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Relative Isoperimetric Inequality in the Plane: The Anisotropic Case

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In this paper we prove a relative isoperimetric inequality in the plane, when the perimeter is defined with respect to a convex, positively homogeneous function of degree one $H\colon \mathbb{R}^2 \to [0,+\infty[$. Under suitable assumptions on Ω and H, we also characterize the minimizers.

Keywords: Anisotropic perimeter, relative isoperimetric inequalities, Wulff shape

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1. Introduction

Let Ω be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. The classical relative isoperimetric inequality states that

$$P^2(E;\Omega) \geq C \min\{|E|, |\Omega \setminus E|\}, \tag{1}$$

for any measurable subset E of Ω (see, for example, [15], [18], [9]). Here $|E|$ is the Lebesgue measure of E, and $P(E;\Omega)$ is the usual perimeter in Ω. Being $P(E;\Omega) = P(\Omega\setminus E;\Omega)$, the inequality (1) can be written as

$$P^2(E;\Omega) \geq C|E|, \tag{2}$$

for any $E \subset \Omega$ such that $|E| \leq |\Omega|/2$.

Natural questions related to the inequality (2) are the following: finding the optimal constant

$$C(\Omega) = \inf\left\{ \frac{P^2(E;\Omega)}{|E|} \colon 0 < |E| \leq \frac{|\Omega|}{2},\ E \subseteq \Omega \right\}, \tag{3}$$

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proving that it is attained, and characterizing the minimizers.

First results in this direction can be found in [9] or [18], where it is proved that $C(\Omega) = \frac{8}{\pi}$ when Ω is the unit disk in $\mathbb{R}^2$, and it is attained at a semicircle. More generally, in [11] the author proves that for an open convex set Ω of the plane, $C(\Omega)$ is actually a minimum. Moreover, there exists a convex minimizer of (3) whose measure equals $\frac{|\Omega|}{2}$, and any minimizer E has the following properties:

(a) $\partial E \cap \Omega$ is either a circular arc or a straight segment. Moreover, neither E nor $\Omega \setminus E$ is a circle.
(b) Let T be one of the terminal points of $\partial E \cap \Omega$. Then T is a regular point of $\partial\Omega$ and $\partial E \cap \Omega$ is orthogonal to $\partial\Omega$. As a consequence, either E or $\Omega \setminus E$ is convex.
(c) If $|E| < \frac{|\Omega|}{2}$, then E is a circular sector having sides on $\partial\Omega$. In such a case, there exists another minimizer F which is a sector with sides on $\partial\Omega$, having the same vertex as E, such that $|F| = \frac{|\Omega|}{2}$.

Furthermore, in [11] $C(\Omega)$ is explicitly computed under the additional assumption that Ω is symmetric about a point and also in special cases of convex domains. If $r(\Omega)$ is the inradius of Ω, then

$$C(\Omega) = \frac{8r^2(\Omega)}{|\Omega|}.$$

We refer the reader to [14] for some extremal problems involving $C(\Omega)$.

The purpose of the present paper is to find analogous results when the Euclidean perimeter is replaced by an "anisotropic" perimeter. More precisely, if H is an arbitrary norm on $\mathbb{R}^2$, the perimeter with respect to H for a set $E \subseteq \mathbb{R}^2$ with sufficiently smooth boundary is given by

$$P_H(E;\Omega) = \int_{\partial E\cap\Omega} H(\nu_E)\, d\mathcal{H}^1,$$

where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure and ν_E is the unit outer normal to E (see Section 2 for the precise definition).

We recall that in this setting it is well-known that the following isoperimetric inequality holds for any $E \subseteq \mathbb{R}^2$

$$P_H^2(E;\mathbb{R}^2) \geq 4|W||E|, \tag{4}$$

where $W = \{(x,y)\colon H^o(x,y) < 1\}$ and H^o is polar to H (see [10], [12], [16], [2], [21]). Moreover, the equality in (4) holds if and only if E is homothetic to W. We refer to W as the Wulff shape. For some classes of isoperimetric inequalities which involve other kind of weighted measures, we refer to [8] and the references therein.

Our results can be summarized as follows. Under suitable assumptions on H, we first show that an anisotropic relative isoperimetric inequality holds. That is:

when Ω is an open, bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary, then there exists $C_H(\Omega) > 0$ such that

$$C_H(\Omega) = \inf\left\{ \frac{P_H^2(E;\Omega)}{|E|} : 0 < |E| \le \frac{|\Omega|}{2},\ E \subseteq \Omega \right\}. \tag{5}$$

Then we prove that, for a convex set Ω, $C_H(\Omega)$ is actually a minimum, there exists a convex minimizer of (5) whose measure equals $\frac{|\Omega|}{2}$, and any minimizer E has the following properties:

(α) $\partial E \cap \Omega$ is either homothetic to a Wulff arc (that is an arc of ∂W) or a straight segment. Moreover, neither E nor $\Omega \setminus E$ is homothetic to a Wulff shape.

(β) Let T be one of the terminal points of $\partial E \cap \Omega$. Then T is a regular point of $\partial\Omega$ and $\partial E \cap \Omega$ verifies the following contact angle condition with $\partial\Omega$:

$$\langle \nabla H(\nu_E), \nu_\Omega \rangle = 0,$$

where ν_Ω and ν_E are the usual unit outer normal vectors to $\partial\Omega$ and ∂E at T respectively.

(γ) If $|E| < \frac{|\Omega|}{2}$, then E is homothetic to a Wulff sector (see Section 2 for the precise definition) having sides on $\partial\Omega$. In such a case, there exists another minimizer F which is a sector with sides on $\partial\Omega$, having the same vertex as E, such that $|F| = \frac{|\Omega|}{2}$.

Furthermore, we explicitly compute $C_H(\Omega)$ under the additional assumption that Ω is symmetric about a point. Indeed,

$$C_H(\Omega) = \frac{8r_H^2(\Omega)}{|\Omega|},$$

where $r_H(\Omega)$ is defined in Theorem 3.11. For example, if Ω is obtained by a rotation of $\frac{\pi}{2}$ of a level set of H, that is $\Omega = \{(x,y) \colon H(-y,x) < r\}$, then

$$C_H(\Omega) = \frac{8r^2}{|\Omega|} = \frac{8}{\kappa_H},$$

where $\kappa_H = |\{(x,y) \colon H(x,y) < 1\}|$. We recover immediately the classical result $C_H = 8/\pi$ when H is the Euclidean norm.

We point out that some applications of the above results can be found in [13].

The paper is organized as follows. In Section 2 we give the precise definitions of anisotropic perimeter and some basic properties. In Section 3 we prove the main result. A fundamental argument is to study problem (5) by considering the area $|E|$ fixed. Finally, we give some examples.

2. Notation and preliminaries

Let $H : \mathbb{R}^2 \to [0, +\infty[$ be a $C^2(\mathbb{R}^2 \setminus \{0\})$ function such that $H^2(\xi)$ is strictly convex and

$$H(t\xi) = |t| H(\xi), \quad \forall \xi \in \mathbb{R}^2,\ \forall t \in \mathbb{R}. \tag{6}$$

Moreover, suppose that there exist two positive constants $\alpha \le \beta$ such that

$$\alpha|\xi| \le H(\xi) \le \beta|\xi|, \quad \forall \xi \in \mathbb{R}^2. \tag{7}$$

We define the polar function $H^o\colon \mathbb{R}^2 \to [0,+\infty[$ of H as

$$H^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v\rangle}{H(\xi)}$$

where $\langle \cdot,\cdot\rangle$ is the usual scalar product of $\mathbb{R}^2$. It is easy to verify that also H^o is a convex function which satisfies properties (6) and (7). Furthermore,

$$H(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v\rangle}{H^o(\xi)}.$$

The set

$$W = \{\xi \in \mathbb{R}^2 \colon H^o(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin.

We will call Wulff sector with vertex at the origin the set $A \cap W$, where A is an open cone with vertex at $(0,0)$.

The following properties of H and H^o hold true (see for example [6]):

$$H(\nabla H^o(\xi)) = H^o(\nabla H(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \tag{8}$$

$$H^o(\xi)\nabla H(\nabla H^o(\xi)) = H(\xi)\nabla H^o(\nabla H(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}. \tag{9}$$

Definition 2.1 (Anisotropic relative perimeter). Let Ω be an open bounded set of $\mathbb{R}^2$. In [3], the perimeter of $F \subset \mathbb{R}^2$ in Ω with respect to H is defined as the quantity

$$P_H(F;\Omega) = \sup\left\{ \int_F \operatorname{div}\sigma dx \colon \sigma \in C_0^1(\Omega;\mathbb{R}^2),\ H^o(\sigma) \le 1 \right\}.$$

The equality

$$P_H(F;\Omega) = \int_{\Omega\cap\partial^* F} H(\nu_F) d\mathcal{H}^1$$

holds, where $\partial^* F$ is the reduced boundary of F and ν_F is the unit outer normal to F (see [3]).

The anisotropic perimeter of a set F is finite if and only if the usual Euclidean perimeter $P(F;\Omega)$

$$P(F;\Omega) = \sup\left\{ \int_F \operatorname{div}\sigma dx \colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ |\sigma| \le 1 \right\}.$$

is finite. Indeed, by properties (6) and (7) we have that

$$\frac{1}{\beta}|\xi| \le H^o(\xi) \le \frac{1}{\alpha}|\xi|,$$

and then

$$\alpha P(E;\Omega) \le P_H(E;\Omega) \le \beta P(E;\Omega). \tag{10}$$

Remark 2.2. We observe that when $\partial E \cap \Omega$ is the image of a smooth curve $\gamma(t) = (x(t), y(t))$, $t \in [a,b]$, then $P_H(E;\Omega)$ coincides with the value

$$\mathcal{L}_H(\gamma) = \int_a^b H(-y'(t), x'(t))\, dt. \tag{11}$$

By regularity of H, the curve joining two points P_0 and P_1 which minimizes $\mathcal{L}_H$ is the straight segment P_0P_1. This can be shown by classical argument of Calculus of Variations. We consider, for sake of simplicity, the curves $\gamma(t) = (t, u(t))$. Denoting by $\mathcal{L}_H(u) = \mathcal{L}_H(\gamma)$, the minimum of the problem

$$\begin{cases} \min \mathcal{L}_H(u), \\ u(a) = u_a, \ u(b) = u_b, \end{cases}$$

is the solution to

$$\begin{cases} \frac{d}{dt} H_x(-u'(t), 1) = 0, \\ u(a) = u_a, \ u(b) = u_b. \end{cases}$$

Such solution is the linear function passing through $P_0 = (a, u_a)$ and $P_1 = (b, u_b)$.

Definition 2.3 (Anisotropic curvature ([1], [6])). Let $F \subset \mathbb{R}^2$ be a bounded open set with smooth boundary, $\nu_F(x,y)$ the unit outer normal at $(x,y) \in \partial F$, in the usual Euclidean sense. Let u be a C^2 function such that $F = \{u > 0\}$, $\partial F = \{u = 0\}$ and $\nabla u \neq (0,0)$ on ∂F. Hence, $\nu_F = -\frac{\nabla u}{|\nabla u|}$ on ∂F. The anisotropic outer normal n is defined as

$$n_F(x,y) = \nabla H(\nu_F(x,y)) = \nabla H\left(-\frac{\nabla u}{|\nabla u|}\right), \quad (x,y) \in \partial F,$$

and, by the properties of H,

$$H^o(n_F) = 1.$$

The anisotropic curvature k_H of ∂F is

$$k_H(x,y) = \operatorname{div} n_F(x,y) = \operatorname{div}\left[\nabla H\left(-\frac{\nabla u}{|\nabla u|}\right)\right], \quad (x,y) \in \partial F.$$

Let $(x_0, y_0) \in \partial F$. Without loss of generality, we can locally describe ∂F with a C^2 function $v \colon]x_0 - \delta, x_0 + \delta[\to \mathbb{R}$, that is F is the epigraph of v near $(x_0, y_0) = (x_0, v(x_0))$. By properties of H, the anisotropic curvature $k_H(x_0, y_0)$ of ∂F at (x_0, y_0) can be written as

$$k_H(x_0, y_0) = -\left.\frac{d}{dt} H_x(-v'(t), 1)\right|_{t = x_0}.$$

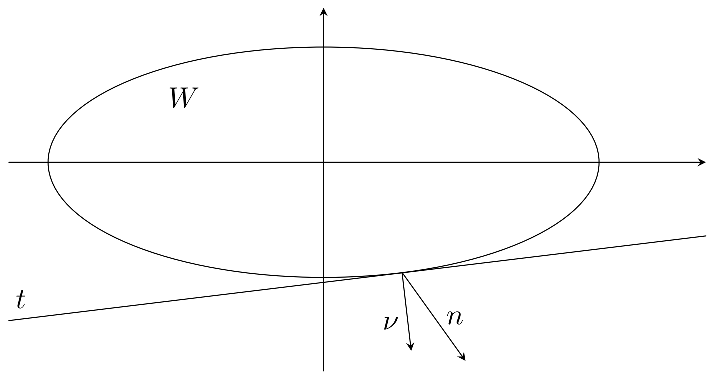

Figure 2.1: Here $H(x,y) = (x^2/a^2 + y^2/b^2)^{1/2}$ and $H^o(x,y) = (a^2x^2 + b^2y^2)^{1/2}$. When $a \neq b$, the usual and the anisotropic outer normal are, in general, different.

Remark 2.4. We stress that if F is homothetic to the Wulff shape W and centered at (x_0, y_0), the anisotropic outer normal at $(x,y) \in \partial F$ has the direction of $(x - x_0, y - y_0)$. Indeed, being $F = \{(x,y)\colon H^o(x - x_0, y - y_0) = \lambda\}$, for some positive λ, by property (9) it follows that

$$n_F(x,y) = \nabla H\big(\nabla H^o(x - x_0, y - y_0)\big) = \frac{1}{\lambda}(x - x_0, y - y_0).$$

See Figure 2.1 for an example.

Remark 2.5. Let be $F = \frac{1}{\lambda}W$, with $\lambda > 0$. It is not difficult to show (see, for instance, [5], [6]) that the anisotropic curvature at $(x,y) \in \partial F$ is

$$k_H(x,y) = \lambda.$$

3. An anisotropic relative isoperimetric inequality

Theorem 3.1. *Let Ω be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. Then an anisotropic relative isoperimetric inequality holds. Namely, there exists a constant $C > 0$ such that*

$$P_H^2(E;\Omega) \geq C \min\{|E|, |\Omega \setminus E|\}, \tag{12}$$

for every measurable set $E \subseteq \Omega$.

Proof. The hypotheses made on Ω guarantee that a relative isoperimetric inequality holds when we consider the usual perimeter $P(E;\Omega)$ (see [15], [18], [11]). Hence the inequality (12) follows immediately from property (10). $\square$

Our aim is to study, for Ω bounded and convex, the best constant in the inequality (12), that is to find the infimum

$$C_H = \inf\left\{Q(E)\colon 0 < |E| \leq \frac{|\Omega|}{2},\ E \subseteq \Omega\right\}, \tag{13}$$

where

$$Q(E) = \frac{P_H^2(E;\Omega)}{|E|},$$

to prove that C_H is actually a minimum, and to characterize the minimizers. Furthermore, we will find the explicit value of C_H in some special case.

If E is a minimizer of (13), then E solves also the following problem under volume constraint:

$$\min\{P_H(F;\Omega),\ F\subset\Omega \text{ and } |F|=|E|\}.$$

The following result characterizes the minimizers of the above problem.

Theorem 3.2. *Let Ω be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. Then there exists a minimizer E of the problem*

$$\min\{P_H(F;\Omega),\ F\subset\Omega \text{ and } |F|=k\}, \tag{14}$$

with $0<k\leq|\Omega|/2$ fixed. Moreover, $\partial E\cap\Omega$ is either homothetic to an arc of ∂W, or a straight segment. Hence a minimizer of (13), if exists, has the same characterization.

Proof. The existence of a minimizer of (14) follows by the lower semicontinuity of P_H (see [3]) using standard methods of Calculus of Variations.

To prove the result, we proceed by steps.

Step 1. First, we show that a minimizer E is locally homothetic to an arc of ∂W, or a straight segment.

Fixed $(x_0,y_0)\in\partial E\cap\Omega$, we can locally describe $\partial E\cap\Omega$ with a C^2 function u (see [1], [7], [4], [19]). That is, without loss of generality, there exists a rectangle $R=]x_0-\delta,x_0+\delta[\times I$ where $E\cap R$ is the epigraph of $u:]x_0-\delta,x_0+\delta[\to I$. Moreover, there exists λ such that u is the minimum of the functional

$$J(v)=\int_{x_0-\delta}^{x_0+\delta}H(-v'(t),1)dt+\lambda\int_{x_0-\delta}^{x_0+\delta}v(t)dt,$$

with boundary conditions $v(x_0+\delta)=u(x_0+\delta)$ and $v(x_0-\delta)=u(x_0-\delta)$. The corresponding Euler equation associated to J is

$$\begin{cases} -\dfrac{d}{dt}H_x(-v'(t),1)=\lambda, & t\in]x_0-\delta,x_0+\delta[,\\ v(x_0\pm\delta)=u(x_0\pm\delta). \end{cases} \tag{15}$$

If $\lambda=0$, there exists a linear function u_0 which solves (15). If $\lambda\neq0$, by Remark 2.5, the function $u_\lambda(t)$, which describes $\frac{1}{\lambda}\partial W$ (up to translation) near x_0, is a solution of (15). On the other hand, for any $\lambda\in\mathbb{R}$, the regularity on H guarantees that the functional J is strictly convex. Hence, $u_\lambda=u$ is the unique solution of (15) (see also [5], [19]).

Step 2. Now we show that the minimizer has the same anisotropic curvature at any point.

Let us take (x_1, y_1) and (x_2, y_2) in $\partial E \cap \Omega$. As in the Step 1, let us consider $u_1 \colon B_1 =]x_1 - \delta_1, x_1 + \delta_1[\to I_1$ and $u_2 \colon B_2 =]x_2 - \delta_2, x_2 + \delta_2[\to I_2$ two functions which locally describe $\partial E \cap \Omega$. Moreover, there exist λ_1 and λ_2 such that u_i, for $i = 1, 2$, minimizes the functional

$$J_i(v) = \int_{B_i} H(-v'(t), 1)dt + \lambda_i \int_{B_i} v(t)dt, \quad i = 1, 2,$$

with boundary conditions $v(x_i \pm \delta_i) = u_i(x_i \pm \delta_i)$. We claim that $\lambda_1 = \lambda_2$. This can be shown by arguing as in [17], Theorem 2. We briefly describe the idea, and we refer to the quoted paper for the precise details.

We assume that $0 \le \lambda_1 < \lambda_2$. A similar argument can be repeated in the other cases.

For every $\lambda \in]\lambda_1, \lambda_2[$ there exists a function $u_{\rho,i}$ which is the unique minimizer to

$$\int_{B_{\rho,i}} H(-v'(t), 1)dt + \lambda \int_{B_{\rho,i}} v(t)dt, \quad i = 1, 2,$$

where $0 < \rho < \min_i \delta_i$ and $B_{\rho,i} =]x_i - \rho, x_i + \rho[$, with boundary conditions $v(x_i \pm \rho) = u_i(x_i \pm \rho)$.

By convexity of H, a comparison argument shows that $u_{\rho,1} \le u_1$ in $B_{\rho,1}$, and $u_{\rho,2} \ge u_2$ in $B_{\rho,2}$. Defining

$$V_{\rho,i} = \int_{B_{\rho,i}} |u_i - u_{\rho,i}| dt,$$

it is possible to prove that there exist two suitable positive numbers r_1 and r_2 such that

$$V_{r_1,1} = V_{r_2,2}. \tag{16}$$

This implies that, defining the set E^* as

$$E^* = [E \cup (\operatorname{epi} u_{r_1,1} \cap C_1)] \setminus [C_2 \cap (E \setminus \operatorname{epi} u_{r_2,2})],$$

where $C_i = B_i \times I_i$, we have that $|E^*| = |E|$.

Finally, we get that $E \Delta E^* \Subset \Omega$ and

$$\begin{aligned}
&P_H(E; \Omega) - P_H(E^*; \Omega) \\
&= \int_{B_{r_1,1}} H(-u_1', 1)dt + \int_{B_{r_2,2}} H(-u_2', 1)dt - \int_{B_{r_1,1}} H(-u_{r_1,1}', 1)dt \\
&\qquad - \int_{B_{r_2,2}} H(-u_{r_2,2}', 1)dt + \lambda \int_{B_{r_1,1}} (u_1 - u_{r_1,1})dt + \lambda \int_{B_{r_2,2}} (u_2 - u_{r_2,2})dt,
\end{aligned}$$

where last line in the above equality vanishes, by (16).

By minimality of $u_{r_1,1}$ and $u_{r_2,2}$, $P_H(E;\Omega) > P_H(E^*;\Omega)$, and this contradicts the minimality of E. Hence, $\lambda_1 = \lambda_2$.

Step 3. We point out that the claim of Step 2 assure that $\partial E \cap \Omega$ consists of Wulff arcs, all with the same curvature, or straight segments. To conclude the proof of the Theorem, we have to prove that E and $\partial E \cap \Omega$ are connected. This can be shown by repeating line by line the proof of Theorem 2 in [11]. □

The following property of the minimizers is a direct consequence of Remark 2.2.

Proposition 3.3. *Let Ω be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. Suppose that E is a minimizer of* (13)*. If $|E| < |\Omega|/2$ and $\partial E \cap \Omega$ is not a straight segment, $\partial E \cap \Omega$ is concave towards E.*

Proof. If $|E| < |\Omega|/2$ and $\partial E \cap \Omega$ is strictly concave towards $\Omega \setminus E$, we can consider a new set E^* by adding to E the region of Ω between $\partial E \cap \Omega$ and a straight segment joining two suitable points of $\partial E \cap \Omega$. Choosing the two points sufficiently near, we get that $|E| \le |E^*| \le |\Omega|/2$ and, by Remark 2.2, $P_H(E^*;\Omega) < P_H(E;\Omega)$. This contradicts the minimality of E. □

Theorem 3.4. *Let Ω be an open bounded convex set of $\mathbb{R}^2$. Suppose that E is a minimizer of* (13)*, and let T be a terminal point of ∂E. Then $\partial\Omega$ at T is C^1, and*

$$\langle n_E, \nu_\Omega \rangle = 0 \tag{17}$$

where n_E is the anisotropic outer normal to ∂E and ν_Ω is the usual unit outer normal to $\partial\Omega$ at T.

Remark 3.5. The angle condition is justified by the following natural geometric argument.

Let $s\colon \alpha_s x + \beta_s y + q_s = 0$ be a straight line, $P_0 = (x_0, y_0) \in \mathbb{R}^2 \setminus s$. By an immediate calculation, the straight segment which minimizes L_H between P_0 and s is parallel to the straight line $r\colon \alpha_r x + \beta_r y = 0$ which has to satisfy the following orthogonality condition:

$$\langle \nabla H(\beta_r, \alpha_r), (\beta_s, \alpha_s) \rangle = 0. \tag{18}$$

Using the notation of Theorem 3.4, if we consider as r the tangent line to $\partial\Omega$ at a terminal point T of $\partial E \cap \Omega$, and as s the tangent straight line to ∂E at T, then (17) and (18) coincide.

Proof of Theorem 3.4. We first assume that $\partial\Omega$ is C^1 at T.

Let us suppose, by contradiction, that (17) is not verified. The idea is to construct a new set E^* such that $Q(E^*) < Q(E)$. This will contradict the minimality of E. To do that, we need to distinguish three cases.

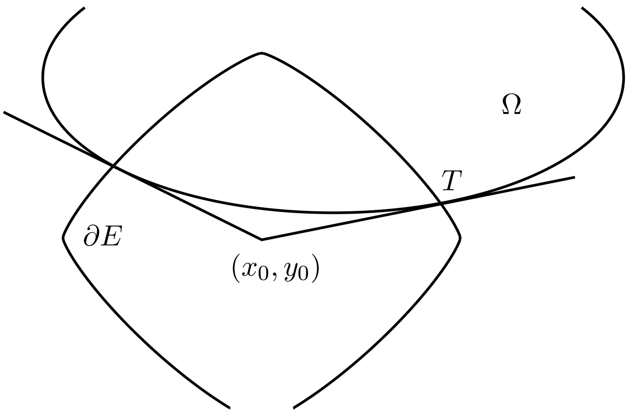

Figure 3.1: Contact angle condition, with $H(x,y)=(x^4+y^4)^{\frac{1}{4}}$ and E is homothetic to the Wulff shape W and centered at (x_0, y_0). The tangent lines to $\partial\Omega$ at the contact points have the same direction of the anisotropic normal to ∂E at the same points.

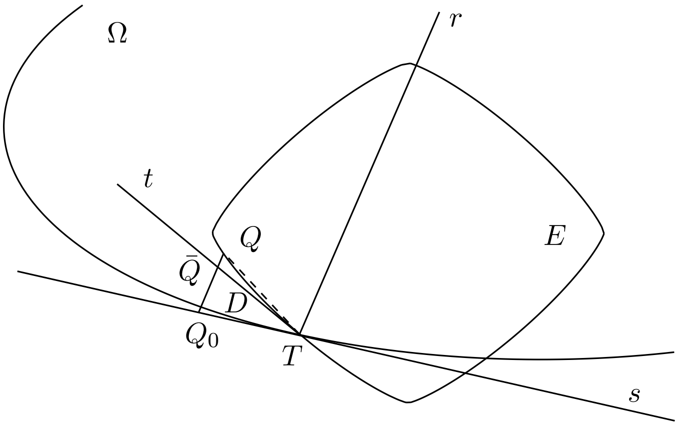

Figure 3.2: Case 1, construction of E^*.

First of all, we denote with s the tangent line to $\partial\Omega$ at T, and with t and r two half line with vertex at T and towards Ω such that t is tangent to ∂E at T and r satisfies the angle condition (18) with respect to s.

Case 1. We first assume that $|E| < |\Omega|/2$ and the angle between s and t towards E is greater than the one between s and r towards E. We construct the straight segment QQ_0 parallel to r joining a suitable point $Q \in \partial E \cap \Omega$ and $Q_0 \in s$. Being Ω convex, we can consider the point $\bar{Q} = QQ_0 \cap \partial\Omega$. Denoted by D the closed region delimited by $Q\bar{Q}$, the arc of ∂E joining Q and T and the arc of $\partial\Omega$ between T and $\bar{Q}$, let be $E^* = E \cup D$ (see Figure 3.2).

We choose Q sufficiently near to T such that $|E^*| < |\Omega|/2$. Hence E^* has larger area than E and, by Remark 3.5 and Lemma 2.2 also smaller anisotropic perimeter.

Case 2. Now we still suppose that $|E| < |\Omega|/2$, and the angle between s and t towards E is smaller that the one between s and r. We construct the straight segment QQ_0 parallel to r joining a suitable point $Q \in t$ and $Q_0 \in s$, the point

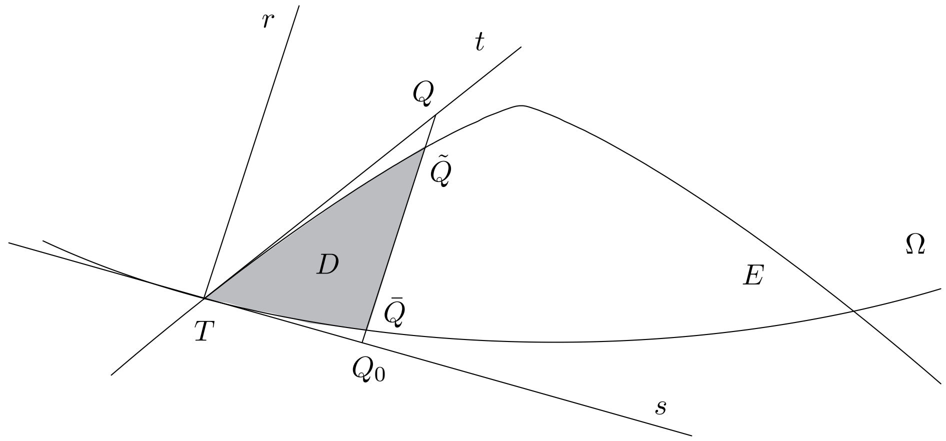

Figure 3.3: Case 2, construction of E^*.

$\bar{Q} = QQ_0 \cap \partial\Omega$ and the set D as the intersection between the triangle QTQ_0 and E (see Figure 3.3). We define $E^* = E \setminus D$.

We show that, for a suitable choice of Q,

$$\frac{P_H^2(E;\Omega)}{|E|} > \frac{P_H^2(E^*;\Omega)}{|E^*|}. \tag{19}$$

Differently from the Case 1, inequality (19) is not obvious because E^* has both smaller perimeter and area. Hence, we explicitly calculate the right-hand side in (19). Denoted by $A = |E|$, $P = P_H(E;\Omega)$, $\delta A = |D| = |E| - |E^*|$, $\delta P = P_H(E;\Omega) - P_H(E^*;\Omega)$, the inequality (19) becomes

$$\frac{P^2}{A} > \frac{(P-\delta P)^2}{A-\delta A}. \tag{20}$$

Denoting by $l_{1,H} = \mathcal{L}_H(\gamma_1)$ and $l_{2,H} = \mathcal{L}_H(\gamma_2)$, where γ_1 and γ_2 are the curves which represent TQ and QQ_0 respectively, it is easy to prove that

$$l_{1,H} = l_1 \cdot H(-\beta,\alpha) = l_1 C_1,$$

where l_1 and (α,β) are respectively the usual lenght and the direction of TQ, and

$$l_{2,H} = l_2 \cdot H(-\beta_r,\alpha_r) = l_2 C_2,$$

where l_2 and (α_r,β_r) are respectively the usual lenght and the direction of QQ_0. Observe that by construction, $l_{1,H} > l_{2,H}$.

We first show (20) replacing δP with $\delta\tilde{P} = l_{1,H} - l_{2,H}$ and δA with $\delta\tilde{A} = \delta A + A_1 + A_2$, where A_1 and A_2 are the measures of the sets as in Figure 3.4.

By elementary properties of triangles,

$$\frac{(P-\delta\tilde{P})^2}{A-\delta\tilde{A}} = \frac{(P - l_1C_1 + l_2C_2)^2}{A - l_1 l_2 \sin(\gamma+\vartheta)} = \frac{\left(P - l_1\left(C_1 - \frac{\sin\gamma}{\sin\theta}C_2\right)\right)^2}{A - l_1^2\frac{\sin\gamma}{\sin\theta}\sin(\gamma+\vartheta)} = f(l_1)$$

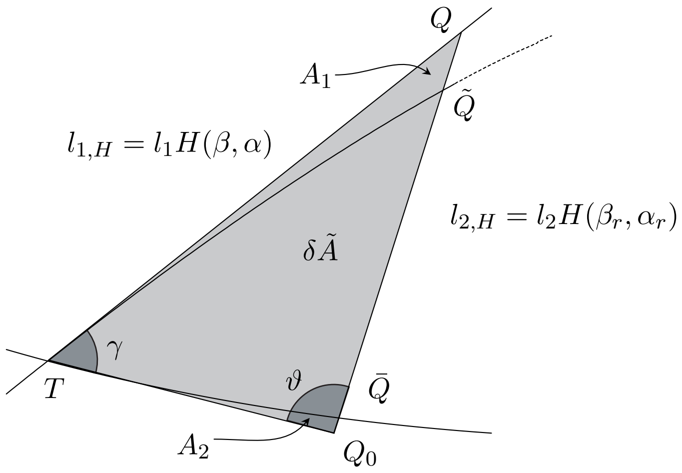

Figure 3.4: Approximation in Case 2.

The function f is strictly decreasing in the interval $[0, \bar{C}]$, with

$$\bar{C} = \frac{A}{P} \frac{C_1 - C_2 \frac{\sin\gamma}{\sin\theta}}{\frac{\sin\gamma}{\sin\theta} \sin(\gamma + \theta)}$$

which is strictly positive, being $l_{1,H} > l_{2,H}$. This implies that, for $l_1 < \bar{C}$,

$$\frac{P^2}{A} > \frac{(P - \delta\tilde{P})^2}{A - \delta\tilde{A}}. \tag{21}$$

On the other hand, by Remark 2.2 we get

$$\delta P \geq \delta\tilde{P}.$$

Hence, being obviously $\delta\tilde{A} \geq \delta A$, by (21), it follows (20) for a suitable choice of Q.

Case 3. Finally, if $|E| = |\Omega|/2$, we can both consider, as minimum sets, E and $\Omega \setminus E$. Hence, if the angle condition is not verified, we can suppose, without loss of generality, that the lines r, s and t verify the hypotheses of Case 2.

If $\partial E \cap \Omega$ is a straight segment, or it is strictly concave towards E, we can repeat line by line the same argument of Case 2. Otherwise, if $\partial E \cap \Omega$ is strictly concave towards $\Omega \setminus E$, proceeding as in Case 1 we construct the straight segment QQ_0, and another straight segment BC joining two suitable points of $\partial E \cap \Omega$. Let D_1 and D_2 be as in Figure 3.5, and define $E^* = (E \setminus D_1) \cup D_2$. Choosing B, C and Q in such a way that $|E| = |E^*|$, since $P_H(E^*; \Omega) < P_H(E; \Omega)$ we obtain a contradiction, and the proof of the Theorem is completed when T is a regular point of $\partial\Omega$. Finally, we show that $\partial E \cap \Omega$ cannot join $\partial\Omega$ at a non regular point.

By contradiction, suppose that $\partial\Omega$ is not regular at T. By convexity it has different right and left tangent straight lines, that we denote by s_1 and s_2 respectively.

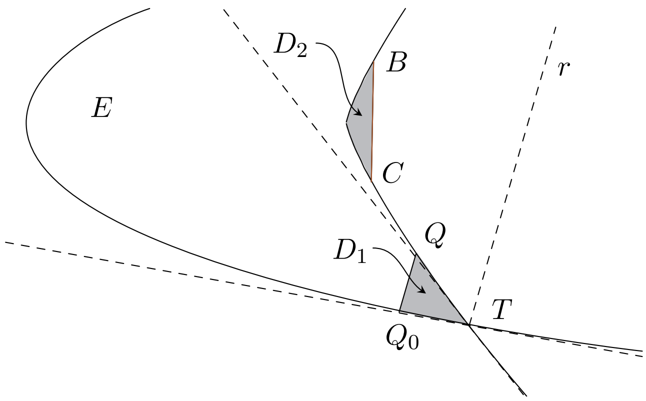

Figure 3.5.

Clearly, the tangent line t does not satisfy the contact angle condition with both s_1 and s_2. So we can repeat the arguments just considered by replacing the straight line s with s_1 or s_2, and obtaining a contradiction with the minimality of E. $\square$

Proposition 3.6. *Let Ω be an open bounded convex set of $\mathbb{R}^2$, $0 < k \leq |\Omega|/2$, and set E_k be a minimizer of problem*

$$\min\{P_H(F;\Omega),\ F \subset \Omega \text{ and } |F| = k\}.$$

We have the following properties:

(*1*) *neither E_k nor $\Omega \setminus E_k$ is homothetic to a Wulff shape;*

(*2*) *if $k < |\Omega|/2$, and T_1 and T_2 are the terminal points of $\partial E_k \cap \Omega$ on $\partial\Omega$, then the left and right tangent straight lines at T_1 to $\partial\Omega$ do not make a cone towards $\Omega \setminus E_k$ with the analogous lines at T_2.*

(*3*) *if $k < |\Omega|/2$ and $\partial E_k \cap \Omega$ is not a straight segment, $\partial E_k \cap \Omega$ is concave towards E.*

Proof. We prove the three properties by contradiction with the minimality of E_k, finding a set with same area and smaller perimeter.

Let E_k or $\Omega \setminus E_k$ be homothetic to a Wulff shape. Since the perimeter $P_H(E_k;\Omega)$ is invariant up to translations in Ω, we can suppose that ∂E_k touches at least at one (regular) point $P \in \partial\Omega$, and there exists a small ball B_P centered at P such that $B_P \cap \partial E_k \not\subset \partial\Omega$.

We stress that in P the contact angle condition cannot hold. Indeed $\nu_{E_k}(P) = \nu_\Omega(P)$, and by (17) and the homogeneity of H we should have that

$$0 = \langle n_{E_k}(P), \nu_\Omega(P)\rangle = \langle \nabla H(\nu_\Omega(P)), \nu_\Omega(P)\rangle = H(\nu_\Omega(P)),$$

so $\nu_\Omega = 0$ and this is absurd. Then arguing as in Case 3 of the proof of Theorem 3.4, being E_k (or $\Omega\setminus E_k$) strictly convex we can add and subtract two small regions in order to get a new set with the same area and smaller perimeter (see Figure 3.5. This proves (*1*).

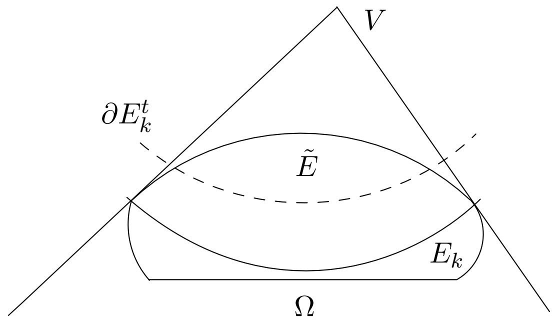

Figure 3.6.

Property (*2*) easily follows by the convexity of Ω. Indeed, if E_k has measure smaller than $|\Omega|/2$ and does not verify (*2*), we can do a suitable translation ∂E_k^t of ∂E_k towards the vertex V of the cone in $\mathbb{R}^2$, in such a way that the set $\tilde{E}$ bounded by $\partial E_k^t \cap \Omega$ towards V and $\partial\Omega$, has measure k and smaller perimeter than E_k in Ω (see Figure 3.6). This contradicts the minimality of E_k.

Finally, suppose that ∂E_k is concave towards $\Omega \setminus E_k$. By property (*2*), the tangent straight lines at terminal points of $\partial E_k \cap \Omega$ either make a cone towards E_k or are parallel. As in Proposition 3.3, in both cases we can add a small region to E_k in order to decrease the perimeter and, similarly as in the proof of property (*2*), with a suitable translation of $\partial E_k \cap \Omega$ towards the vertex of the cone, keep fixed the area $|E_k|$. This proves property (*3*). □

In order to prove the existence of a minimizer of (13), we need the following technical lemma.

Lemma 3.7. *Let $\mu\colon]0,+\infty[\to \mathbb{R}$ be a lower semicontinuous function. Suppose that for any $k > 0$ there exists $\delta_k > 0$ such that*

$$\mu(k+\delta) \le \mu(k), \quad \text{for any } \delta \in [0,\delta_k]. \tag{22}$$

Then μ is decreasing in $]0,+\infty[$.

Proof. By contradiction, suppose that there exist $k_1 < k_2$ such that

$$\mu(k_1) < \mu(k_2). \tag{23}$$

Define $\varphi(k)$ as

$$\varphi(k) = \begin{cases} \mu(k_1) & \text{if } k \le k_1, \\ \mu(k) & \text{if } k_1 < k < k_2, \\ \mu(k_2) & \text{if } k \ge k_2. \end{cases}$$

The function φ is lower semicontinuous, and for any k there exists $\delta_k > 0$ such that $\varphi(k+\delta) \le \varphi(k)$, for any $\delta \in [0,\delta_k]$. Hence, we can define $\bar{\delta} > 0$ as

$$\bar{\delta} = \sup\{\delta > 0 \colon \varphi(k_1+\delta) \le \varphi(k_1)\}.$$

If $\bar{\delta} = +\infty$, then $\varphi(k_2) \leq \varphi(k_1)$, and this contradicts (23). Hence, suppose that $\bar{\delta} < +\infty$. Being φ lower semicontinuous, $\bar{\delta}$ is actually a maximum:

$$\varphi(k_1 + \bar{\delta}) \leq \liminf_{\delta \to \bar{\delta}} \varphi(k_1 + \delta) \leq \varphi(k_1).$$

But this contradicts the definition of $\bar{\delta}$. Indeed, by the property of φ we can take $\tilde{\delta} > \bar{\delta}$ such that $\varphi(k_1 + \tilde{\delta}) \leq \varphi(k_1 + \bar{\delta}) \leq \varphi(k_1)$. Hence, necessarily $\mu(k_1) \geq \mu(k_2)$, and the proof is concluded. □

Theorem 3.8. *Let Ω be an open bounded convex set of $\mathbb{R}^2$. Let $\mu(k)$ be the function defined in $]0, |\Omega|/2]$ as*

$$\mu(k) = \min \left\{ \frac{P_H^2(F;\Omega)}{k},\ F \subset \Omega \text{ and } |F| = k \right\}. \tag{24}$$

Then, we have the following results hold:

(*1*) *$\mu(k)$ is a decreasing lower semicontinuous function in $]0, |\Omega|/2]$,*
(*2*) *the sets which minimize* (24) *verify the contact angle condition. More precisely, they verify the thesis of Theorem* 3.4.

Proof. We first prove that the function μ is lower semicontinuous in $]0, |\Omega|/2]$.

Let be $k \in]0, |\Omega|/2]$, and take a positive sequence k_n such that $k_n \to k$. Consider $E_n \subset \Omega$ such that $|E_n| = k_n$ and $\mu(k_n) = Q(E_n) = k_n^{-1} P_H^2(E_n;\Omega)$. By Proposition 3.6, E_n is convex. Hence, by the Blaschke selection Theorem (see [20], page 50) E_n converges (up to a subsequence) to a set E in the Hausdorff metric. Being E_n convex and bounded, then $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$ strongly, and $|E| = k$. Using the lower semicontinuity of $P_H(\,\cdot\,;\Omega)$ (see [3]) we get

$$\mu(k) \leq Q(E) \leq \liminf_n \frac{P_H^2(E_n;\Omega)}{k_n} = \liminf_n \mu(k_n).$$

In order to prove that μ is decreasing, let be $k \in]0, |\Omega|/2[$ fixed and consider E_k, $|E_k| = k$ such that $\mu(k) = Q(E_k)$.

We claim that there exists a positive number δ_k and a family of sets $E_k(\delta)$, $0 < \delta \leq \delta_k$ with continuously increasing area and $Q(E_k(\delta)) \leq Q(E_k)$. Then

$$\mu(|E_k(\delta)|) \leq Q(E_k(\delta)) \leq \mu(k), \quad \delta \in]0, \delta_k]. \tag{25}$$

Being μ lower semicontinuous in $]0, |\Omega|/2]$, by Lemma 3.7 this is sufficient to show that μ is decreasing.

By Theorem 3.2, $\partial E_k \cap \Omega$ is a straight segment or a Wulff arc, and by property (*1*) of Proposition 3.6, it has two terminal points T_i on $\partial\Omega$. We suppose that such points are regular for $\partial\Omega$, so that by property (*2*) of Proposition 3.6, the tangent lines to $\partial\Omega$, s_i at T_i either are parallel or make a cone A towards E_k. In the first case, the claim follows immediately by the convexity of Ω and making a suitable translation of ∂E_k. Hence, we consider the second case, and suppose without loss

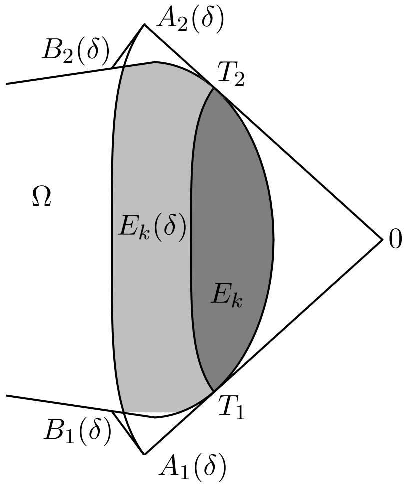

Figure 3.7.

of generality that $s_1 \cap s_2 = (0,0)$. Moreover, by property *(3)* of Proposition 3.6, $\partial E_k \cap \Omega$ is a straight segment, or concave towards E_k.

We need to distinguish two cases for the shape of Ω.

Case 1. $\partial E_k \cap \partial\Omega$ is not contained in ∂A.

We set $C(\delta)$, $\delta \geq 0$, the region bounded by $(1+\delta)\partial E_k$ and ∂A, and $E_k(\delta) = C(\delta) \cap \Omega$. For sake of simplicity, we define $C(0) = C$.

Let $A_i(\delta)$ be the boundary point of $\partial C(\delta) \cap A$ on s_i. Moreover, let be $B_i(\delta) = \partial\Omega \cap w_i$, where w_i is the tangent line to $\partial C(\delta)$ at $A_i(\delta)$. (see Figure 3.7).

Now we compute area and relative perimeter of $E_k(\delta)$. Observe that the triangles D_i of vertex $A_i(\delta)$, $B_i(\delta)$ and T_i have area $|D_i| = o(\delta)$. We have:

$$|E_k(\delta)| \geq |E_k| + (|C(\delta)| - |C|) + o(\delta) = |E_k| + 2\delta|C| + o(\delta)$$

and

$$P_H(E_k(\delta);\Omega) \leq P_H(C(\delta);A) = (1+\delta)P_H(C;A) = (1+\delta)P_H(E_k;\Omega).$$

It follows that

$$\begin{aligned} &\frac{1}{\delta}\left[Q(E_k(\delta)) - Q(E_k)\right] \\ &\leq \frac{1}{\delta} Q(E_k)\left[\frac{(1+\delta)^2}{1 + 2\delta\frac{|C|}{|E_k|} + o(\delta)} - 1\right] \\ &= Q(E_k)\left[\frac{2\left(1 - \frac{|C|}{|E_k|}\right) + o(1)}{1 + o(1)}\right]. \end{aligned} \tag{26}$$

Since $|E_k| < |C|$, then for δ sufficiently small we obtain that the left-hand side of (26) is negative. This proves (25), and hence *(1)*, if T_i are regular points of $\partial\Omega$. If, for example, T_1 is not a regular point, we can repeat the arguments just considered by replacing s_1 with the left or right tangent straight line.

Now we prove (*2*). In order to fix the ideas, we consider the regular point T_1 and the straight line r which verifies the contact angle condition with s_1. Let α_{opt} be the angle between s_1 and r towards E_k, and α the corresponging angle between $\partial E_k \cap \Omega$ and s_1 towards E_k. Suppose by contradiction that $\alpha \neq \alpha_{opt}$.

If $\alpha < \alpha_{opt}$, then the construction made in the proof of Case 2 of Theorem 3.4 allows to take E^* such that $|E^*| < |E_k|$ and $Q(E^*) < Q(E_k)$, and this contradicts the monotonicity of μ. If $\alpha > \alpha_{opt}$, and $\partial E_k \cap \Omega$ is a Wulff arc, as in Case 3 of Theorem 3.4 we can add and subtract two sets in order to decrease the perimeter and to preserve the area, contradicting the minimality of E_k. In the case that $\partial E_k \cap \Omega$ is a straight segment, we can add a small region to E_k in order to decrease the perimeter and with a suitable translation, keep fixed the area $|E_k|$.

Finally, T_1 cannot be a singular point for $\partial\Omega$. Otherwise, similarly as observed at the end of the proof of Theorem 3.4 and proceeding as above, we get a contradiction with the minimality of the minimizer.

Case 2. $\partial E_k \cap \partial\Omega$ is contained in ∂A, that is $E_k = C$.

Define $E_k(\lambda) = \lambda E_k$, $\lambda \geq 0$, and $r \geq 0$ such that

$$\lambda_{max} = \max\{\lambda \geq 0\colon E_k(\lambda) \cap \partial\Omega \subset \partial A\}.$$

First, we prove that at the terminal points of $\partial E_k \cap \Omega$ it holds the contact angle condition (17).

In order to fix the ideas, we consider the regular point $T_1 \in \partial\Omega$ and the straight line r which verifies the contact angle condition with s_1. Let α_{opt} be the angle between s_1 and r towards E_k, and α the corresponging angle between $\partial E_k \cap \Omega$ and s_1 towards E_k. Suppose by contradiction that $\alpha \neq \alpha_{opt}$.

Case 2-a. Let be $\lambda_{max} > 1$. Reasoning as in the proof of Theorem 3.4, we find E^* such that $Q(E^*) < Q(E_k)$, with $|E_k| - |E^*|$ sufficiently small. Then there exists $\rho > 0$ such that $|\rho E^*| = \rho^2 |E^*| = |E_k|$, and $Q(\rho E^*) = Q(E^*) < Q(E_k)$. This contradicts the minimality of E_k.

Repeating the same argument for T_2, we have that the terminal points of $\partial E_k \cap \Omega$ have to verify the angle condition, that is E_k is homothetic to a Wulff sector $W \cap A$.

Case 2-b. Let be $\lambda_{max} = 1$. Then, as $0 < \lambda < \lambda_{max}$, the set λE_k is such that $Q(\lambda E_k) = Q(E_k)$. Thanks to Case 2-a, we have that $\mu(|\lambda E_k|)$ is attained at a Wulff sector, namely the set $(\tilde{\lambda} W) \cap A = (\tilde{\lambda} W) \cap \Omega$, for $\tilde{\lambda} > 0$ such that $|\lambda E_k| = |(\tilde{\lambda} W) \cap A|$. Hence $\mu(|E_k|) = Q((\tilde{\lambda} W) \cap \Omega) < Q(E_k)$. Define

$$\gamma_{max} = \max\{\gamma \geq 0\colon (\gamma W) \cap \Omega \text{ is homothetic to a Wulff sector}\}. \tag{27}$$

We have that γ_{max} is finite and $|\gamma_{max} W \cap \Omega| < |E_k|$. Otherwise, there exists $\gamma \leq \gamma_{max}$ such that $|\gamma W \cap \Omega| = |E_k|$ and $Q(\gamma W \cap \Omega) = Q(\tilde{\lambda} W \cap \Omega) < Q(E_k)$, and this is a contradiction.

As matter of fact, the homogeneity of H and (9) imply, for $\xi \in \partial W$, that $H(\nu_W(\xi)) = \langle \nu_W(\xi), \xi \rangle$. Moreover, for $\xi \in \partial A$, $\langle \nu_A(\xi), \xi \rangle = 0$. Hence by the divergence Theorem we get that, for $\gamma > 0$,

$$P_H(\gamma W; A) = 2\gamma |W \cap A|. \tag{28}$$

Define $E(\delta) = \Omega \cap [(\gamma_{max} + \delta)W]$, and A_δ the cone made by the two half-straight lines s_i^δ, $i = 1, 2$ with origin at $(0, 0)$ and passing through one of the two terminal points of $\partial[E(\delta) \cap \Omega]$.

By (28) and the convexity of Ω, we get, for an appropriate δ, $|E(\delta)| = k$ and

$$Q(E(\delta)) \leq 4|W \cap A_\delta| < 4|W \cap A| = Q(\gamma_{max} W) < Q(E_k).$$

Then ∂E_k must verify the contact angle condition at each T_i, and this concludes the Case 2-b, and (*2*) is proved.

In order to prove (25), and hence (*1*), we observe that from (*2*), $E_k = (\lambda W) \cap \Omega$, for some $\lambda > 0$. Let γ_{max} as in (27), and suppose that $\gamma_{max} = \lambda$, otherwise (25) is immediate, being $Q(E_k) = Q(\gamma W \cap \Omega)$, for any $0 < \gamma < \gamma_{max}$. Defining $E(\delta) = \Omega \cap [(\gamma_{max} + \delta)W]$ and reasoning as in Case 2-b, we get (25).

Finally, the regularity of T_i on $\partial\Omega$ follows exactly as in the Case 1, and the proof is completed. □

Remark 3.9. We observe that if E is a minimizer of (13), and $|E| < |\Omega|/2$, then E is homothetic to a Wulff sector with sides on $\partial\Omega$. Otherwise, arguing as in Case 1 of the proof of Theorem 3.8, we construct a new set E^* with $Q(E^*) < Q(E)$. Hence, $E = E(\lambda) = A \cap (\lambda W)$ with sides on $\partial\Omega$. Being

$$Q(E(\rho)) = 4|W \cap A|, \quad \forall \rho \colon |E(\rho)| \leq \frac{|\Omega|}{2},$$

where $E(\rho) = A \cap (\rho W)$, there exists another minimizer F which is a Wulff sector with sides on $\partial\Omega$ and $|F| = |\Omega|/2$.

Now we are able to prove the main result.

Theorem 3.10. *Let Ω be an open bounded convex set of $\mathbb{R}^2$. Then there exists a convex minimizer of problem* (13) *whose measure is equal to $|\Omega|/2$. More precisely, either a minimizer E of* (13) *has measure $|\Omega|/2$, or E is homothetic to a Wulff sector with sides on $\partial\Omega$. Finally, it verifies the contact angle condition.*

Proof. Let μ defined as in the above theorem and, being μ decreasing in $]0, |\Omega|/2]$, it attains its minimum at $k = |\Omega|/2$.

Now we are able to prove that (13) has a minimum. Let $\tilde{E}$ be such that $|\tilde{E}| = |\Omega|/2$ and $\mu(|\Omega|/2) = Q(\tilde{E})$. Let E_n, $n \in \mathbb{N}$ be a minimizing sequence of problem (13), that is

$$\lim_n Q(E_n) = C_H, \qquad 0 < |E_n| \leq |\Omega|/2.$$

Without loss of generality, we may suppose that, for any $n \in \mathbb{N}$, $Q(E_n) = \mu(|E_n|)$. Otherwise, we replace E_n with the minimizer of problem (14) with volume constraint $k = |E_n|$. Then

$$C_H \le Q(\tilde{E}) = \mu(|\Omega|/2) \le \mu(|E_n|) = Q(E_n).$$

Passing to the limit,

$$C_H = \mu\left(\frac{|\Omega|}{2}\right),$$

and $\tilde{E}$ is a minimizer of (13), whose boundary in Ω is a straight segment or a Wulff arc. From the proof of Theorem 3.8 it follows that if E is another minimizer of (13) with $|E| < |\Omega|/2$, then it is a Wulff sector with sides on $\partial\Omega$. Recalling Theorem 3.4, the result is completely proved. □

In the following theorem, we characterize the minimizers for centrosymmetric sets, and find the constant C_H in (13).

For sake of simplicity, if T is a point in $\mathbb{R}^2$, we put $L_H(T) = \mathcal{L}_H(\gamma)$, where $\mathcal{L}_H$ is defined in (11), and γ is a curve which represent the straight segment OT joining T with the origin O. We observe that if $T = (x, y)$, then $L_H(T) = H(-y, x)$.

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^2$ be a convex bounded set, symmetric about the origin O. Then a minimizer of* (13) *is a set E whose boundary in Ω is a straight segment passing through the origin and such that $P_H(E;\Omega) = 2r_H$, where $r_H = r_H(\Omega) = \min_{T\in\partial\Omega} L_H(T)$. Hence,*

$$C_H = \frac{8r_H^2}{|\Omega|}.$$

Proof. The first step is to prove the existence of a set E enjoying the properties of the statement. Let us consider the set

$$B(r_H) = \{(x, y) \in \mathbb{R}^2 \colon L_H(x, y) < r_H\}.$$

Then $\partial B(r_H)$ meets $\partial\Omega$ at least at two symmetric regular points T_1, T_2. We observe that in T_i the contact angle condition is satisfied. Indeed, the anisotropic outer normal to the straight segment OT_i is $n_E(T_i) = \nabla H(-y_i, x_i)$, where $T_i = (x_i, y_i)$, $i = 1, 2$. Denoted by $\nu_\Omega(T_i)$ the unit outer normal to $\partial\Omega$ at T_i, being $\nu_\Omega(T_i) = (H_y(-y_i, x_i), -H_x(-y_i, x_i))$, we have $\langle n_E(T_i), \nu_\Omega(T_i)\rangle = 0$.

We show that T_1T_2 is the boundary in Ω of the required set E, and $Q(E) = \frac{8r_H^2}{|\Omega|}$.

By Theorem 3.10, there exists a convex minimizer of (13) whose measure is $|\Omega|/2$, which is a straight segment or a Wulff arc. If we show that $P_H(E;\Omega) \le P_H(F;\Omega)$, where F is a open convex subset of Ω such that $|F| = |\Omega|/2$ and $\partial F \cap \Omega$ is a straight segment or a Wulff arc, we have done.

Clearly, any straight segment passing through the origin bounds in Ω a set with greater perimeter than E and with same area $|\Omega|/2$. We do not consider the straight segments which not contain the origin, because they bounds in Ω sets with measure different from $|\Omega|/2$. Hence we can suppose that $\partial F \cap \Omega$ is a Wulff arc.

Obviously, $O \notin \partial F$, otherwise $|F| \neq |\Omega|/2$. More precisely, denoted by P_1 and P_2 the terminal points of $\partial F \cap \Omega$, we get that $O \in F \setminus \bar{G}$, where $G \subset F$ is bounded by $\partial\Omega$ and P_1P_2, otherwise $|\Omega|/2| \leq |G| < |F|$, and this is impossible. Hence we can consider the straight segments in F, OP_1 and OP_2, and it is not difficult to show that

$$P_H(F;\Omega) > L_H(P_1) + L_H(P_2) \geq 2r_H = P_H(E;\Omega),$$

and this concludes the proof. $\square$

Remark 3.12. If $\Omega = \{(x,y)\colon H(-y,x) < r\}$, i.e. Ω is obtained by a rotation of $\frac{\pi}{2}$ the $r-$level set of H, then Theorem 3.11 gives

$$C_H = \frac{8r^2}{|\Omega|} = \frac{8}{\kappa_H},$$

where $\kappa_H = |\{(x,y)\colon H(x,y) < 1\}|$. Observe that any straight segment passing through the origin and joining the boundary of Ω bounds a minimizer.

In particular, if $H(x,y) = H^o(x,y) = (x^2+y^2)^{1/2}$, we recover the classical result $C_H = \frac{8}{\pi}$ (see for instance [18], [11]).

4. Some examples

Here we apply the results just obtained to some particular function H.

Example 4.1. Let $H(x,y)$ defined as

$$H(x,y) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}.$$

An immediate calculation gives that

$$H^o(x,y) = \left(a^2x^2 + b^2y^2\right)^{\frac{1}{2}}.$$

If Ω is the ellipse $\Omega = \{(x,y)\colon H^o(x,y) < r\}$, then $\Omega = \{(x,y)\colon H(-y,x) < \frac{r}{ab}\}$, and $|\Omega| = \frac{\pi r^2}{ab}$. By Theorem 3.11 and Remark 3.12 we have

$$P_H^2(E;\Omega) \geq \frac{8}{\pi ab}|E|, \quad \forall E \subset \Omega\colon |E| \leq \frac{\pi r^2}{2ab}. \tag{29}$$

Moreover, the equality in (29) holds if and only if $\partial E \cap \Omega$ is any straight segment passing through the origin (see Figure 4.1).

We observe that if we compute C_H for the ellipse $\Omega_1 = \{(x,y)\colon H(x,y) < r\}$, with for example, $a > b$, then the smaller axis of the ellipse (in the usual sense) is the boundary of the only minimizer of (13) (see Figure 4.1), and the constant C_H is

$$C_H = \frac{8}{\pi ab}\frac{b^2}{a^2}.$$

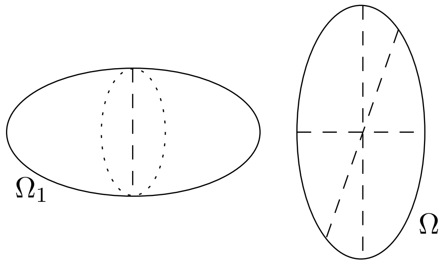

Figure 4.1: In the first figure, Ω_1 is a level set of H, and the straight segment is the boundary of the only minimizer of (13). In the second figure, Ω is a level set of H^o, and any straight segment passing through the origin is the boundary of a minimizer.

We point out that the above result for Ω can be obtained directly by the classical relative isoperimetric inequality for the Euclidean perimeter. Indeed, the anisotropic relative perimeter of a smooth set E, whose boundary is described by $(u(t), v(t))$, with $t \in [\alpha, \beta]$, is

$$P_H(E;\Omega) = \int_\alpha^\beta H(-v', u')\, dt = \int_\alpha^\beta \left(\frac{(v')^2}{a^2} + \frac{(u')^2}{b^2} \right)^{\frac{1}{2}} dt. \tag{30}$$

Defining $w = au$ and $z = bv$, the curve $(w(t), z(t))$ describe the boundary of the unit Euclidean disk B_r with radius r and centered at the origin. By changing the variables in (30), we get

$$\int_\alpha^\beta \left(\frac{(z')^2}{a^2b^2} + \frac{(w')^2}{a^2b^2} \right)^{\frac{1}{2}} dt = \frac{1}{ab} P(\tilde{E}, B_1) \geq \frac{1}{ab}\sqrt{\frac{8}{\pi}} |\tilde{E}|^{\frac{1}{2}} = \sqrt{\frac{8}{\pi ab}} |E|^{\frac{1}{2}},$$

where $\tilde{E}$ is the set obtained by E after the change of variables. Being $|\tilde{E}| = ab|E|$, we get (29).

Finally, the characterization of the minimizers is a direct consequence of the fact that in the classical relative isoperimetric inequality, the minimizers are the diameters. Hence in this case we get the relative anisotropic isoperimetric inequality by a linear trasformation, as a consequence of the classical relative isoperimetric inequality.

Example 4.2. Now suppose that

$$H(x, y) = (|x|^p + |y|^p)^{\frac{1}{p}}.$$

where $2 \leq p < +\infty$ and $p' = \frac{p}{p-1}$. Hence, we have $H^o(x, y) = (|x|^{p'} + |y|^{p'})^{\frac{1}{p'}}$.

Let us consider $\Omega = \{(x, y) \colon |x|^p + |y|^p < r^p\}$. Being Ω invariant by $\frac{\pi}{2}-$rotations, by Theorem 3.11 and Remark 3.12 we have

$$P_H^2(E;\Omega) \geq \frac{8}{\kappa_H}|E|, \quad \forall E \subset \Omega \colon |E| \leq \frac{r^2 \kappa_H}{2},$$

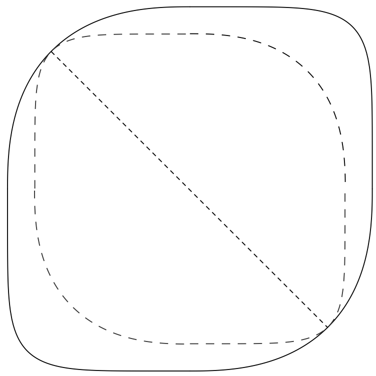

Figure 4.2: Example 4.3. The solid line represents a level set of H, while the straight segment is the boundary of the only minimizer of (13).

where $\kappa_H = |\{(x,y)\colon H(x,y) < 1\}|$, and any straight segment passing through the origin bounds a minimizer.

Example 4.3. Let H be defined as follows:

$$H(x,y) = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } xy \geq 0, \\ (|x|^q + |y|^q)^{1/q} & \text{if } xy \leq 0, \end{cases}$$

with $p > 2$, $q > 2$ and $p > q$. Let us consider $\Omega = \{(x,y)\colon H(-y,x) < r\}$. Then

$$C_H = C_H(\Omega) = \frac{8}{\kappa_H}.$$

We stress that if $\Omega_1 = r\{(x,y)\colon H(x,y) < r\}$, then easy computations give that

$$C_H = C_H(\Omega_1) = \frac{8}{\kappa_H} 4^{\frac{1}{p}-\frac{1}{q}}.$$

Observe that $C_H(\Omega) > C_H(\Omega_1)$ (compare Figure 4.2).

Example 4.4 (A non-regular case). Let us consider $H(x,y) = \max\{|x|,|y|\}$. The singular behavior of H does not allow to apply the previous results. Then, in order to prove the anisotropic isoperimetric inequality relative to Ω with respect to H, we argue by approximation.

Let be $\Omega = \{(x,y)\colon \max\{|x|,|y|\} < r\}$, and $H_p(x,y) = (|x|^p + |y|^p)^{1/p}$. For any set $E \subset \Omega$ such that $|E| \leq 2r^2$, we have

$$P^2_{H_p}(E;\Omega) \geq 2|E|, \tag{31}$$

and the best constant is reached by a rectangle whose boundary in Ω is the straight segment joining $(-r,0)$ and $(r,0)$ (or $(0,-r)$ and $(0,r)$). We can pass to the limit as $p \to +\infty$ in (31), obtaining

$$P^2_H(E;\Omega) \geq 2|E|, \quad \forall E \subset \Omega\colon |E| \leq 2r^2. \tag{32}$$

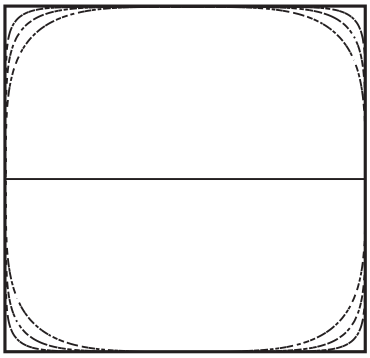

Figure 4.3: Example 4.4

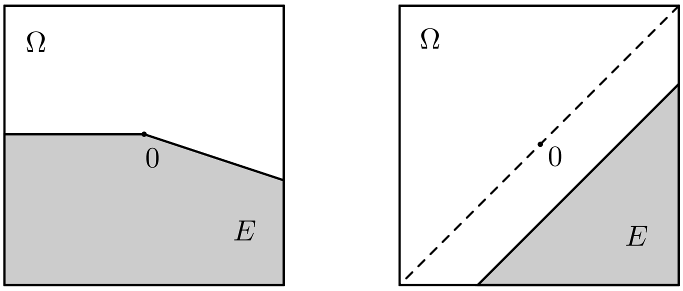

Figure 4.4:

Any straight segment passing through the origin and joining the boundary of Ω bounds a minimizer. Unlike the case of H smooth (compare Remark 3.12), such sets are not the only minimizers.

For example, in Figure 4.4 some minimizer is represented. Indeed, if ∂E is described by a Lipschitz function $u(t)$, $t \in [a,b]$, the perimeter is

$$P_H(E;\Omega) = \int_a^b \max\{1, |-u'(t)|\}dt.$$

Then in the picture on the left-hand side of Figure 4.4, the perimeter of E is $2r$ and $|E| = 2r^2$. Moreover, in the other picture any triangle E such that $\partial E \cap \Omega$ is a straight segment parallel to a diagonal is a minimizer.

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