follows that all their points are extreme in $\mathsf{B}_\square$. Also, it is easy to construct support hyperplanes to $\mathsf{B}_\square$ at each of the points $\pm(1,-1,1)$, $\pm(1,-3,1)$, $\pm(1,0,0)$ and $\pm(0,0,1)$ that contain no other points of $\mathsf{B}_\square$. This concludes the proof. $\square$

From now on we will use the following notations for the extreme points of $\mathsf{B}_\square$:

$$\begin{aligned}
P_t(x,y) &= \pm(tx^2 + 2\sqrt{1-t}xy - y^2),\\
Q_t(x,y) &= \pm(-x^2 + 2\sqrt{1-t}xy + ty^2),\\
R_1(x,y) &= \pm(x^2 - 3xy + y^2),\\
R_2(x,y) &= \pm(x^2 - xy + y^2),\\
R_3(x,y) &= \pm x^2,\\
R_4(x,y) &= \pm y^2,
\end{aligned}$$

for $t \in [0,1]$.

3. Markov and Bernstein inequalities in $\mathcal{P}(^2\square)$

Bernstein and Markov type inequalities in $\mathcal{P}(^2\square)$ are estimates on the length of $\nabla P(x,y)$ for $P \in \mathcal{P}(^2\square)$ and $(x,y) \in \square$. If the estimate is uniform for all $(x,y) \in \square$, then we talk about a Markov type estimate. On the contrary, if $(x,y) \in \square$ is fixed, then what we obtain is a Bernstein type estimate. Notice that the derivative of P at (x,y) is defined as the linear form $DP(x,y)$ given by

$$DP(x,y)(h_1,h_2) = \langle \nabla P(x,y), (h_1,h_2) \rangle,$$

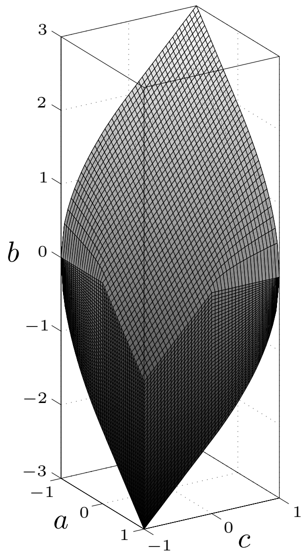

Figure 2.2: The unit sphere $\mathsf{B}_\square$.

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^2$. Therefore the norm of $DP(x,y)$ as a linear form on ℓ_2^2, denoted by $\|DP(x,y)\|_2$, coincides with the length of the gradient $\nabla P(x,y)$, denoted by $\|\nabla P(x,y)\|_2$. This justifies the simultaneous use of the notations $\|DP(x,y)\|_2$ and $\|\nabla P(x,y)\|_2$.

Theorem 3.1. *If $P \in \mathcal{P}(^2\square)$ then we have*

$$\|DP(x,y)\|_2 \le \mathcal{M}(x,y) \cdot \|P\|_\square, \tag{3}$$

for every $(x,y) \in \square$, where

$$\mathcal{M}(x,y) = \begin{cases} \sqrt{\frac{24y^4+12x^2y^2+x^4+x(8y^2+x^2)^{\frac{3}{2}}}{8y^2}} & \text{if } 0 < \alpha_0 x \le y \le x, \\ \sqrt{\frac{24x^4+12x^2y^2+y^4+y(8x^2+y^2)^{\frac{3}{2}}}{8x^2}} & \text{if } 0 < x \le y \le \frac{x}{\alpha_0}, \\ \sqrt{13x^2-24xy+13y^2} & \text{otherwise,} \end{cases}$$

and $\alpha_0 \approx 0.4029036618$ is the unique root of the equation

$$80\alpha^4 - 192\alpha^3 + 92\alpha^2 - 1 = (8\alpha^2+1)^{\frac{3}{2}}$$

in the interval $[\frac{3-\sqrt{5}}{2}, \frac{12-3\sqrt{3}}{13}]$. Moreover, the inequality is sharp.

Proof. The Krein-Milman approach yields, for $(x,y) \in \square$,

$$\begin{aligned} &\mathcal{M}(x,y) \\ &= \max\{\|\nabla P_t(x,y)\|_2, \|\nabla Q_t(x,y)\|_2, \|\nabla R_k(x,y)\|_2 : \ t \in [0,1], \ k = 1,2,3,4\}. \end{aligned}$$

The result is easy to prove if $x = 0$ or $y = 0$, hence we can assume throughout the proof that $xy \neq 0$.

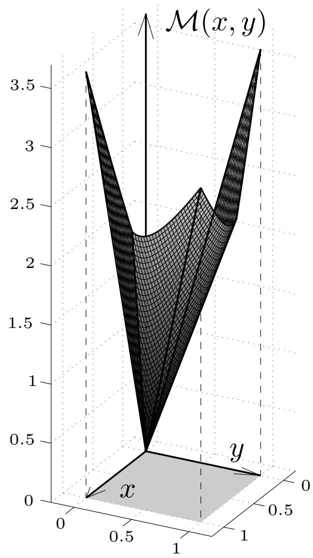

Figure 3.1: The Bernstein mapping $\mathcal{M}(x,y)$ for $\square$.

Let us investigate first the polynomials P_t and Q_t at $(x,y)\in\square$. Using the change of variable $u=\sqrt{1-t}$ with $t\in[0,1]$ (and therefore $u\in[0,1]$ too), then it is easy to check that

$$\begin{aligned}&\max\{\|\nabla P_t(x,y)\|_2, \|\nabla Q_t(x,y)\|_2 :\ t\in[0,1]\}\\ &=\begin{cases}\max\{2\sqrt{f_1(u)} : u\in[0,1]\} & \text{if } y\le x,\\ \max\{2\sqrt{f_2(u)} : u\in[0,1]\} & \text{if } y\ge x,\end{cases}\end{aligned}$$

where $f_1(u)=y^2u^4-2xyu^3-(y^2-x^2)u^2+x^2+y^2$ and $f_2(u)=x^2u^4-2xyu^3+(y^2-x^2)u^2+x^2+y^2$ for $u\in[0,1]$. Actually, using the obvious symmetry of the problem with respect to the line $y=x$, we can restrict the calculations to points $(x,y)\in\square$ with $0\le\alpha:=y/x\le 1$. Hence, we just need to maximize $f_1(u)=x^2[\alpha^2u^4-2\alpha u^3+(1-\alpha^2)u^2+1+\alpha^2]$, where $u\in[0,1]$. Notice that f_1 is nonnegative in $[0,1]$. Also, the polynomial $f_1(u)$ has one critical point at 0 and at most one more critical point in $[0,1]$ at $u_0=\frac{3-\sqrt{8\alpha^2+1}}{4\alpha}$. Actually $u_0\in[0,1]$ if and only if $\frac{3-\sqrt{5}}{2}\le\alpha\le 1$. A simple reasoning lets us deduce that f_1 attains its maximum at u_0 whenever $u_0\in[0,1]$. Otherwise f_1 is increasing. The reader can easily check that

$$\max\{2\sqrt{f_1(u)} : u\in[0,1]\}=\begin{cases}x\sqrt{4\alpha^2-8\alpha+8} & \text{if } 0\le\alpha\le\frac{3-\sqrt{5}}{2},\\ x\sqrt{\frac{24\alpha^4+12\alpha^2+1+(8\alpha^2+1)^{\frac{3}{2}}}{8\alpha^2}} & \text{if } \frac{3-\sqrt{5}}{2}\le\alpha\le 1.\end{cases}$$

As for the other extreme points in $\mathsf{B}_\square$, if $(x,y)\in\square$ with $x\neq 0$, then

$$\begin{aligned}&\max\{\|\nabla R_k(x,y)\|_2 : k=1,2,3,4\}\\ &=x\max\{\sqrt{13\alpha^2-24\alpha+13},\sqrt{5\alpha^2-8\alpha+5},2,2\alpha\},\end{aligned}$$

where α stands for y/x as before. Now, restricting ourselves again to the points $(x,y) \in \square$ with $0 \le \alpha = y/x \le 1$, it is easy to check that $\sqrt{13\alpha^2 - 24\alpha + 13} \ge \sqrt{5\alpha^2 - 8\alpha + 5}$ for $0 \le \alpha \le 1$. Also, one can easily prove that $\sqrt{13\alpha^2 - 24\alpha + 13} \le 2$ whenever $\frac{12-3\sqrt{3}}{13} \le \alpha \le 1$. Hence

$$\max\{\|\nabla R_k(x,y)\|_2 : k = 1,2,3,4\} = \begin{cases} x\sqrt{13\alpha^2 - 24\alpha + 13} & \text{if } 0 \le \alpha \le \frac{12-3\sqrt{3}}{13}, \\ 2x & \text{if } \frac{12-3\sqrt{3}}{13} \le \alpha \le 1. \end{cases}$$

Notice that $\frac{3-\sqrt{5}}{2} \approx 0.3820 < 0.5234 \approx \frac{12-3\sqrt{3}}{13}$.

Finally, the reader can easily verify that for $0 \le \alpha \le \frac{3-\sqrt{5}}{2}$ we have

$$\sqrt{4\alpha^2 - 8\alpha + 8} \le \sqrt{13\alpha^2 - 24\alpha + 13}$$

and for $\frac{12-3\sqrt{3}}{13} \le \alpha \le 1$

$$2 \le \sqrt{\frac{24\alpha^4 + 12\alpha^2 + 1 + (8\alpha^2+1)^{\frac{3}{2}}}{8\alpha^2}}.$$

Since the mappings $\sqrt{13\alpha^2 - 24\alpha + 13}$ and $\sqrt{\frac{24\alpha^4+12\alpha^2+1+(8\alpha^2+1)^{\frac{3}{2}}}{8\alpha^2}}$ meet at just one point α_0 in the interval $[\frac{3-\sqrt{5}}{2}, \frac{12-3\sqrt{3}}{13}]$, we have that for $(x,y) \in \square$ with $x \ne 0$ and $0 \le \alpha \le 1$

$$\mathcal{M}(x,y) = \begin{cases} x\sqrt{13\alpha^2 - 24\alpha + 13} & \text{if } 0 \le \alpha \le \alpha_0, \\ x\sqrt{\frac{24\alpha^4+12\alpha^2+1+(8\alpha^2+1)^{\frac{3}{2}}}{8\alpha^2}} & \text{if } \alpha_0 \le \alpha \le 1. \end{cases}$$

In order to find $\mathcal{M}(x,y)$ for $(x,y) \in \square$ with $y \ge x$, we just need to swap x and y in the previous formula thanks to the symmetry of the problem. This concludes the proof. $\square$

Corollary 3.2. *If $P \in \mathcal{P}(^2\square)$ then for any $(x,y) \in \square$ we have*

$$\|DP(x,y)\|_2 \le \sqrt{13} \cdot \|P\|_\square.$$

Equality occurs for the polynomials $\pm(x^2 - 3xy + y^2)$ at the points $(0,1)$ and $(1,0)$.

Proof. A sketch of the graph of $\mathcal{M}$ can be of help for this proof (see in Figure 3.1). The result follows straightforwardly from the fact that $\mathcal{M}(x,y)$ is convex in $\square$. Thus

$$\begin{aligned} \max\{\mathcal{M}(x,y) : (x,y) \in \square\} &= \max\{\mathcal{M}(0,0), \mathcal{M}(1,0), \mathcal{M}(0,1), \mathcal{M}(1,1)\} \\ &= \max\{0, \sqrt{13}, 2\sqrt{2}\} = \sqrt{13}. \end{aligned}$$ $\square$

4. Polarization constant of $\mathcal{P}(^2\square)$

First we obtain the following Bernstein type inequality where only the norm $\|\cdot\|_\square$ is considered.

Theorem 4.1. *If for each $(x,y)\in\square$, $\Psi(x,y)$ represents the best constant in*

$$\|DP(x,y)\|_\square \le \Psi(x,y)\|P\|_\square, \quad \textit{for every } P\in\mathcal{P}(^2\square), \tag{4}$$

then

$$\Psi(x,y)=\begin{cases} 3x-2y & \textit{if } y\le(\sqrt{2}-1)x,\\ \frac{5}{2}x-y+\frac{y^2}{2x} & \textit{if } x\ne 0 \textit{ and } (\sqrt{2}-1)x\le y\le\frac{1}{2}x,\\ 2x+\frac{y^2}{2x} & \textit{if } x\ne 0 \textit{ and } \frac{1}{2}x\le y\le x,\\ 2y+\frac{x^2}{2y} & \textit{if } y\ne 0 \textit{ and } x\le y\le 2x,\\ \frac{5}{2}y-x+\frac{x^2}{2y} & \textit{if } y\ne 0 \textit{ and } 2x\le y\le(\sqrt{2}+1)x,\\ 3y-2x & \textit{if } (\sqrt{2}+1)x\le y. \end{cases}$$

Proof. If $(x,y)\in\square$, the Krein-Milman approach shows that

$$\begin{aligned}&\Psi(x,y)\\ &=\max\{\|DP_t(x,y)\|_\square,\|DQ_t(x,y)\|_\square,\|DR_k(x,y)\|_\square:\ t\in[0,1],\ k=1,2,3,4\}.\end{aligned}$$

The result is easy to check if $x=0$ or $y=0$. Hence we assume throughout the proof that $xy\ne 0$. Let us start by considering the polynomials P_t with $t\in[0,1]$. Since

$$\nabla P_t(x,y)=(2tx+2\sqrt{1-t}y,2\sqrt{1-t}x-2y),$$

and the maximum of $|DP_t(x,y)h|$ for each $h=(h_1,h_2)\in\square$ is attained at the extreme points of $\square$, we have, for $t\in[0,1]$,

$$\begin{aligned}&\|DP_t(x,y)\|_\square\\ &=\max\{2tx+2\sqrt{1-t}y,|2\sqrt{1-t}x-2y|,|2tx+2\sqrt{1-t}(x+y)-2y|\}.\end{aligned}$$

Let us define

$$\begin{aligned}g_1(t)&=2tx+2\sqrt{1-t}y,\\ g_2(t)&=2\sqrt{1-t}x-2y,\\ g_3(t)&=2tx+2\sqrt{1-t}(x+y)-2y,\end{aligned}$$

for $t\in[0,1]$. The mapping g_1 has at most one critical point in $[0,1]$, which occurs at $t_0=1-\left(\frac{y}{2x}\right)^2$ if and only if $y\le 2x$. Moreover, t_0 is a local maximum. We deduce that

$$\max\{g_1(t):t\in[0,1]\}=\begin{cases} g_1(t_0) & \text{if } y\le 2x,\\ g_1(1) & \text{if } y\ge 2x,\end{cases}=\begin{cases} 2x+\frac{y^2}{2x} & \text{if } y\le 2x,\\ 2y & \text{if } y\ge 2x.\end{cases}$$

The mapping g_2 is monotone. Thus

$$\begin{aligned}\max\{g_2(t) : t \in [0,1]\} &= \max\{|g_2(0)|, |g_2(1)|\}\\ &= \max\{2y, 2|x-y|\} = \begin{cases} 2(x-y) & \text{if } y \le \frac{x}{2},\\ 2y & \text{if } y \ge \frac{x}{2}.\end{cases}\end{aligned}$$

As for g_3, it can be easily verified that g_3 has at most one critical point in $[0,1]$, which occurs at $t_1 = 1 - \frac{1}{4}\left(1 + \frac{y}{x}\right)^2$ if and only if $y \le x$. Actually t_1 is a local maximum. Since g_3 is non negative on $[0,1]$ when $y \le x$ we deduce

$$\begin{aligned}\max\{|g_3(t)| : t \in [0,1]\} &= \begin{cases} |g_3(t_1)| & \text{if } y \le x,\\ \max\{|g_3(0)|, |g_3(1)|\} & \text{if } y \ge x,\end{cases}\\ &= \begin{cases} \frac{5}{2}x - y + \frac{y^2}{2x} & \text{if } y \le x,\\ \max\{2x, 2(y-x)\} & \text{if } y \ge x,\end{cases}\\ &= \begin{cases} \frac{5}{2}x - y + \frac{y^2}{2x} & \text{if } y \le x,\\ 2x & \text{if } x \le y \le 2x,\\ 2(y-x) & \text{if } y \ge 2x.\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}&\max\{\|DP_t(x,y)\|_\Box : t \in [0,1]\}\\ &= \begin{cases} \max\left\{2x + \frac{y^2}{2x}, 2(x-y), \frac{5}{2}x - y + \frac{y^2}{2x}\right\} & \text{if } y \le \frac{x}{2},\\ \max\left\{2x + \frac{y^2}{2x}, 2y, \frac{5}{2}x - y + \frac{y^2}{2x}\right\} & \text{if } \frac{x}{2} \le y \le x,\\ \max\left\{2x + \frac{y^2}{2x}, 2y, 2x\right\} & \text{if } x \le y \le 2x,\\ \max\left\{2y, 2(y-x)\right\} & \text{if } y \ge 2x,\end{cases}\\ &= x\begin{cases} \max\left\{2 + \frac{\alpha^2}{2}, 2(1-\alpha), \frac{5}{2} - \alpha + \frac{\alpha^2}{2}\right\} & \text{if } 0 \le \alpha \le \frac{1}{2},\\ \max\left\{2 + \frac{\alpha^2}{2}, 2\alpha, \frac{5}{2} - \alpha + \frac{\alpha^2}{2}\right\} & \text{if } \frac{1}{2} \le \alpha \le 1,\\ \max\left\{2 + \frac{\alpha^2}{2}, 2\alpha, 2\right\} & \text{if } 1 \le \alpha \le 2,\\ \max\left\{2\alpha, 2(\alpha-1)\right\} & \text{if } \alpha \ge 2,\end{cases}\end{aligned}$$

where $\alpha = y/x$. Now, a simple exercise shows that

$$\begin{aligned}\frac{5}{2} - \alpha + \frac{\alpha^2}{2} &\ge \max\left\{2 + \frac{\alpha^2}{2}, 2(1-\alpha)\right\} && \text{if } 0 \le \alpha \le \frac{1}{2},\\ 2 + \frac{\alpha^2}{2} &\ge \max\left\{2\alpha, \frac{5}{2} - \alpha + \frac{\alpha^2}{2}\right\} && \text{if } \frac{1}{2} \le \alpha \le 1,\\ 2\alpha &\ge \max\left\{2 + \frac{\alpha^2}{2}, 2\right\} && \text{if } 1 \le \alpha \le 2,\\ 2\alpha &\ge 2(\alpha - 1) && \text{if } \alpha \ge 2.\end{aligned}$$

Hence

$$\max\{\|DP_t(x,y)\|_\square : t\in[0,1]\} = x\begin{cases}\frac{5}{2}-\alpha+\frac{\alpha^2}{2} & \text{if } 0\le\alpha\le\frac{1}{2},\\ 2+\frac{\alpha^2}{2} & \text{if } \frac{1}{2}\le\alpha\le 2,\\ 2\alpha & \text{if } \alpha\ge 2,\end{cases}$$

$$= \begin{cases}\frac{5}{2}x-y+\frac{y^2}{2x} & \text{if } y\le\frac{x}{2},\\ 2x+\frac{y^2}{2x} & \text{if } \frac{x}{2}\le y\le 2x,\\ 2y & \text{if } y\ge 2x.\end{cases}$$

By symmetry

$$\max\{\|DQ_t(x,y)\|_\square : t\in[0,1]\} = \begin{cases}2x & \text{if } y\le\frac{x}{2},\\ 2y+\frac{x^2}{2y} & \text{if } \frac{x}{2}\le y\le 2x,\\ \frac{5}{2}y-x+\frac{x^2}{2y} & \text{if } y\ge 2x,\end{cases}$$

from which it follows easily that

$$\begin{aligned}&\max\{\|DP_t(x,y)\|_\square, \|DQ_t(x,y)\|_\square : t\in[0,1]\}\\ &= \begin{cases}\frac{5}{2}x-y+\frac{y^2}{2x} & \text{if } y\le\frac{x}{2},\\ 2x+\frac{y^2}{2x} & \text{if } \frac{x}{2}\le y\le x,\\ 2y+\frac{x^2}{2y} & \text{if } x\le y\le 2x,\\ \frac{5}{2}y-x+\frac{x^2}{2y} & \text{if } y\ge 2x.\end{cases}\end{aligned} \tag{5}$$

On the other hand

$$\begin{aligned}\|DR_1(x,y)\|_\square &= \max\{|2x-3y|, |x+y|, |-3x+2y|\}\\ &= \begin{cases}3x-2y & \text{if } 0\le y\le\frac{2x}{3},\\ x+y & \text{if } \frac{2x}{3}\le y\le\frac{3x}{2},\\ 3y-2x & \text{if } \frac{3x}{2}\le y\le 1.\end{cases}\\ \|DR_2(x,y)\|_\square &= \max\{|2x-y|, |x+y|, |-x+2y|\}\\ &= \begin{cases}2x-y & \text{if } 0\le y\le\frac{x}{2},\\ x+y & \text{if } \frac{x}{2}\le y\le 2x,\\ 2y-x & \text{if } 2x\le y\le 1.\end{cases}\\ \|DR_3(x,y)\|_\square &= 2x,\\ \|DR_4(x,y)\|_\square &= 2y.\end{aligned}$$

The reader can easily check that

$$\max\{\|DR_k(x,y)\|_\square : k=1,2,3,4\} = \begin{cases}3x-2y & \text{if } 0\le y\le\frac{x}{2},\\ 2x & \text{if } \frac{x}{2}\le y\le x,\\ 2y & \text{if } x\le y\le 2x,\\ 3y-2x & \text{if } y\ge 2x.\end{cases} \tag{6}$$

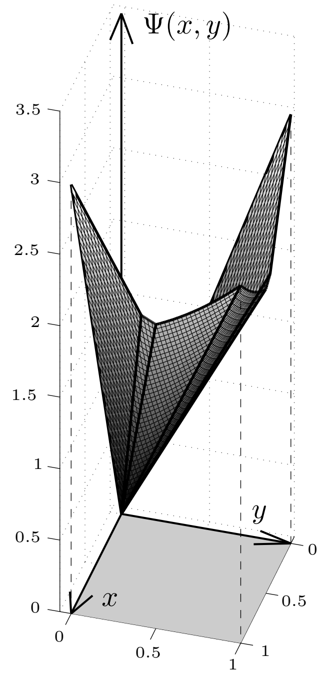

Figure 4.1: $\Psi(x,y)$ for $(x,y)\in\Box$.

In order to compare (5) and (6) we restrict attention to the case $y \le x$. Notice that the surfaces $z = 3x - 2y$ and $z = \frac{5}{2}x - y + \frac{y^2}{2x}$ meet along the line $y = (\sqrt{2}-1)x$. In fact we have

$$3x - 2y \ge \frac{5}{2}x - y + \frac{y^2}{2x} \quad \text{if } 0 \le y \le (\sqrt{2}-1)x,$$
$$\frac{5}{2}x - y + \frac{y^2}{2x} \ge 3x - 2y \quad \text{if } (\sqrt{2}-1)x \le y \le \frac{x}{2},$$

whereas

$$2x + \frac{y^2}{2x} \ge 2x \quad \text{if } \frac{x}{2} \le y \le x.$$

Then the result follows from the latter by symmetry with respect to the line $y = x$. $\Box$

Corollary 4.2. *If $P \in \mathcal{P}(^2\Box)$ then*

$$\max_{(x,y)\in\Box} \|DP(x,y)\|_\Box \le 3\|P\|_\Box. \tag{7}$$

Furthermore, 3 is optimal in (7) since equality holds for the polynomials $P(x,y) = \pm(x^2 - 3xy + y^2)$.

Proof. We just need to observe that the best constant in (7) is nothing but the maximum of Ψ over $\Box$. At this point it can be of help to have a sketch of the graph of Ψ on $\Box$ (see Figure 4.1). Using the convexity of Ψ:

$$\begin{aligned}\max\{\Psi(x,y) : (x,y) \in \Box\} &= \max\{\Psi(0,0), \Psi(1,0), \Psi(0,1), \Psi(1,1)\}\\ &= \max\left\{0, 3, \frac{5}{2}\right\} = 3,\end{aligned}$$

which concludes the proof. $\Box$

Corollary 4.3. *If $P \in \mathcal{P}(^2\Box)$ and $L \in \mathcal{L}^s(^2\Box)$ is the polar of P, then*

$$\|L\|_\Box \leq \frac{3}{2}\|P\|_\Box. \tag{8}$$

Furthermore, $\frac{3}{2}$ is optimal in (8) since equality holds for the polynomials $P(x,y) = \pm(x^2 - 3xy + y^2)$.

Proof. The result follows from Corollary 4.2 taking into consideration that $DP(x,y)(h_1,h_2) = 2L((x,y),(h_1,h_2))$ for all $(x,y),(h_1,h_2) \in \mathbb{R}^2$. $\Box$

5. Unconditional constants in $\mathcal{P}(^2\Box)$

Theorem 5.1. *If $P \in \mathcal{P}(^2\Box)$ then*

$$|||P|||_\Box \leq 5\|P\|_\Box.$$

Equality is attained for the polynomials $P(x,y) = \pm(x^2 - 3xy + y^2)$. Therefore the unconditional constant of the canonical basis of $\mathcal{P}(^2\Box)$ is 5.

Proof. The Krein-Milman approach shows that the unconditional constant of the canonical basis of $\mathcal{P}(^2\Box)$ is given by

$$\max\{|||P_t|||_\Box, |||Q_t|||_\Box, |||R_k(x,y)|||_\Box : t \in [0,1],\ k = 1,2,3,4\}.$$

On the one hand we have $|||P_t|||_\Box = \|tx^2 + 2\sqrt{1-t}xy + y^2\|_\Box = t + 2\sqrt{1-t} + 1$. Similarly $|||Q_t|||_\Box = t + 2\sqrt{1-t} + 1$. Since the mapping $h(t) = t + 2\sqrt{1-t} + 1$ is decreasing in $[0,1]$, it follows that

$$\max\{|||P_t|||_\Box, |||Q_t|||_\Box\} = h(0) = 3.$$

On the other hand

$$\begin{aligned}&\max\{|||R_1(x,y)|||_\Box, |||R_2(x,y)|||_\Box, |||R_3(x,y)|||_\Box, |||R_4(x,y)|||_\Box\}\\ =\ &\max\{5,3,1\} = 5,\end{aligned}$$

which proves the result. $\Box$

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