Journal of Convex Analysis
Volume 20 (2013), No. 1, 199–220

Pointwise Estimates and Monotonicity Formulas without Maximum Principle

Marcelo Montenegro*
Departamento de Matemática, Universidade Estadual de Campinas – IMECC, Rua Sérgio Buarque de Holanda, 651, Campinas-SP, Brazil, CEP 13083-859
msm@ime.unicamp.br

Enrico Valdinoci†
Dipartimento di Matematica, Università di Milano, Via Cesare Saldini, 50, 20133 Milano, Italy; and: Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche, Via Ferrata, 1, 27100 Pavia, Italy
enrico@math.utexas.edu

Received: July 27, 2011

We study a second order elliptic partial differential equation for which a maximum principle is not available and whose nonlinearity is not C^1.

We discuss the role of a pointwise gradient bound and we derive a monotonicity estimate near flat points of the free boundary.

Keywords: Elliptic PDEs, a-priori estimates, monotonicity formula, free boundary

1991 Mathematics Subject Classification: 35B09, 35B45, 35J25, 35J61, 35R35

1. Introduction

Given $p \in (-1, 1)$ and an open set $\Omega \subseteq \mathbb{R}^n$, not necessarily bounded, we consider a solution $u \in C(\overline{\Omega}) \cap C^2(\Omega \cap \{u > 0\})$ of the elliptic partial differential equation

$$\Delta u(x) = (u(x))^p \quad \text{for every } x \in \Omega \cap \{u > 0\}. \tag{1}$$

The study of the PDE in (1) is a classical topic in both the pure and the applied mathematics settings, since it arises in reaction-diffusion processes and it is related to the variational analysis of some free boundary problems (see [14, 13, 6, 1]). The range $p \in (-1, 0)$ is sometimes called the "quenching regime", and the range $p \in (0, 1)$ the "strong absorption regime". In particular, it is of interest the "dead core" of the solution, i.e. the set $\Omega \cap \{u = 0\}$ and its free boundary $\Omega \cap (\partial\{u > 0\})$. We remark that if p were bigger or equal than 1, then the strong maximum principle

*Marcelo Montenegro has been partially supported by *CNPq* and *FAPESP*.
†Enrico Valdinoci has been supported by *FIRB Analysis and Beyond* and *GNAMPA Equazioni nonlineari su varietà: proprietà qualitative e classificazione delle soluzioni*.

ISSN 0944-6532 / $ 2.50

would apply to (1) and then the dead core would be empty (see, e.g., [3, 15]). On the contrary, the absence of the maximum principle when $p \in (-1,1)$ may lead to plateaus, as shown by the onedimensional solution

$$u_o(x_1,\dots,x_n) = \left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n^+\right)^{2/(1-p)}, \tag{2}$$

where, as usual,

$$x_n^+ := \begin{cases} 0 & \text{if } x_n \leqslant 0, \\ x_n & \text{if } x_n > 0. \end{cases}$$

In this paper, a central role will be played by the following pointwise gradient estimate:

$$\frac{|\nabla u(x)|^2}{2} \leqslant M\frac{(u(x))^{p+1}}{p+1}, \tag{3}$$

for some $M > 0$.

It is worth to note that (3) may be seen as a partition of energy, in the sense that the kinetic part of the energy functional associated to (1), i.e. the left hand side of (3), is bounded by M times the potential part of the energy, i.e. the right hand side of (3).

So, our first result is that (3) holds away from $\{u = 0\} \cup \partial\Omega$:

Theorem 1.1 (Pointwise gradient bound). *Let* $p \in (-1,1)$. *Let* $u \in C(\overline{\Omega}) \cap C^2(\Omega \cap \{u > 0\})$ *be a solution of* (1), *with* $0 \in \Omega \cap \{u > 0\}$.

Let $d_0 := \mathrm{dist}(0, \{u = 0\} \cup \partial\Omega)$.

Then there exists $M > 0$ *such that* (3) *holds for all* $x \in B_{d_0/2}$.

Here, M *depends only on* n, p, d_0 *and* $\|u\|_{L^\infty(B_{d_0/2})}$.

The proof of Theorem 1.1, which is contained in Section 2, uses some techniques of [4], where a parabolic equation without maximum principle was considered. Indeed, though (3) seems reminiscent of the classical a-priori estimate in [10], the setting here is different and the assumptions and the techniques of [10] seem to be not directly applicable, since it is required in [10] that the PDE holds in the whole of $\mathbb{R}^n$.

Such an assumption has been weakened in [7] in order to deal with Dirichlet problems in possibly unbounded domains with nonnegative mean curvature (see also Lemma 3.2 in [1], and references therein, for the case of bounded domains) and, in general, the mean curvature assumption cannot be removed, see Remark 2(i) in [7]. Nevertheless, we take here no assumption at all on the domain Ω and no boundary condition along $\partial\Omega$ is involved in our framework.

We also recall that the maximum principle, which is not available in our setting, is an essential ingredient for the proofs in [10, 7].

Also, Theorem 1.1 here improves Theorem 1 of [12] in the range $p \in (-1, 0)$.

The constant M in (3) is quite important for the applications: for instance, it plays a crucial role in the regularity of the free boundary (see, e.g., Remark 1.4 in [13]). In Appendix A we will point out that, when (3) holds near the free boundary, then a precise estimate on M becomes available, namely M can be chosen to be as close to 1 as we wish (see Theorem A.1).

Such estimate with a sharp constant also implies the following monotonicity formula near the free boundary points. For this, we define the local rescaled energy functional as

$$\mathscr{E}(u,r) := \frac{1}{r^{n-1}} \left(\int_{B_r} |\nabla u|^2 + \frac{2u^{p+1}}{p+1} \right).$$

We also define the following (truncated) cone, for every $r > 0$:

$$\mathscr{C}_r := \left\{ x \in B_r \setminus \{0\} \text{ s.t. } x_n \geqslant \frac{|x|}{2} \right\} = \{ty,\, t \in (0,r),\, y \in \mathscr{D}\}, \tag{4}$$

where

$$\mathscr{D} := \{ y \in \partial B_1 \text{ s.t. } y_n \geqslant 1/2 \}.$$

Theorem 1.2 (Monotonicity formula). *Let $p \in (0,1)$. Let u be a solution of (1) in Ω.*

Suppose that $0 \in \Omega \cap \partial\{u > 0\}$ and that

$$\begin{gathered} \text{for every } \eta > 0 \text{ there exists } \epsilon(\eta) \in (0,R) \text{ such that} \\ |\nabla u(x)|^2 \leqslant \left(\frac{2}{p+1} + \eta \right) (u(x))^{p+1} \text{ for every } x \in B_{\epsilon(\eta)}. \end{gathered} \tag{5}$$

Then, for every $\eta > 0$ there exists $\epsilon_o(\eta) > 0$ such that, for every $r \in (0, \epsilon_o(\eta))$, we have that

$$\frac{\partial \mathscr{E}}{\partial r}(u,r) \geqslant \frac{2}{r^{n-1}} \int_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2 - \frac{\eta}{r^n} \int_{B_r} u^{p+1}. \tag{6}$$

Moreover, if there exist $C > 0$ and $\tilde{\epsilon} > 0$ such that, for every $r \in (0, \tilde{\epsilon})$

$$\fint_{B_r} u^{p+1} \leqslant C \fint_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2, \tag{7}$$

then there exists $\epsilon' > 0$ such that for every $r \in (0, \epsilon')$

$$\frac{\partial \mathscr{E}}{\partial r}(u,r) \geqslant 0. \tag{8}$$

In particular, (7) (and so (8)) holds true if

$$\begin{gathered} \text{there exist } \epsilon'' > 0 \text{ and } g \in C^1(\overline{\mathscr{C}_{\epsilon''}}) \text{ such that } g(0) = 0, \\ \lim_{s \to 0^+} |\nabla g(sy) \cdot y| = 0 \text{ uniformly for every } y \in \mathscr{D}, \text{ and} \\ u(x) = \left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n^+ + g(x) \right)^{2/(1-p)} \quad \text{for every } x \in B_{\epsilon''}. \end{gathered} \tag{9}$$

For different monotonicity formulas in related frameworks, see [16, 8, 5], and also [9] for several applications. As customary, in (6) and (7), ν denotes the exterior normal of ∂B_r. We notice that the function in (9) reduces to the one in (2) when $g = 0$, that condition (9) may be seen as an asymptotic expansion of u near the nice points of the free boundary. Indeed, as we will see more precisely in the forthcoming Theorem 1.3, the expression of u in (9) is reminiscent of the asymptotic expression of the minimizers (see, e.g., Lemmata 4.5 and 7.2 in [1]).

We observe that no boundary conditions are assumed in Theorem 1.2, but only that (5) holds. The proof of Theorem 1.2, which is contained in Section 3, follows the technique of [11]: the role played in [11] by the monotonicity formula of [10], which is not available in our setting, is played here by the estimate given in (5).

As a concrete application of Theorem 1.2, we deal with the associated variational free boundary problem. For this, we recall some classical terminology (see, e.g., [1]). Given a bounded domain $U \subset \mathbb{R}^n$, we define

$$\mathscr{F}_U(v) := \int_U \left(\frac{|\nabla v|^2}{2} + \frac{|v|^{p+1}}{p+1} \right).$$

We define u to be a minimum for $\mathscr{F}_U$ if $\mathscr{F}_U(u) \leqslant \mathscr{F}_U(u+\zeta)$ for every $\zeta \in C_0^\infty(U)$. That is, roughly speaking, u minimizes the functional $\mathscr{F}_U$ with respect to its own boundary Dirichlet data on ∂U.

In this framework, the results of [1] make possible to apply Theorem 1.2 to minimizers, near the points around which the free boundary is flat, as it is stated in the next result:

Theorem 1.3 (Monotonicity formula near flat free boundary points). *Let $p \in (0,1)$ and $R > 0$. Let $u \geqslant 0$ be a minimizer of $\mathscr{F}_{B_R}$, with $0 \in \partial\{u > 0\}$.*

Then, there exist $\ell > 0$, a_o, $c_o \in (0,1)$ such that the following statement holds.

Suppose that, for some $a \in [0, a_o]$ and $\rho \in (0, c_o a^\ell)$, we have that

$$u(x) = 0 \quad \textit{for every } x = (x', x_n) \in B_\rho \textit{ with } x_n \leqslant -a\rho. \tag{10}$$

Then, then there exists $\rho_o \in (0, \rho)$ such that for every $r \in (0, \rho_o)$

$$\frac{\partial \mathscr{E}}{\partial r}(u, r) \geqslant 0. \tag{11}$$

We think it would be an interesting problem to investigate whether similar results hold in further generality; for instance, it would be desirable to obtain (11) under weaker assumptions, or to classify the points for which (11) holds true. The proof of Theorem 1.3 is contained in Section 3. Next are the proofs of the results stated above.

2. Proof of Theorem 1.1

Theorem 1.1 will follow from a more general estimate:

Lemma 2.1. *Let $Z \in C^2((0,+\infty))$ and $f \in C^1((0,+\infty))$. Let $L > 0$. Suppose that there exists $C > 0$, possibly depending on n, p and L, such that*

$$Z'(r) \geqslant 0 \tag{12}$$

and

$$\begin{aligned} & Z'(r)Z(r)^{1/2} + |Z(r)f'(r)| + |Z'(r)f(r)| + Z(r) \\ \leqslant\ & C(\tfrac{1}{2}Z'(r)^2 - Z''(r)Z(r)), \end{aligned} \tag{13}$$

for all $r \in (0, L]$.

Let $u \in C^3(\Omega_\star) \cap C(\overline{\Omega}_\star)$ be a solution of

$$\Delta u = f(u) \tag{14}$$

in $\Omega_\star$, with $0 < u \leqslant L$ in $\Omega_\star$.

Assume that

$$\inf_{x\in\Omega_\star} Z(u(x)) > 0. \tag{15}$$

Suppose also that there exists a continuous, compactly supported function

$$\begin{gathered} \psi \in C^2(\Omega_\star, (0,+\infty)) \quad \textit{with } \psi = 0 \textit{ on } \partial\Omega_\star, \\ \textit{such that } \frac{|\nabla\psi|^2}{\psi} \textit{ is bounded in } \Omega_\star. \end{gathered} \tag{16}$$

Then, there exists $P > 0$ such that

$$\psi(x)|\nabla u(x)|^2 \leqslant PZ(u(x)) \quad \textit{for every } x \in \Omega_\star. \tag{17}$$

Here, P depends only on $\Omega_\star$, n, p, ψ and L.

Proof. We follow and extend some computations developed in [4] for parabolic problems. Let

$$w := \frac{|\nabla u|^2}{Z(u)}, \qquad v := w\psi. \tag{18}$$

Assume by contradiction that the estimate is false, i.e.

$$\sup_{\Omega_\star} v > P, \tag{19}$$

where $P > 0$ will be conveniently chosen later.

Notice that v is continuous in $\overline{\Omega}_\star$, thanks to (15), so it attains a maximum point $x_0 \in \overline{\Omega}_\star$. Hence, by (19)

$$v(x_0) > P. \tag{20}$$

Then $x_0 \in \Omega_\star$, because $v = 0$ on $\partial\Omega_\star$. Accordingly,

$$\nabla v(x_0) = 0 \tag{21}$$

and

$$\Delta v(x_0) \leqslant 0. \tag{22}$$

Now, we compute Δv and evaluate it at x_0. This will lead to the absurd $\Delta v(x_0) > 0$ if one fixes P sufficiently large. We have, using the repeated indexes convention:

$$\Delta v = \psi \Delta w + w \Delta \psi + 2 \nabla w \cdot \nabla \psi. \tag{23}$$

Also,

$$\partial_i w = \frac{2 \partial_j u \, \partial_{ij} u \, Z(u) - |\nabla u|^2 Z'(u) \partial_i u}{Z(u)^2} \tag{24}$$

and

$$\begin{aligned} \Delta w &= \partial_{ii} w \\ &= \frac{2(\partial_{ij} u)^2 Z(u) + 2 \partial_j u \, \partial_j (\Delta u) \, Z(u) - |\nabla u|^4 Z''(u) - |\nabla u|^2 Z'(u) \Delta u}{Z(u)^2} \\ &\qquad - 2 \frac{Z'(u)}{Z(u)} \partial_i u \partial_i w. \end{aligned} \tag{25}$$

On the other hand, using equation (14), we get

$$\partial_j u \partial_j (\Delta u) = (\partial_j u)^2 f'(u),$$

which, plugged into (25), gives

$$\begin{aligned} \Delta w = &\frac{2(\partial_{ij} u)^2 Z(u) + 2 |\nabla u|^2 Z(u) f'(u) - |\nabla u|^4 Z''(u) - |\nabla u|^2 Z'(u) f(u)}{Z(u)^2} \\ &\qquad - 2 \frac{Z'(u)}{Z(u)} \partial_i u \partial_i w. \end{aligned} \tag{26}$$

Henceforth all functions are evaluated at the point x_0. So, relation (21) yields

$$\psi \nabla w + w \nabla \psi = 0$$

and so

$$\nabla w \cdot \nabla \psi = -w \frac{|\nabla \psi|^2}{\psi}.$$

Inserting in (23),

$$\Delta v = \psi \Delta w + w \left(\Delta \psi - 2 \frac{|\nabla \psi|^2}{\psi} \right). \tag{27}$$

Replacing (26) in (27),

$$\begin{aligned} \Delta v = \psi \Bigg[&\frac{2(\partial_{ij} u)^2 Z(u) + 2 |\nabla u|^2 Z(u) f'(u) - |\nabla u|^4 Z''(u) - |\nabla u|^2 Z'(u) f(u)}{Z(u)^2} \\ &- 2 \frac{Z'(u)}{Z(u)} \partial_i u \partial_i w \Bigg] + w \left(\Delta \psi - 2 \frac{|\nabla \psi|^2}{\psi} \right) \end{aligned}$$

which is equivalent to

$$\begin{aligned}\Delta v = \frac{1}{Z(u)} \big[&2\psi(\partial_{ij}u)^2 + 2\psi Z(u)f'(u)w - \psi Z(u)Z''(u)w^2 - \psi w f(u)Z'(u) \\ &- 2\psi Z'(u)\partial_i u \partial_i w\big] + w\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right).\end{aligned} \tag{28}$$

We remark that

$$\nabla u(x_0) \neq 0, \tag{29}$$

otherwise $v(x_0) = 0$, against (20).

Now, we assume, without loss of generality, that $\nabla u(x_0)$ is parallel to the first coordinate axis. Then, we obtain from (24) that

$$\partial_1 w = \frac{2\partial_1 u\, \partial_{11} u\, Z(u) - |\nabla u|^2 Z'(u)\, \partial_1 u}{Z(u)^2}$$

and so, by (29),

$$\begin{aligned}\partial_{11} u &= \frac{|\nabla u|^2 Z'(u)\partial_1 u + \partial_1 w Z(u)^2}{2\partial_1 u\, Z(u)} \\ &= \frac{1}{2}\left[w Z'(u) + \frac{\partial_1 w\, Z(u)}{\partial_1 u}\right].\end{aligned} \tag{30}$$

Also, from (21),

$$0 = \partial_1 v = w\partial_1\psi + \psi\partial_1 w. \tag{31}$$

By plugging (31) into (30), we obtain

$$\partial_{11} u = \frac{1}{2} w \left(Z'(u) - \frac{\partial_1\psi}{\psi\partial_1 u} Z(u)\right).$$

This, combined with (28), furnishes

$$\begin{aligned}\Delta v \geqslant \frac{1}{Z(u)} \Bigg[&\frac{1}{2}\psi w^2 \left(Z'(u)^2 + \frac{(\partial_1\psi)^2}{\psi^2(\partial_1 u)^2} Z(u)^2 - 2Z(u)Z'(u)\frac{\partial_1\psi}{\psi\partial_1 u}\right) \\ &+ 2\psi Z(u) f'(u) w \\ &- \psi Z(u) Z''(u) w^2 - \psi w f(u) Z'(u) - 2\psi Z'(u)\partial_1 u \partial_1 w\Bigg] \\ &+ w\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right).\end{aligned} \tag{32}$$

Now we estimate some terms involving ψ and $\nabla\psi$. By (16), we have

$$|\partial_1 u\, w\, \partial_1\psi| \leqslant |\nabla u|\, w\, |\nabla\psi| = Z(u)^{1/2}\psi^{1/2} w^{3/2} \frac{|\nabla\psi|}{\psi^{1/2}}. \tag{33}$$

Thus, from (31), (33) and (12) we obtain

$$\begin{aligned}2\psi Z'(u)\partial_1 u\partial_1 w &= -2Z'(u)\partial_1 u w\partial_1\psi \\ &\leqslant 2Z'(u)Z(u)^{1/2}\psi^{1/2}w^{3/2}\sup_{\Omega_\star}\frac{|\nabla\psi|}{\psi^{1/2}}. \end{aligned} \tag{34}$$

On the other hand

$$\begin{aligned}\frac{1}{2}w^2\frac{(\partial_1\psi)^2}{\psi(\partial_1 u)^2}Z(u)^2 &= \frac{1}{2}\frac{(\partial_1\psi)^2}{\psi}Z(u)w \\ &\geqslant -\frac{1}{2}\left(\sup_{\Omega_\star}\frac{|\nabla\psi|^2}{\psi}\right)Z(u)w. \end{aligned} \tag{35}$$

We also know that

$$-w^2Z(u)Z'(u)\frac{\partial_1\psi}{\partial_1 u} \geqslant -\left(\sup_{\Omega_\star}\frac{|\nabla\psi|}{\psi^{1/2}}\right)Z'(u)Z(u)^{1/2}\psi^{1/2}w^{3/2}. \tag{36}$$

The last term to estimate is

$$w\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right) \geqslant -w\sup_{\Omega_\star}\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right). \tag{37}$$

Bringing (34)–(37) into (32) we obtain the following expression evaluated at the point x_0:

$$\begin{aligned}\Delta v \geqslant\ & \frac{1}{Z(u)}\left[\frac{1}{2}\psi w^2 Z'(u)^2 - \frac{1}{2}\left(\sup_{\Omega_\star}\frac{|\nabla\psi|^2}{\psi}\right)Z(u)w\right. \\ & - 4\left(\sup_{\Omega_\star}\frac{|\nabla\psi|}{\psi^{1/2}}\right)Z'(u)Z(u)^{1/2}\psi^{1/2}w^{3/2} + 2\psi Z(u)f'(u)w \\ & \left. -\psi Z(u)Z''(u)w^2 - \psi w f(u)Z'(u) - wZ\sup_{\Omega_\star}\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right)\right] \\ \geqslant\ & \frac{1}{Z(u)}\left[\psi w^2\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right)\right. \\ & + w\left(2\psi Z(u)f'(u) - \psi f(u)Z'(u) - KZ(u)\right) \\ & \left. - KZ'(u)Z(u)^{1/2}\psi^{1/2}w^{3/2}\right],\end{aligned}$$

where

$$K := 4\left(\sup_{\Omega_\star}\frac{|\nabla\psi|^2}{\psi} + \sup_{\Omega_\star}\frac{|\nabla\psi|}{\psi^{1/2}} + \sup_{\Omega_\star}|\Delta\psi|\right).$$

As a consequence, by (13),

$$\Delta v \geqslant \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)}\left[\psi w^2 - 2(1+\|\psi\|_{L^\infty(\Omega_\star)} + K)Cw - KC\psi^{1/2}w^{3/2}\right].$$

Therefore, up to renaming constant C, we conclude that

$$\begin{aligned}\Delta v &\geqslant \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)}\left(\psi w^2 - C(w + \psi^{1/2}w^{3/2})\right)\\ &= \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)\psi}\left(v^2 - C(v + v^{3/2})\right).\end{aligned}$$

Thus if (20) holds for some large enough P we obtain

$$\Delta v \geqslant \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)\psi} > 0,$$

which is in contradiction with (22). $\square$

Now, we complete the proof of Theorem 1.1. For this, we check that (12) and (13) are satisfied when $Z(u) := u^{p+1}/(p+1)$ and $f := Z'$, for every $u \in (0, L]$ and every fixed $L > 0$, and then we apply Lemma 2.1.

Indeed: we note that $Z'(u) = u^p$ and so (12) holds true; moreover,

$$\begin{aligned}\frac{1}{2}Z'(u)^2 - Z''(u)Z(u) &= \left(\frac{1-p}{2(p+1)}\right)u^{2p},\\ Z'(u)Z(u)^{1/2} &= \frac{u^{\frac{3p+1}{2}}}{(p+1)^{1/2}}\\ Z''(u)Z(u) &= \frac{p}{p+1}u^{2p}\\ \text{and} \qquad Z'(u)Z'(u) &= u^{2p}.\end{aligned}$$

Since $p \in (-1, 1)$, the above equalities easily imply (13). Furthermore, $u \in C^3(B_{d_0/2})$ by elliptic estimates. As a consequence, Theorem 1.1 follows from Lemma 2.1, by choosing $\Omega_\star := B_{d_0/2}$ and $\psi = d^2$, where $d \in C^2(B_{d_0/2}, (0, +\infty))$ is a function agreeing with $\mathrm{dist}(x, \partial B_{d_0/2})$ in a $(d_0/4)$-neighborhood of $\partial B_{d_0/2}$.

3. Proofs of Theorems 1.2 and 1.3

We start with the following Pohozaev-type identity:

Lemma 3.1. *Let $R > 0$ and $f \in C(\mathbb{R})$. Let F be a primitive of f. Let $v \in C^2(B_R)$ be a solution of $\Delta v = f(v)$ in B_R. Then, for every $r \in (0, R)$,*

$$\begin{aligned}&\int_{B_r}\left((n-2)|\nabla v|^2 + 2nF(v)\right)\\ &= r\int_{\partial B_r}\left[|\nabla v|^2 + 2F(v) - 2\left(\frac{\partial v}{\partial \nu}\right)^2\right].\end{aligned}$$

Proof. The argument in the proof of Lemma 2.2 of page 846 of [11] may be repeated verbatim. $\square$

We remark that we can use Lemma 3.1 in the proof of Theorem 1.2, since in this case $p \in (0,1)$, and so $f(u) := u^p \in C(\mathbb{R})$.

Now, we perform some elementary geometric observations on the level sets of a function that can be written in the form requested in (9) (recall also the notation in (4)):

Lemma 3.2. *Let* $\epsilon'' > 0$, $g : \overline{\mathscr{C}_{\epsilon''}} \to \mathbb{R}$, *with*

$$\lim_{\mathscr{C}_{\epsilon''} \ni x \to 0} \frac{g(x)}{|x|} = 0. \tag{38}$$

Let

$$v(x) := \left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n^+ + g(x) \right)^{2/(1-p)} \qquad \textit{for every } x \in B_{\epsilon''}. \tag{39}$$

Then, there exists $\epsilon_\star \in (0, \epsilon'')$ *such that*

$$\mathscr{C}_{\epsilon_\star} \subseteq \{v > 0\}.$$

Proof. We take $x \in \mathscr{C}_{\epsilon_\star}$ and we compute:

$$\begin{aligned} \sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n^+ + g(x) &\geqslant |x| \left[\sqrt{\frac{1}{2(p+1)}}\,\frac{1-p}{2} - \frac{|g(x)|}{|x|} \right] \\ &\geqslant \sqrt{\frac{1}{2(p+1)}}\,\frac{1-p}{4}, \end{aligned}$$

where we have used (38) in the last step, assuming $\epsilon_\star$ small enough. Then, the desired claim follows from (39). $\square$

Now, we complete the proof of Theorem 1.2. We can prove (6), by using the argument on page 847 of [11], by replacing the estimate of [10], which is not available here, with (5). In fact, this will cause a remainder term that we will take into account when we prove (8) below.

Indeed, using Lemma 3.1, we see that

$$\begin{aligned} \frac{\partial \mathscr{E}}{\partial r}(u,r) &= (1-n)r^{-n} \int_{B_r} \left(|\nabla u|^2 + \frac{2u^{p+1}}{p+1} \right) + r^{1-n} \int_{\partial B_r} \left(|\nabla u|^2 + \frac{2u^{p+1}}{p+1} \right) \\ &= r^{-n} \int_{B_r} \left(\frac{2u^{p+1}}{p+1} - |\nabla u|^2 \right) + 2r^{1-n} \int_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2 . \end{aligned}$$

This and (5) easily gives (6).

Also, (8) easily follows from (6) and (7) by choosing η suitably small with respect to C, and then setting $\epsilon' := \min\{\epsilon(\eta), \tilde{\epsilon}\}$.

Now, we show that (9) implies (7). For this, we recall Lemma 3.2 and we compute

$$\nabla u(x) = \frac{2}{1-p}\left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n + g(x)\right)^{(p+1)/(1-p)} \cdot\left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)e_n + \nabla g(x)\right) \quad \text{for every } x \in \mathscr{C}_{\epsilon''} \text{ with } x_n > 0.$$

As a consequence, we have that if $y \in \partial B_1$ and $y_n \geqslant 1/2$ and $r > 0$ is sufficiently small

$$\begin{aligned}
&\nabla u(ry)\cdot y \\
&= \frac{2}{1-p} r^{(p+1)/(1-p)} \left(\sqrt{\frac{1}{(p+1)}}\,(1-p)y_n + \frac{g(ry)}{r}\right)^{(p+1)/(1-p)} \\
&\qquad \cdot\left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)y_n + \nabla g(ry)\cdot y\right) \\
&\geqslant c_1 r^{(p+1)/(1-p)},
\end{aligned}$$

for a suitable $c_1 > 0$, thanks to the behavior of g near 0 given in (9).

Consequently,

$$\begin{aligned}
\fint_{\partial B_r}\left(\frac{\partial u}{\partial \nu}\right)^2 &= \fint_{y\in\partial B_1} (\nabla u(ry)\cdot y)^2 \\
&\geqslant \fint_{y\in\partial B_1\cap\{y_n\geqslant 1/2\}} (\nabla u(ry)\cdot y)^2 \geqslant c_1 r^{2(p+1)/(1-p)},
\end{aligned} \tag{40}$$

for a suitable $c_1 > 0$. On the other hand,

$$\begin{aligned}
\fint_{B_r} u^{p+1} &= \fint_{y\in B_1} (u(ry))^{p+1} \\
&= r^{2(p+1)/(1-p)} \fint_{y\in B_1}\left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)y_n^+ + \frac{g(ry)}{r}\right)^{2(p+1)/(1-p)} \\
&\leqslant c_2 r^{2(p+1)/(1-p)},
\end{aligned} \tag{41}$$

for a suitable $c_2 > 0$. Then, (7) is a consequence of (40) and (41). This ends the proof of Theorem 1.2.

Now, we deal with the proof of Theorem 1.3. First of all, we note that (5) holds true under the assumptions of Theorem 1.3, thanks to Lemma 1.2 on pages 1420–1421 of [13].

Furthermore, we show that (9) is satisfied. This will imply the result of Theorem 1.3, thanks to Theorem 1.2.

For this scope, we recall the notion of flat free boundary points, as described in Definition 5.1 on page 82 of [1] (in fact, with respect to the notation of [1], we reverse the direction of x_n). Given $\rho > 0$, a, $b \in [0,1]$ and $\tau \in [0,+\infty]$, we say that u belongs to the class $F(a,b;\tau)$ in B_ρ if u is a solution of (1) in B_ρ, with $0 \in \partial\{u > 0\}$, such that

- $u(x) = 0$ for every $x = (x', x_n) \in B_\rho$ with $x_n \leqslant -a\rho$,
- $\sqrt{2(p+1)}\,\dfrac{1}{1-p}(u(x))^{(1-p)/2} \geqslant x_n + b\rho$ for every $x = (x', x_n) \in B_\rho$ with $x_n \geqslant b\rho$,
- $|\nabla(x)|^2 \leqslant \dfrac{2(1+\tau)}{p+1}(u(x))^{p+1}$ for every $x \in B_\rho$.

With this notation, we observe that (10) implies that

$$u \text{ belongs to the class } F(a,1;+\infty) \text{ in } B_\rho, \tag{42}$$

Accordingly, (42) and Theorem 6.1 on page 101 of [1] imply that

$$B_{\rho/8} \cap \partial\{u > 0\} \text{ is a } C^1 \text{ graph} \tag{43}$$

and

$$u^{(1-p)/2} \in C^1(\overline{\{u > 0\}} \cap B_{\rho/8}). \tag{44}$$

Indeed, the quantities ℓ, a_o and c_o in the statement of Theorem 1.3 are determined here by the use of Theorem 6.1 of [1].

Then, (43) makes possible to use Lemma 4.5 of [1] from which we conclude that there exists $\rho' \in (0, \rho/8)$ and $g : B_{\rho'} \to \mathbb{R}$ such that, for every $x \in B_{\rho'}$,

$$u(x) = \left(\sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n^+ + g(x)\right)^{2/(1-p)}, \tag{45}$$

with

$$\lim_{x\to 0} \frac{g(x)}{|x|} = 0. \tag{46}$$

Also,

$$x_n^+ \text{ belongs to } C^1(\overline{\{x_n > 0\}} \cap B_{\rho'}), \tag{47}$$

and Lemma 3.2 implies that

$$\mathscr{C}_r \subseteq \{x_n > 0\} \cap \{u > 0\}, \tag{48}$$

if $r > 0$ is sufficiently small. From (44), (45), (47) and (48), we obtain that the function

$$g(x) = (u(x))^{(1-p)/2} - \sqrt{\frac{1}{2(p+1)}}\,(1-p)x_n^+ \text{ belongs to } C^1(\overline{\mathscr{C}_r}). \tag{49}$$

Accordingly, for every $y \in \mathscr{D}$, we may define

$$\ell(y) := \lim_{s \to 0^+} \nabla g(sy) \cdot y.$$

Then, from (46), we have that $g(0) = 0$ and that, for every $y \in \mathscr{D}$ (see definition (4))

$$\begin{aligned} 0 = \lim_{s \to 0^+} \frac{g(sy) - g(0)}{s} &= \lim_{s \to 0^+} \int_0^1 \nabla g(\theta s y) \cdot y \, d\theta \\ = \int_0^1 \ell(y) \, dt &= \ell(y). \end{aligned}$$

This says that

$$\lim_{s \to 0^+} \nabla g(sy) \cdot y = 0. \tag{50}$$

Thus, in order to check that (9) holds true, we need to show that the limit in (50) is uniform for $y \in \mathscr{D}$. For this, we fix $\epsilon > 0$ and, by the compactness of $\mathscr{D}$, we take $y_1, \ldots, y_m \in \mathscr{D}$ in such a way that for every $y \in \mathscr{D}$ there exists $i(y) \in \{1, \ldots, m\}$ such that

$$|y - y_{i(y)}| \leqslant \epsilon. \tag{51}$$

Then, we use (50) to say that, for every $i \in \{1, \ldots, m\}$, there exists $s_i > 0$ such that if $s \in (0, s_i)$ then

$$|\nabla g(sy_i) \cdot y_i| \leqslant \epsilon. \tag{52}$$

Moreover, we know from (49) that ∇g is uniformly continuous on $\overline{\mathscr{C}_r}$. Thus, since $|sy - sy_{i(y)}| \leqslant |sy| + |sy_{i(y)}| \leqslant 2s$, we have that there exists $s_o > 0$ such that, if $s \in (0, s_o)$, then

$$|\nabla g(sy) - \nabla g(sy_{i(y)})| \leqslant \epsilon. \tag{53}$$

So, we define $s_\star := \min\{s_o, s_1, \ldots, s_m\} > 0$. In this way, if $s \in (0, s_\star)$,

$$\begin{aligned} &|\nabla g(sy) \cdot y| \\ &= |\nabla g(sy) \cdot y - \nabla g(sy_{i(y)}) \cdot y + \nabla g(sy_{i(y)}) \cdot (y - y_{i(y)}) + \nabla g(sy_{i(y)}) \cdot y_{i(y)}| \\ &\leqslant |(\nabla g(sy) - \nabla g(sy_{i(y)})) \cdot y| + |\nabla g(sy_{i(y)})| \, |y - y_{i(y)}| + |\nabla g(sy_{i(y)}) \cdot y_{i(y)}| \\ &\leqslant \epsilon + \sup_{\overline{\mathscr{C}_r}} |\nabla g| \, \epsilon + \epsilon \end{aligned}$$

for every $y \in \mathscr{D}$, thanks to (51), (52) and (53). This says that the limit in (50) holds uniformly in $y \in \mathscr{D}$, hence (9) is satisfied, and so Theorem 1.3 is a consequence of Theorem 1.2.

A. The optimal constant in (3) and some of its consequences

We would like to recall that, in many cases of interest, the constant M in (3) may be determined explicitly in a sharp way, leading to interesting consequences. For instance, the following result holds:

Theorem A.1. *Let $p \in (0,1)$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of (1). Suppose that $0 \in \Omega \cap \partial\{u > 0\}$.*

Let $R > 0$ be such that $B_R \subseteq \Omega$ and suppose that (3) holds for all $x \in B_R$.

Then,

$$\text{for every } \eta > 0 \text{ there exists } \epsilon(\eta) \in (0,R) \text{ such that}$$
$$|\nabla u(x)|^2 \leqslant \left(\frac{2}{p+1} + \eta\right)(u(x))^{p+1} \quad \text{for every } x \in B_{\epsilon(\eta)}.$$

We remark that the constant $2/(p+1)$ in (5) is optimal, since it is attained by the onedimensional example in (2).

The proof of Theorem A.1 makes use of many ideas developed by [13] in the context of minimizers. A similar result was also obtained in [1], see Lemma 2.1 and Remark 1.10 there. We think it is useful to provide in this appendix a self-contained proof, also to stress the relation with the estimate in (5).

Below, we develop some tools towards the proof of Theorem A.1, which will be completed at the end of this appendix.

A.1. Growth from the free boundary

It is interesting to point out that when (3) holds near the free boundary, one obtains, as a consequence, an optimal bound on the growth from the free boundary itself, as pointed out by the following observation (see also page 67 of [1] and page 1060 of [12] for related results):

Lemma A.2. *Let $q > -1$ and $R, C > 0$. Let $v \in C(B_R, [0,+\infty)) \cap C^1(B_R \cap \{u > 0\})$. Suppose that*

$$v(0) = 0 \tag{54}$$

and

$$|\nabla v(x)|^2 \leqslant C(v(x))^{q+1} \quad \text{for every } x \in B_R \cap \{v > 0\}.$$

Then, for every $x \in B_{R/2}$,

$$|v(x)| \leqslant C_o|x|^{2/(1-q)}, \tag{55}$$

for a suitable $C_o > 0$ depending only on C and q.

Proof. Fix $x_o \in B_{R/2}$: we prove (55) for such x_o. If $v(x_o) = 0$ we are done, so we suppose $v(x_o) > 0$. Notice that

$$B_\rho(x_o) \subseteq B_R \quad \text{for every } \rho \in [0, R/2]. \tag{56}$$

By (54) and (56) there exists $d \in (0, |x_o|]$ such that $B_d(x_o) \subseteq B_R \cap \{v > 0\}$ and there exists $p \in B_R \cap \partial B_d$ such that $v(p) = 0$. For every $x \in B_R$, let $w(x) := (v(x))^{(1-q)/2}$. Then, if $x \in B_d(x_o)$,

$$|\nabla w(x)| = \frac{1-q}{2}|v(x)|^{-(q+1)/2}|\nabla v(x)| \leqslant \frac{(1-q)\sqrt{C}}{2}.$$

Then

$$(v(x_o))^{2/(1-q)} = w(x_o) = w(x_o) - w(p) \leqslant \frac{(1-q)\sqrt{C}}{2}\,|x_o - p|$$
$$\leqslant \frac{(1-q)\sqrt{C}}{2}\,d \leqslant \frac{(1-q)\sqrt{C}}{2}\,|x_o|,$$

which implies the desired result. □

A.2. Algebraic computations

Lemma A.3. *Let $q \in (-1,1)$, $\ell > 0$. Let $U \subseteq \mathbb{R}^n$ be open, $v \in C^3(U, [0,+\infty))$ and*

$$P(x) := |\nabla v(x)|^2 - \ell(v(x))^{q+1}.$$

Then

$$\Delta P = 2\sum_{i,j=1}^{n}(\partial_{ij}v)^2 + 2(\nabla(\Delta v))\cdot\nabla v - \ell(1-q)v^q\Delta v - \ell(q+1)qv^{q-1}|\nabla v|^2. \quad (57)$$

Moreover, if we assume that

$$\Delta v = v^q \quad \textit{in } U, \quad (58)$$

that

$$x_o \in U \quad \textit{is a critical point of } P \quad (59)$$

and that

$$v(x_o) > 0 \quad \textit{and} \quad P(x_o) = 0, \quad (60)$$

then

$$\Delta P(x_o) \geqslant \left[\frac{\ell(q+1)}{2} - 1\right]\ell\,(1-q)\,(v(x_o))^{2q}. \quad (61)$$

Proof. By a direct computation, or making use of the Bochner-Weitzenböck formula (see, for instance, [2]), we have

$$\Delta|\nabla v|^2 = 2\sum_{i,j=1}^{n}(\partial_{ij}v)^2 + 2(\nabla(\Delta v))\cdot\nabla v.$$

This easily implies (57).

Now, we assume (58), (59) and (60), and we prove (61). For this, we can choose a coordinate frame in which

$$D^2v(x_o) \text{ is diagonal}, \quad (62)$$

so we plug (58) into (57) and we use (60) to obtain

$$\begin{aligned}
&\Delta P(x_o)\\
&= 2\sum_{i=1}^{n}(\partial_{ii}v(x_o))^2 - \ell(q+1)(v(x_o))^{2q} + (2-\ell(q+1))q(v(x_o))^{q-1}|\nabla v(x_o)|^2\\
&= 2\sum_{i=1}^{n}(\partial_{ii}v(x_o))^2 - \ell(q+1)(v(x_o))^{2q} - 2q\ell\left[\frac{\ell(q+1)}{2} - 1\right](v(x_o))^{2q}.
\end{aligned} \quad (63)$$

Also, we deduce from (59) and (62) that, for every $1 \leqslant i \leqslant n$,

$$0 = \partial_i P(x_o) = (2\partial_{ii} v(x_o) - \ell(q+1)(v(x_o))^q)\, \partial_i v(x_o). \tag{64}$$

Since, from (60), we have that

$$|\nabla v(x_o)| > 0, \tag{65}$$

it follows that there exists $i_\star \in \{1, \dots, n\}$ for which

$$\partial_{i_\star} v(x_o) \neq 0 \tag{66}$$

and so (64) says that

$$\partial_{i_\star i_\star} v(x_o) = \frac{\ell(q+1)}{2} (v(x_o))^q. \tag{67}$$

By plugging this into (63), we obtain

$$\begin{aligned}
\Delta P(x_o) &\geqslant \left\{ 2\,\frac{\ell^2(q+1)^2}{4} - \ell(q+1) - 2q\ell \left[\frac{\ell(q+1)}{2} - 1 \right] \right\} (v(x_o))^{2q} \\
&= \left\{ \ell(q+1) \left[\frac{\ell(q+1)}{2} - 1 \right] - 2q\ell \left[\frac{\ell(q+1)}{2} - 1 \right] \right\} (v(x_o))^{2q} \\
&= \left\{ \ell(q+1-2q) \left[\frac{\ell(q+1)}{2} - 1 \right] \right\} (v(x_o))^{2q},
\end{aligned}$$

that is (61). □

A.3. Some remarks on spherical averages

We collect here some general results on spherical averages. The arguments are mostly taken from [13], but we give the details for the reader's convenience.

Lemma A.4. *Let $q \in [0,1]$, and $v \in C(B_2, [0, +\infty)) \cap C^2(B_2 \cap \{u > 0\})$ be such that $\Delta v = v^q$ in $B_2 \cap \{v > 0\}$. Then, there exist $C > 1 > c > 0$, depending only on n, such that if*

$$\fint_{\partial B_1} v \geqslant C \tag{68}$$

then

$$v(0) \geqslant c \fint_{\partial B_1} v. \tag{69}$$

Moreover, there exists $C_\star > C$ such that if (68) holds with $C_\star$ instead of C, then, for every $x \in B_{1/4}$ and every $\sigma \in (0, 1/4)$

$$c_\star \fint_{\partial B_\sigma} v \leqslant v(x) \leqslant \fint_{\partial B_\sigma} v \tag{70}$$

for a suitable $c_\star \in (0,1)$.

Proof. Given a ball $B_r(P) \subset B_2$, we define $h_{B_r(P)}$ to be the harmonic function in $B_r(P)$ such that $h_{B_r(P)}(x) = v(x)$ for every $x \in \partial B_r(P)$. We observe that, since v is subharmonic, $h_{B_r(P)}(y) \geqslant v(y) \geqslant 0$ for every $y \in B_r(P)$. Accordingly, for every $r \in (0,1)$,

$$\fint_{\partial B_r} v \leqslant \fint_{\partial B_r} h_{B_1} = h_{B_1}(0) = \fint_{\partial B_1} h_{B_1} = \fint_{\partial B_1} v. \tag{71}$$

Also, for every $x \in \overline{B_{r/2}(P)}$ we have that $B_{r/4}(P) \subseteq B_r(x)$. As a consequence,

$$\begin{aligned} &\text{if } B_{2r}(x) \subseteq B_2 \text{ and } |x-P| \leqslant r/2, \text{ then} \\ &v(P) \leqslant \fint_{B_{r/4}(P)} v(y)\,dy \leqslant \frac{|B_r|}{|B_{r/4}|} \fint_{B_r(x)} h_{B_r(x)} \\ &\qquad = \frac{|B_1|}{|B_{1/4}|} h_{B_r(x)}(x) = c_1 \fint_{\partial B_r(x)} h_{B_r(x)} = c_1 \fint_{\partial B_r(x)} v, \end{aligned} \tag{72}$$

for a suitable $c_1 > 0$, depending only on n.

Moreover, we observe that

$$\fint_{\partial B_\rho(q)} v^q \leqslant \left[\fint_{\partial B_\rho(q)} v \right]^q, \tag{73}$$

as long as $B_\rho(q) \subset B_2$. Indeed, if $q \in \{0,1\}$ this is obvious, while, if $q \in (0,1)$, Jensen's inequality and the convexity of the function $\Phi(r) := |r|^{1/q}$ gives that

$$\fint_{\partial B_\rho(q)} v^q = \left[\Phi\left(\fint_{\partial B_\rho(q)} v^q \right) \right]^q \leqslant \left[\fint_{\partial B_\rho(q)} \Phi(v^q) \right]^q = \left[\fint_{\partial B_\rho(q)} v \right]^q,$$

as desired.

Now, for every $x \in \mathbb{R}^n \setminus \{0\}$, we consider the fundamental solution

$$G(x) := \begin{cases} -\log|x| & \text{if } n = 2, \\ |x|^{2-n} - 1 & \text{if } n \geqslant 3, \end{cases}$$

and we observe that $G \geqslant 0$ in B_1, $G = 0$ on ∂B_1, $\Delta G = -c_2 \delta_0$, for some $c_2 > 0$, and $\nabla G(x) \cdot x < 0$ for every $x \in \partial B_1$.

Accordingly, Green's identity gives, for some $c_3 > 0$,

$$-\int_{B_1} \Delta v G = c_2 v(0) + \int_{\partial B_1} \frac{\partial G}{\partial \nu} v = c_2 v(0) - c_3 \fint_{\partial B_1} v. \tag{74}$$

On the other hand, since $|x|^{n-1} G(x)$ is bounded in B_1, we obtain, by exploiting polar coordinates, and recalling (73) and (71),

$$\int_{B_1} \Delta v G = \int_{B_1} v^q G = \int_0^1 \int_{\partial B_\rho} v^q G \leqslant c_4 \int_0^1 \fint_{\partial B_\rho} v^q \rho^{n-1} G$$
$$\leqslant c_5 \int_0^1 \fint_{\partial B_\rho} v^q \leqslant c_5 \int_0^1 \left[\fint_{\partial B_\rho} v \right]^q \leqslant c_5 \left[\fint_{\partial B_1} v \right]^q,$$

for suitable c_4, $c_5 > 0$. By inserting this estimate into (74), we conclude that

$$c_2 v(0) \geqslant c_3 \fint_{\partial B_1} v - c_5 \left[\fint_{\partial B_1} v \right]^q.$$

Since $q < 1$, we obtain that (69) holds under assumption (68).

Now, we prove (70). Given $x \in B_{1/4}$, we use (72) with $r := 1/2$ to see that

$$\sup_{P \in B_{1/2}(x)} v(P) \leqslant c_1 \fint_{\partial B_{1/2}(x)} v. \tag{75}$$

Thus, we exploit (68) with $C_\star$ instead of C, (69) and (75) to estimate

$$c\, C_\star \leqslant c \fint_{\partial B_1} v \leqslant v(0) \leqslant c_1 \fint_{\partial B_{1/2}(x)} v.$$

That is, the function $\tilde{v}(y) := 4^{1/(1-q)} v(x + (y/2))$ satisfies $\Delta \tilde{v} = \tilde{v}^q$ in B_2 and

$$\fint_{\partial B_1} \tilde{v} \geqslant C,$$

as long as $C_\star$ is suitably large. As a consequence, (69) applied to $\tilde{v}$ yields that

$$v(x) \geqslant c_6 \fint_{\partial B_{1/2}(x)} v, \tag{76}$$

for a suitable $c_6 > 0$.

Now, if $\sigma \in (0, 1/4)$, the fact that $\overline{B_\sigma} \subseteq B_{1/2}(x)$ and (75) give that

$$\fint_{\partial B_\sigma} v \leqslant c_1 \fint_{\partial B_{1/2}(x)} v,$$

which, combined with (76), furnishes

$$v(x) \geqslant \frac{c_6}{c_1} \fint_{\partial B_\sigma} v.$$

This and the fact that v is subharmonic imply (70). $\square$

A.4. Improved gradient estimates

Lemma A.5. *Let $q \in (-1,1)$, $M > 0$, $u \in C(B_R, [0,+\infty)) \cap C^2(B_R \cap \{u > 0\})$ be such that $\Delta u = u^q$ in $B_R \cap \{u > 0\}$. Assume that $0 \in \partial\{u > 0\}$ and that*

$$|\nabla u(x)|^2 \leqslant M(u(x))^{q+1} \quad \text{for every } x \in B_R \cap \{u > 0\}. \tag{77}$$

Then, for every $\eta > 0$ there exists $\epsilon(\eta) \in (0,R)$ such that

$$|\nabla u(x)|^2 \leqslant \left(\frac{2}{q+1} + \eta\right) (u(x))^{q+1} \quad \text{for every } x \in B_{\epsilon(\eta)}. \tag{78}$$

Proof. We will make use of a technique developed in [13] (see, in particular, pages 1420–1423 there). For all $x \in B_R$, we let

$$R(x) := \frac{|\nabla u(x)|^2}{(u(x))^{q+1}}$$

and, for every $\epsilon \in (0,R)$, we consider

$$\Psi(\epsilon) := \sup_{x \in B_\epsilon \cap \{u>0\}} R(x).$$

Clearly, Ψ is monotone, and bounded by M thanks to (77), so we can define

$$\ell := \lim_{\mathbb{N} \ni m \to +\infty} \Psi(1/m).$$

Let $x_m \in B_{1/m} \cap \{u > 0\}$ be such that

$$\Psi(1/m) \leqslant R(x_m) + \frac{1}{m}.$$

We fix $K > 1$, to be conveniently chosen in what follows, and we define

$$\rho_m := \left(\frac{u(x_m)}{K}\right)^{(1-q)/2}.$$

Notice that, if K is large enough,

$$\rho_m \leqslant \frac{1}{10m},$$

thanks to Lemma A.2. Consequently,

$$\text{for all } y \in B_2, \text{ we have } x_m + \rho_m y \in B_{2/m}, \tag{79}$$

and we may define

$$v_m(y) := \frac{u(x_m + \rho_m y)}{\rho_m^{2/(1-q)}}.$$

We observe that $v_m \geqslant 0$, that

$$v_m(0) = K, \tag{80}$$

that

$$\Delta v_m = v_m^q \geqslant 0 \text{ in } B_2 \cap \{v_m > 0\} \tag{81}$$

and that

$$\frac{|\nabla v_m(y)|^2}{(v_m(y))^{q+1}} = \frac{|\nabla u(x_m + \rho_m y)|^2}{(u(x_m + \rho_m y))^{q+1}} = R(x_m + \rho_m y). \tag{82}$$

This and (79) give that

$$\sup_{y \in B_2 \cap \{v_m > 0\}} \frac{|\nabla v_m(y)|^2}{(v_m(y))^{q+1}} \leqslant \Psi(2/m). \tag{83}$$

Consequently, if $\zeta_m := (v_m)^{(1-q)/2}$, we have that $|\nabla \zeta_m| \leqslant (1-q)\sqrt{M}/2$ and so $|\zeta_m(y)| \leqslant |\zeta_m(0)| + (1-q)\sqrt{M}$ for all $y \in B_2$. Therefore, by (80),

$$\sup_{y \in B_2} v(y) \leqslant \tilde{C}, \tag{84}$$

for a suitable $\tilde{C}$ depending on M and K. Moreover, by (80) and the subharmonicity of v_m,

$$\fint_{\partial B_1} v_m \geqslant v_m(0) = K \tag{85}$$

and so, if K is chosen suitably large, we can apply Lemma A.4 to v_m: thus, recalling also (84), we obtain that there exists $\tilde{c} \in (0,1)$ such that

$$\tilde{c} < v_m(y) < \frac{1}{\tilde{c}}$$

for all $y \in B_{1/2}$. We remark that $\tilde{c}$ depends on K, but K will be fixed now, once and for all.

Since the standard elliptic Calderón–Zygmund and Schauder estimates apply to (81), up to subsequence, v_m converges to some v in $C^1(B_{1/2})$, with

$$\tilde{c} < v(y) < \frac{1}{\tilde{c}} \text{ for all } x \in B_{1/2}.$$

As a result, if $y \in B_{1/2}$,

$$\frac{|\nabla v(y)|^2}{(v(y))^{q+1}} = \lim_{m \to +\infty} \frac{|\nabla v_m(y)|^2}{(v_m(y))^{q+1}} \leqslant \lim_{m \to +\infty} \Psi(2/m) = \ell,$$

thanks to (83), while

$$\frac{|\nabla v(0)|^2}{(v(0))^{q+1}} = \lim_{m \to +\infty} \frac{|\nabla v_m(0)|^2}{(v_m(0))^{q+1}} = \lim_{m \to +\infty} R(x_m) = \ell,$$

thanks to (82).

That is, the function $P : B_{1/2} \to \mathbb{R}$ defined by

$$P(y) := |\nabla v(y)|^2 - \ell(v(y))^{q+1}$$

has a maximum at $y = 0$. Therefore, noticing that $v(0) = K$, due to (80), we are in the position of applying Lemma A.3: we obtain from (61) that

$$0 \geqslant \Delta P(0) \geqslant \left[\frac{\ell(q+1)}{2} - 1\right] \ell\,(1-q)\,(v(0))^{2q}.$$

This gives that $\ell \leqslant 2/(q+1)$, from which the desired result plainly follows. $\square$

The proof of Theorem A.1 now follows easily from Lemma A.5.

Acknowledgements. Part of this work was delivered while Enrico Valdinoci was visiting the Universidade Estadual de Campinas, whose warm hospitality and pleasant atmosphere was greatly appreciated.

References

[1] H. W. Alt, D. Phillips: A free boundary problem for semilinear elliptic equations, J. Reine Angew. Math. 368 (1986) 63–107.

[2] M. Berger, P. Gauduchon, E. Mazet: Le Spectre d'une Variété Riemannienne, Lecture Notes in Mathematics 194, Springer, Berlin (1971).

[3] C. Bandle, R. P. Sperb, I. Stakgold: Diffusion and reaction with monotone kinetics, Nonlinear Anal. 8(4) (1984) 321–333.

[4] J. Dávila, M. Montenegro: Existence and asymptotic behavior for a singular parabolic equation, Trans. Amer. Math. Soc. 357(5) (2005) 1801–1828.

[5] J. Dávila, A. C. Ponce: Hausdorff dimension of ruptures sets and removable singularities, C. R., Math., Acad. Sci. Paris 346(1–2) (2008) 27–32.

[6] A. Friedman, D. Phillips: The free boundary of a semilinear elliptic equation, Trans. Amer. Math. Soc. 282(1) (1984) 153–182.

[7] A. Farina, E. Valdinoci: A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature, Adv. Math. 225(5) (2010) 2808–2827.

[8] Z. Guo, J. Wei: Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity, Manuscr. Math. 120(2) (2006) 193–209.

[9] E. Lindgren, A. Petrosyan: Regularity of the free boundary in a two-phase semilinear problem in two dimensions, Indiana Univ. Math. J. 57(7) (2008) 3397–3417.

[10] L. Modica: A gradient bound and a Liouville theorem for nonlinear Poisson equations, Commun. Pure Appl. Math. 38(5) (1985) 679–684:

[11] L. Modica: Monotonicity of the energy for entire solutions of semilinear elliptic equations, in: Partial Differential Equations and the Calculus of Variations, Vol. II, F. Colombini et al. (eds.), Progr. Nonlinear Differential Equations Appl. 2, Birkhäuser, Boston (1989) 843–850.

[12] Li Ma, J. C. Wei: Properties of positive solutions to an elliptic equation with negative exponent, J. Funct. Anal. 254(4) (2008) 1058–1087.

[13] D. Phillips: Hausdorff measure estimates of a free boundary for a minimum problem, Commun. Partial Differ. Equations 8(13) (1983) 1409–1454.

[14] D. Phillips: A minimization problem and the regularity of solutions in the presence of a free boundary, Indiana Univ. Math. J. 32(1) (1983) 1–17.

[15] J. L. Vázquez: A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12(3) (1984) 191–202.

[16] G. S. Weiss: Partial regularity for weak solutions of an elliptic free boundary problem, Commun. Partial Differ. Equations 23(3–4) (1998) 439–455.