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Characterizations of Pointwise Additivity of Subdifferential

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In this paper we prove that the additivity of subdifferential in a given point of a locally convex space X is equivalent to other important optimality properties of an associated family of optimization problems. As a consequence, the subdifferential additivity is characterized by a dual closedness condition in $X^* \times \mathbb{R}$ endowed with the weak-star topology. Also, some special cases in which this closedness condition can be given in X^* are presented.

Keywords: Lower-semicontinuous function, conjugate function, subdifferential, additivity of subdifferential, convolution, normal cone

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1. Introduction

Let $f : X \to \overline{\mathbb{R}}$ be an extended real valued proper convex function on a locally convex space X and its *subdifferential* defined by

$$\partial f(x) = \{x^* \in X^*; f(x) \leq f(u) + x^*(x-u), \text{ for all } u \in X\}, \quad x \in X, \tag{1}$$

where X^* is the dual space of X.

We recall that the function f is said to be *proper* if $f(x) > -\infty$ for every $x \in X$ and there exists $\overline{x} \in X$ such that $f(\overline{x}) \in \mathbb{R}$.

We denote by $D(f)$ the *domain* of the function f, that is

$$D(f) = \{x \in X; f(x) < \infty\}. \tag{2}$$

Obviously, $\partial f(x) = \emptyset$ for all $x \notin D(f)$.

Given a closed convex set $C \subset X$ we denote by I_C its *indicator function* defined by $I_C(x) = 0$ if $x \in X$ and $I_C(x) = \infty$ if $x \notin C$. It is well known that

$$\partial I_C(x) = N_C(x), \quad x \in C, \tag{3}$$

where $N_C(x)$ is the *normal cone* of C at x.

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If f_1, f_2 are two proper convex functions on X, then

$$\partial f_1(x) + \partial f_2(x) \subset \partial(f_1 + f_2)(x), \quad x \in X.$$

Generally, this inclusion is strict, but adding some *regularity conditions* we can obtain just an equality, that is, the *additivity property of subdifferential*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x), \quad x \in X, \tag{4}$$

holds. Some additivity formulas without any regularity condition were established by Attouch and Théra [1], Hirriart-Urruty and Phelps [8], Jules and Lassonde [11], Revalski and Théra [17], Thiboult [21], by different extended forms of the sum in the second hand of (4). This additivity properties have a crucial role in convex analysis, specially in convex optimization. A well-known simple and remarkable regularity condition for the additivity property (4) is the following: *the functions f_1, f_2 are proper convex and finit at a point in which at least one is continuous* [2, 12, 20]. In the case of Banach space (generally, in a barreled space), if f_1, f_2 are also lower-semicontinuous, this condition becomes an *interiority condition*, namely

$$(\operatorname{int} D(f_1)) \cap D(f_2) \neq \emptyset \quad \text{or} \quad (\operatorname{int} D(f_2)) \cap D(f_2) \neq \emptyset, \tag{5}$$

since any lower-semicontinuous function is continuous on the interior of its effective domain (see, for example, [2, 9, 12, 20]). The literature is quite rich in such regularity conditions (see, for example, [1, 3, 4, 5, 6, 13, 17, 18]). Some efficient conditions are obtained using criteria for maximality of the sum of two (maximal) monotone operators since it is well known that the multivalued subdifferential operator is a maximal monotone operator ([1, 17, 19]). Burachik and Jeyacumar has pointed out in [6] the following regularity condition: $\operatorname{epi} f_1^* + \operatorname{epi} f_2^*$ is *weak-star closed in* $X^* \times \mathbb{R}$ (see also [3, 5]). In fact, their result can be also obtained from an earlier result established in [13] (see Theorem 3.3). Moreover, this dual closedness condition was used by Dieter [7] in the study of Fenchel duality results in the theory of optimization and the additivity property of subdifferential is a consequence of Fenchel duality theorem ([2, 20]). The all regularity conditions for additivity property of subdifferential known in the literature assure that the equality (4) holds for all $x \in X$. But, generally, it is possible that the equality (4) be true only for certain elements $x \in X$.

The aim of the present paper is to study the additivity property in a given element $x \in X$. Firstly, we establish the characterization of the equality (4) by other important properties in convex analysis: the optimality of some optimization problems, the equality in Fenchel duality theorem, the stability of some optimization problems and the exactness of certain convolutions (Theorem 3.1). The central idea is to associate a family of convex optimization problems such that their optimality is equivalent to equality (4). Firstly, we establish some results concerning the optimality and lower semi-continuity of optimal value function only on a given set in the parameter space. Consequently, we are in the position to use some similar results to those obtained in [14, 15, 16] to give a characterization of the equality (4) by a closedness condition in $X^* \times \mathbb{R}$ (Theorem 3.3). In particular, we obtain a characterization of additivity property of subdifferential on whole space X (Theorem 3.4). Finally, the special cases of normal cones and Minkowski functionals are presented. Thus, we improve some similar results established in the papers [5], [6], [10].

2. Preliminaries

Let X be a separated locally convex space and X^* its dual space. We denote by $\Gamma_0(X)$ the *set of all proper convex lower-semicontinuous functions* on X. Also, let us denote

$$S_f(\lambda) = \{x \in X; f(x) \leq \lambda\}, \quad \lambda \in \mathbb{R}, \tag{6}$$

called the *level sets* of f and

$$\operatorname{epi} f = \{(x,t) \in X \times \mathbb{R}; f(x) \leq t\}, \tag{7}$$

called the *epigraph* of f.

It is well known that f is lower-semicontinuous on X if and only if $\operatorname{epi} f$ or its level sets are closed in $X \times \mathbb{R}$, X respectively. Moreover, the function f is lower-semicontinous on a set $A \subset X$ if and only if

$$\overline{(\operatorname{epi} f)} \cap (A \times \mathbb{R}) = (\operatorname{epi} f) \cap (A \times \mathbb{R}). \tag{8}$$

If we consider the conjugate f^* of a function f defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x); x \in X\}, \quad x^* \in X^*, \tag{9}$$

then $x^* \in \partial f(x)$ if and only if

$$f(x) + f^*(x^*) = x^*(x). \tag{10}$$

Also, we recall the following consequences of bipolar theorem:

$$p_A^* = I_{A^0}, \tag{11}$$

$$I_A^* = p_{A^0}, \tag{12}$$

where A is a nonvoid convex set containing the origin, ([2,12]). Here we denoted

$$A^\circ = \{x^* \in X^*; x^*(x) \leq 1 \text{ for all } x \in A\}, \tag{13}$$

the *polar* of the set A,

$$p_C(x) = \inf\left\{\lambda; \frac{1}{\lambda}x \in C,\, \lambda \in (0,\infty]\right\}, \quad x \in X, \tag{14}$$

the *Minkowski functional* of a set C containing the origin. Generally, for any nonvoid convex set we have

$$I_A^* = \sigma_A, \tag{15}$$

where σ_A is the *support functional* of the set A, that is,

$$\sigma_A(x^*) = \sup_{x \in A} x^*(x). \tag{16}$$

Let $F : U \times Y \to \overline{\mathbb{R}}$ be a given function, where U, Y are two separated locally convex spaces. We associate the following family of optimization problems

$$(\mathcal{P}_y) \qquad \min\{F(u, y); u \in U\}, \quad y \in Y.$$

Denote by h its *optimal value function*

$$h(y) = \inf\{F(u, y); u \in U\}, \quad y \in Y. \tag{17}$$

If $h(y) = \infty$, then every $u \in U$ can be considered as optimal solution of $\mathcal{P}_y$.

Now, let us consider the following associated set

$$H = \{(y, t) \in Y \times \mathbb{R};\ \text{there exists } u \in U \text{ such that } F(u, y) \leq t\}. \tag{18}$$

It is easy to prove that

$$H \subset \operatorname{epi} h \subset \overline{H}. \tag{19}$$

Thus, if H is closed, then h is lower-semicontinuous. On the other hand, given a set $A \subset Y$ it follows that $\mathcal{P}_y$ has optimal solutions for every $y \in A$, whenever $h(y) > -\infty$, if and only if

$$H \cap (A \times \mathbb{R}) = (\operatorname{epi} h) \times (A \times \mathbb{R}). \tag{20}$$

Consequently, we obtain an easy extension of a result of [15].

Lemma 2.1. *The problems $\mathcal{P}_y$ have optimal solutions for every $y \in A$, whenever $h(y) > -\infty$, and h is lower-semicontinuous on A if and only if*

$$\overline{H} \cap (A \times \mathbb{R}) = H \cap (A \times \mathbb{R}). \tag{21}$$

Proof. Suppose that the equality (21) is fulfilled. Then by (19) it follows (8), that is, h is lower-semcontinuous on A. Moreover, the equality (20) also holds. Therefore, the problems $\mathcal{P}_y$ have optimal solutions for every $y \in A$, whenever $h(y) > -\infty$ and h is lower-semicontinuous.

Conversely, if h is lower-semicontinuous on A, that is, (8) is fulfilled, then by (19) and (20) we obtain (21).

Since H is a epigraph type set ($(y, t') \in H$, whenever $(y, t) \in H$ and $t' > t$), in some special cases the closedness property of H can be considered only for the sections of H in Y. For example, this is the case of the functions F of the form

$$F(u, y) = I_C(y) + \widetilde{F}(u - y), \quad u, y \in Y, \tag{22}$$

where $C \subset Y$, and $\widetilde{F}$ are positively homogeneous functions on Y. Indeed, if $(y,t) \in \overline{H}$, there exists a net $(y_i, t_i)_{i\in I} \subset H$ such that $(y_i, t_i) \to (y,t)$, and so, for every $i \in I$ there exists $u_i \in X$ such that $\widetilde{F}(u_i - y_i) \leq t_i$ and $y_i \in C$. Hence, given $\varepsilon > 0$ we have $i_\varepsilon \in I$ such that

$$\widetilde{F}(u_i - y_i) \leq t + \varepsilon, \quad i \geq i_\varepsilon.$$

We denote

$$H_t = \{y \in Y; (y,t) \in H\}, \quad t \in \mathbb{R}. \tag{23}$$

Obviously, $(y,0) \in H$ for any $y \in C$ since $F(y,y) = 0$. Therefore, if $t = 0$ we get $y \in H_0$. Thus, we can suppose $t \neq 0$. Taking $\varepsilon > 0$ such that $\frac{t}{t+\varepsilon} > 0$, by hypothesis of positive homogeneity we also have

$$\widetilde{F}(u_i - \frac{\varepsilon}{t+\varepsilon}(u_i - y_i) - y_i) \leq t, \quad i \geq i_\varepsilon.$$

Therefore, $y_i \in H_t$ for all $i \geq i_\varepsilon$. Consequently, $y \in \overline{H}_t$, and if H_t are closed in Y for any $t \in \mathbb{R}$, then H is also closed in $Y \times \mathbb{R}$. Generally, we have the following result.

Lemma 2.2. *Let A be a set in Y and suppose that*

$$y \in \overline{H}_{h(y)}, \quad \textit{whenever } y \in A \textit{ and } h(y) \in \mathbb{R}. \tag{24}$$

Then the problems $\mathcal{P}_y$ have optimal solutions for any $y \in A$, $h(y) > -\infty$ and h is lower-semicontinuous on A, if and only if

$$\overline{H}_t \cap A = H_t \cap A \quad \textit{for any } t \in \mathbb{R}. \tag{25}$$

Proof. Obviously, the equality (25) holds if (21) is fulfilled. Conversely, by the hypothesis (24) and the property (25) it follows that $y \in H_{h(y)}$, whenever $y \in A$ and $h(y) \in \mathbb{R}$ and so, the problems $\mathcal{P}_y$ have optimal solutions whenever $y \in A$ and $h(y) > -\infty$. Moreover, if $(y,t) \in \overline{H} \cap (A \times \mathbb{R})$, then $y \in \overline{H}_{t+\varepsilon}$, and so $(y, t+\varepsilon) \in H$, for any $\varepsilon > 0$. Therefore $h(y) \leq t$. Taking an optimal solution of $\mathcal{P}_y$ we obtain that $(y,t) \in H$. Hence (21) holds.

We recall that the *convolution* of two functions f_1 and f_2 on a locally convex space U, denoted by $f_1 \Box f_2$, is defined by

$$(f_1 \Box f_2)(u) = \inf\{f_1(v) + f_2(u-v), v \in U\}, \quad u \in U. \tag{26}$$

The convolution is called *exact* at u if there exists $\overline{v} \in U$ such that

$$(f_1 \Box f_2)(u) = f_1(\overline{v}) + f_2(u - \overline{v}).$$

If $f_1, f_2 \in \Gamma_0(X)$ and $D(f_1) \cap D(f_2) \neq \emptyset$, then

$$(f_1 + f_2)^* = \mathrm{cl}(f_1^* \Box f_2^*),$$

where by $\operatorname{cl} f$ we denote the *closure* of a function f.

In this paper we suppose that X^* is endowed with weak-star topology (or X and X^* are endowed with *compatible topologies* with respect to natural duality between X and X^*).

Therefore, if $(f_1 + f_2)^*$ is a proper function, then

$$(f_1 + f_2)^*(u^*) = (f_1^* \Box f_2^*)(u^*), \tag{27}$$

if and only if the convolution is lower-semicontinuous at $u^* \in U^*$. A detailed treatment of calculus rules of the convolution of convex functions is presented by Laurent [12].

3. Main results

In the sequel we establish some equivalent formulations of the additivity of subdifferential in a given element $x \in X$.

Theorem 3.1. *Given an element* $x \in D(f_1) \cap D(f_2)$, *where* $f_1, f_2 \in \Gamma_0(X)$, *then the following properties are equivalent.*

(*i*) $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$;

(*ii*) *for every* $x^* \in \partial(f_1 + f_2)(x)$ *there exists* $x_1^* \in X^*$ *such that* $(f_1 + f_2)^*(x^*) = f_1^*(x_1^*) + f_2^*(x^* - x_1^*)$;

(*iii*) *the problem*

$$(\mathcal{P}_{x^*}^*) \qquad \min\{f_1^*(u^*) + f_2^*(x^* - u^*); u^* \in X^*\}, \quad x^* \in X^*,$$

has optimal solutions for every $x^* \in \partial(f_1 + f_2)(x)$ *and its optimal value is* $(f_1 + f_2)^*(x^*)$;

(*iv*) *the problem* $\mathcal{P}_{x^*}$ *has optimal solutions for any* $x^* \in \partial(f_1 + f_2)(x)$ *and its optimal values is lower-semicontinuous on* $\partial(f_1 + f_2)(x)$;

(*v*) *the problem*

$$(\mathcal{P}_{x^*}) \qquad \max\{x^*(u) - (f_1 + f_2)(u); u \in X\}$$

is stable for any $x^* \in \partial(f_1 + f_2)(x)$;

(*vi*) *the convolution* $(f_1^* \Box f_2^*)(x^*)$ *is exact and lower-semicontinuous at any* $x^* \in \partial(f_1 + f_2)(x)$.

Proof. Let us denote $h(x^*) = \operatorname{val} \mathcal{P}_{x^*}^*$, $x^* \in X^*$, the optimal value function of the problems $\mathcal{P}_{x^*}^*$.

By (9) it follows that $f_1^*(u^*) + f_2^*(x^* - u^*) \geq u^*(u) + f_1(u) + (x^* - u^*)(u) - f_2(u)$, for all $u \in X$. Therefore, $h(x^*) \geq (f_1 + f_2)^*(x^*)$. On the other hand, by hypothesis, the function $f_1 + f_2$ is proper, and so $(f_1 + f_2)^*$ is proper [2]. Consequently, h is a proper function. Obviously, h is convex. Also, $h(x^*) = (f_1^* \Box f_2^*)(x^*)$. By (27) we have that $h(x^*) = (f_1 + f_2)^*(x^*)$ if and only if h is lower-semicontinuous at $x^* \in X^*$. Thus, the properties (*iii*), (*iv*), (*vi*) are equivalent. Now, by a standard calculus it follows that $\mathcal{P}^*_{x^*}$ is the dual of the problem $\mathcal{P}_{x^*}$. If $\mathcal{P}_{x^*}$ is stable, that is, $\operatorname{val}\mathcal{P}_{x^*} = \operatorname{val}\mathcal{P}^*_{x^*}$ and $\mathcal{P}^*_{x^*}$ has optimal solutions, we obtain that (v) and (iii) are equivalent since obviously $\operatorname{val}\mathcal{P}_{x^*} = (f_1+f_2)^*(x^*)$. Moreover, $\operatorname{val}\mathcal{P}_{x^*} \le \operatorname{val}\mathcal{P}^*_{x^*}$ and so (ii), (iii) are equivalent.

Using (10) for $\partial f_1(x), \partial f_2(x)$ and $\partial(f_1+f_2)(x)$ it follows that $(i) \Rightarrow (ii)$. Now, if $(f_1+f_2^*)(x^*) = f_1^*(x_1^*) + f_2^*(x^*-x_1^*)$ for certain $x_1^* \in X^*$, then by (9) we have

$$x_1^*(u) \le f_1(u) + f_1^*(x_1^*),$$

$$(x^*-x_1^*)(u) \le f_2(u) + f_2^*(x^*-x_1^*),$$

for all $u \in X$. Addiding this inequalities we obtain

$$(f_1+f_2)^*(x^*) \le f_1^*(x_1^*) + f_2^*(x^*-x_1^*).$$

But, by (ii), this inequality is just an equality, and so both previous inequalities become equalities. Therefore, by (10) it follows that $x_1^* \in \partial f_1(x)$ and $x^* - x_1^* \in \partial f_2(x)$, and so, (i) holds. Thus, the proof is complete.

Corollary 3.2. *The convolution of two functions $f_1, f_2 \in \Gamma_0(X)$ is exact at any element $x \in \partial(f_1^*+f_2^*)(x^*)$ if and only if $\partial(f_1^*+f_2^*)(x^*) = \partial f_1^*(x^*) + \partial f_2^*(x^*)$.*

In fact, Theorem 3.1 can be reduced to the equivalence between (i) and (iv). The properties (ii), (iii), (v), (vi) are reformulations of (i) by other important properties in convex analysis. Consequently, by Lemma 2.1 we can obtain a characterization of additivity property of subdifferential with the aid of a closedness condition. Thus, we refind a result established in [13].

Theorem 3.3 ([13]). *Let $f_1, f_2 \in \Gamma_0(X)$. If $x \in D(f_1) \cap D(f_2)$, then*

$$\partial(f_1+f_2)(x) = \partial f_1(x) + \partial f_2(x), \tag{28}$$

if and only if the following closedness condition is fulfilled

$$\overline{(\operatorname{epi} f_1^* + \operatorname{epi} f_2^*)} \cap [\partial(f_1+f_2)(x) \times \mathbb{R}] = (\operatorname{epi} f_1^* + \operatorname{epi} f_2^*) \cap [\partial(f_1+f_2)(x) \times \mathbb{R}], \tag{29}$$

where the closedness is considered with respect to the weak-star topology of X^*.

Proof. We apply Lemma 2.1, where $U = Y = X^*$ endowed with the w^* topology, $A = \partial(f_1+f_2)(x)$ and

$$F(u^*, x^*) = f_1^*(u^*) + f_2^*(x^*-u^*), \quad u^*, x^* \in X^*.$$

Hence, the associated set H defined by (14) is as follows

$$\begin{aligned} H &= \{(x^*,t) \in X^* \times \mathbb{R} \ \text{ there exists } u^* \in X^* \text{ such that } F(u^*,x^*) \le t\} \\ &= \{(x^*,t) \in X^* \times \mathbb{R}; \text{ there exists } u^* \in X^* \text{ such that} \\ &\qquad f_1^*(u^*) + f_2^*(x^*-u^*) \le t\} = \operatorname{epi} f_1^* + \operatorname{epi} f_2^*. \end{aligned} \tag{30}$$

Therefore, the equality (21) becomes the equality (29).

Now, we can establish a characterization of the additivity property of subdifferential on whole X.

Theorem 3.4. *Let us consider $f_1, f_2 \in \Gamma_0(X)$. Then the subdifferential is additive, that is,*

$$\partial(f_1 + f_2) = \partial f_1 + \partial f_2, \tag{31}$$

if and only if

$$\begin{aligned} &\overline{\operatorname{epi} f_1^* + \operatorname{epi} f_2^*} \cap [\operatorname{Range} \partial(f_1 + f_2) \times \mathbb{R}] \\ &= (\operatorname{epi} f_1^* + \operatorname{epi} f_2^*) \cap [\operatorname{Range} \partial(f_1 + f_2) \times \mathbb{R}]. \end{aligned} \tag{32}$$

Proof. Obviously, if $x \notin D(\partial(f_1 + f_2))$, then (31) is trivially fulfilled. Since $\operatorname{Range} \partial(f_1 + f_2) = \bigcup_{x \in D(\partial(f_1+f_2))} \partial(f_1 + f_2)(x)$, if (27) holds for any $x \in D(\partial(f_1 + f_2))$ this means that (32) is fulfilled.

Corollary 3.5. *If $\partial(f_1 + f_2)$ is surjective, then the additivity property holds if and only if* $\operatorname{epi} f_1^* + \operatorname{epi} f_2^*$ *is weak-star closed in $X^* \times \mathbb{R}$.*

Corollary 3.6 ([6]). *If* $\operatorname{epi} f_1^* + \operatorname{epi} f_2^*$ *is weak-star closed in $X^* \times \mathbb{R}$, then the additivity property* (29) *holds.*

Remark 3.7. The closedness condition (29) is fulfilled, for example, if there exists a set A in X^* such that $\partial(f_1 + f_2)(x) \subset A$ and

$$\overline{(\operatorname{epi} f_1^* + \operatorname{epi} f_2^*)} \cap (A \times \mathbb{R}) = (\operatorname{epi} f_1^* + \operatorname{epi} f_2^*) \cap (A \times \mathbb{R}). \tag{33}$$

For instance, if we take $A = D(f_1 + f_2)^*$ and the regularity conditions (33) is fulfilled, then the subdifferential additivity (28) holds on the whole space X.

In the sequel we study some special cases in which the closedness property (29) in $X^* \times \mathbb{R}$ can be replaced by a closedness property in X^*, that is we can apply Lemma 2.2 using the sections of the type (23).

Theorem 3.8. *Let $f_1, f_2 \in \Gamma_0(X)$ such that $x^* \in \overline{H_{h(x^*)}}$, whenever $x^* \in \partial(f_1 + f_2)(x)$, where $x \in D(f_1) \cap D(f_2)$. Then the additivity property* (4) *holds if and only if*

$$\begin{aligned} &\overline{\bigcup_{\tau \in \mathbb{R}} (S_{f_1^*}(\tau) + S_{f_2^*}(t - \tau))} \cap \partial(f_1 + f_2)(x) \\ &= \left(\bigcup_{\tau \in \mathbb{R}} (S_{f_1^*}(\tau) + S_{f_2^*}(t - \tau)) \right) \cap \partial(f_1 + f_2)(x) \end{aligned} \tag{34}$$

for any $t \in \mathbb{R}$.

Proof. Let us take $F(u^*, x^*) = f_1^*(u^*) + f_2^*(x^* - u^*)$, $u^*, x^* \in X^*$. Thus, according to (6) and (23) we obtain $H_t = \bigcup_{\tau \in \mathbb{R}} (S_{f_1^*}(\tau) + S_{f_2^*}(t - \tau))$, $t \in \mathbb{R}$. Since $h(x^*) \in \mathbb{R}$ if $x^* \in \partial(f_1 + f_2)(x)$, applying Theorem 3.1 (*iv*) and Lemma 2.2 we obtain the result.

A typical example was given by the functions F of the form (22). Specifically, we shall present the closedness property (29) if f_1^* and f_2^* are indicator functions of certain weak-star closed convex sets in X^* or proper convex lower-semicontinuous positively homogeneous functions. We recall that the conjugate of a positively homogeneous function is an indicator function, namely if f_1^* is a positively homogenenous function, then $f_1 = I_{C_1}$, where $C_1 = \{x \in X; f_1^*(x^*) \geq x^*(x);$ for all $x^* \in X^*\}$, and so, $\partial f_1(x) = N_{C_1}(x)$, $x \in D(\partial f_1)$. If $f_1^* = I_A$, where A is a weak-star closed convex set in X^* containing the origin, then by (12) we have $f_1 = p_{A^0}$. In the sequel we shall consider two cases of this type.

Theorem 3.9. *Let A, B be two nonvoid closed convex sets in X containing the origin. Then the following properties are equivalent:*

(*i*) $\partial(p_A + p_B)(x) = \partial p_A(x) + \partial p_B(x)$, *for all* $x \in D(p_A) \cap D(p_B)$;
(*ii*) $\partial(p_A + p_B)(0) = \partial p_A(0) + \partial p_B(0)$;
(*iii*) $A^0 + B^0$ *is weak-star closed in* X^*.

Proof. Let us take $f_1 = p_A$, $f_2 = p_B$. Using the equalities (11), (12) we obtain $F(u^*, x^*) = I_{A^0}(u^*) + I_{B^0}(x^* - u^*)$, $h(x^*) = I_{A^0+B^0}(x^*)$, $u^*, x^* \in X^*$. Therefore, the corresponding set $H = \operatorname{epi} f_1^* + \operatorname{epi} f_2^*$ becomes $(A^0 + B^0) \times [0, \infty)$.

On the other hand, it is easy to show that $\partial p_A(0) = A^0$, $\partial p_B(0) = B^0$ and $\partial(p_A + p_B)(0) = \overline{A^0 + B^0}$. Thus, by Theorem 3.3 we easily obtain the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

Remark 3.10. In fact, the equivalence $(i) \Leftrightarrow (iii)$ is contained in an earlier result established in the paper [6]. The authors proved that in the special case of sublinear functions f_1, f_2 the weak-star closedness of $\operatorname{epi} f_1^* + \operatorname{epi} f_2^*$ is necessary and sufficient for the additivity of subdifferential on the whole space X. Moreover, $\operatorname{epi} f_1^* + \operatorname{epi} f_2^* = (\partial f_1(0) + \partial f_2(0)) \times [0, \infty)$. Therefore, this set is weak-star closed in X^* if and only if (iii) holds.

Theorem 3.11. *Let A, B be two closed convex sets in X containing the origin. Then for any $x \in A \cap B$ the following properties are equivalent:*

(*i*) $N_{A\cap B}(x) = N_A(x) + N_B(x)$;
(*ii*) $\overline{\bigcup_{0\leq\tau\leq t}(S_{p_{A^0}}(\tau) + S_{B^0}(t-\tau))} \cap N_{A\cap B}(x) = (\bigcup_{0\leq\tau\leq t}(S_{p_{A^0}}(\tau) + S_{p_{B^0}}(t-\tau))) \cap N_{A\cap B}(x)$, *for any* $t \geq 0$.

Proof. Let us take $f_1 = I_A$ and $f_2 = I_B$. Hence, $\partial(f_1 + f_2)(x) = N_{A\cap B}(x)$. By (9) we also have $f_1^* = p_{A^0}$, $f_2^* = p_{B^0}$. On the other hand, by a standard calculus we obtain that $x^* \in \overline{H_{h(x^*)}}$ if $h(x^*) \in \mathbb{R}$. But, $S_{p_{A^0}}(\tau) + S_{p_{B^0}}(t-\tau) \neq \emptyset$ if and only if $t \geq 0$ and $\tau \in [0, t]$. Consequently, the equality (34) is just the property (ii).

Remark 3.12. For an arbitrary convex closed set C we have $N_C(x) = N_{C-x}(0)$ for any $x \in C$, and so, in the property (ii) we can replace A by $A - \overline{x}$, respectively, B by $B - \overline{x}$ for a certain $\overline{x} \in A \cap B$. By this change, Theorem 3.9 can be applied also in the case in which if $0 \notin A \cap B$. On the other hand, since $I_A^* = \sigma_A$, by Corollary 3.6 it follows that (i) holds for any $x \in X$ if $\operatorname{epi} \sigma_A + \operatorname{epi} \sigma_B$ is weak-star in $X^* \times \mathbb{R}$. This result is established by Burachik and Jeyakumar in [6] (see also [10]). By Theorem 3.7, this closedness condition can be given for its sections in X^*.

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