

Convex Conjugates of Analytic Functions of Logarithmically Convex Functionals

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Let $f_{\mathbf{c}}(r) = \sum_{n=0}^{\infty} e^{c_n} r^n$ be an analytic function; $\mathbf{c} = (c_n) \in l_{\infty}$. We assume that r is some logarithmically convex and lower semicontinuous functional on a locally convex topological space L . In this paper we derive a formula on the Legendre-Fenchel transform of a functional $\hat{\lambda}(\mathbf{c}, \varphi) = \ln f_{\mathbf{c}}(e^{\lambda(\varphi)})$, where $\lambda(\varphi) = \ln r(\varphi)$ ($\varphi \in L$). In this manner we generalize to the infinite case Theorem 3.1 from [7].

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1. Introduction

In Convex analysis, it is natural to consider the Legendre-Fenchel transform of convex composite functions. The general rules of convex conjugate calculus were obtained in [3, 4]. Therein one can find the history of these investigations. The general formulas are not always informative. It appears papers in which authors present forms of convex conjugates for some concrete compositions functions, see for instance [5, 6]. In this paper we would like to present investigations of the spectral radius of weighted composition operators which lead to considerations of compositions with the so-called log-exponential function.

First we present a general result obtained for the spectral radius of weighted composition operators. Let X be a Hausdorff compact space with Borel measure μ , $\alpha : X \mapsto X$ a continuous mapping preserving μ (i.e. $\mu \circ \alpha^{-1} = \mu$) and a be a continuous function on X . Antonevich, Bakhtin and Lebedev constructed a functional τ_{α} , called T -entropy (see [1, 2]), on the set of probability and α -invariant measures M_{α}^1 such that for the spectral radius of the weighted composition operator $(aT_{\alpha})u(x) = a(x)u(\alpha(x))$ acting in L^p -spaces the following variational principle holds

$$\ln r(aT_{\alpha}) = \max_{\nu \in M_{\alpha}^1} \left\{ \int_X \ln |a| d\nu - \frac{\tau_{\alpha}(\nu)}{p} \right\}. \quad (1)$$

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It turned out that τ_α is nonnegative, convex and lower semicontinuous on M_α^1 .

For positive $a \in C(X)$ let $\varphi = \ln a$ and $\lambda(\varphi) = \ln r(e^\varphi T_\alpha)$. The functional λ is convex and continuous on $C(X)$ and the formula (1) states that λ is the Legendre-Fenchel transform of the function $\frac{\tau_\alpha}{p}$, i.e.

$$\lambda(\varphi) = \max_{\nu \in M_\alpha^1} \left\{ \int_X \varphi d\nu - \lambda^*(\nu) \right\}, \quad (2)$$

where

$$\lambda^*(\nu) = \begin{cases} \frac{\tau_\alpha(\nu)}{p}, & \nu \in M_\alpha^1 \\ +\infty, & \text{otherwise.} \end{cases}$$

It means that the effective domain $D(\lambda^*)$ is contained in M_α^1 .

In operators algebras, it is natural to consider functions of operators. In the context of weighted composition operators, a problem arises how the formula (2) changes when instead of aT_α we take some functions of one.

First results was obtained for polynomials of aT_α . Let $\sum_{n=0}^N a_n z^n$ be a polynomial with positive coefficients a_n . The convex conjugate of $\tilde{\lambda}(\varphi) = \ln r(\sum_{n=0}^N a_n (e^\varphi T_\alpha)^n)$ is equal to

$$\tilde{\lambda}^*(m) = m(X) \lambda^* \left(\frac{m}{m(X)} \right) + \min_{\mathbf{t} \in S_{m(X)}} \sum_{n=0}^N t_n \ln \frac{t_n}{a_n}, \quad (3)$$

where $S_{m(X)} = \{(t_n)_{n=0}^N : t_n \geq 0, \sum_{n=0}^N t_n = 1 \text{ and } \sum_{n=0}^N n t_n = m(X)\}$ and the effective domain of $\tilde{\lambda}^*$ is contained in the set $\{m \in C(X)^* : m \in M_\alpha \text{ and } m(X) \in [0, N]\}$; M_α is the set of all α -invariant measures on X (see [7, 8] for more details).

Let c_n denote $\ln a_n$. If we consider dependence of the logarithm of the spectral radius also on the vector $(c_n)_{n=0}^N$, that is if we consider the functional $\hat{\lambda}((c_n), \varphi) = \ln r(\sum_{n=0}^N e^{c_n} r(e^\varphi T_\alpha)^n)$, then the convex conjugate $\hat{\lambda}$ has the form

$$\hat{\lambda}^*((t_n), \bar{\mu}) = \bar{\mu}(X) \lambda^* \left(\frac{\bar{\mu}}{\bar{\mu}(X)} \right) + \sum_{n=0}^N t_n \ln t_n \quad (4)$$

and the effective domain of $\hat{\lambda}^*$ is contained in the set

$$\mathcal{M} = \left\{ ((t_n), \bar{\mu}) : t_n \geq 0, \sum_{n=0}^N t_n = 1, \bar{\mu} \in M_\alpha \text{ and } \bar{\mu}(X) = \sum_{n=0}^N n t_n \right\}$$

(see [7] for more details).

The generalization of the formula (3) on the case of analytic functions was obtained in [9] where authors proved that for $\tilde{\lambda}(\varphi) = \ln r(\sum_{n=0}^\infty a_n (e^\varphi T_\alpha)^n)$ (we present this result only in the case of all $a_n > 0$) the convex conjugate has the form

$$\tilde{\lambda}^*(m) = m(X) \lambda^* \left(\frac{m}{m(X)} \right) + \min_{(t_n) \in S_{m(X)}} \liminf_{N \rightarrow \infty} \sum_{n=0}^N t_n \ln \frac{t_n}{a_n},$$

where $S_{m(X)} = \{(t_n)_{n=0}^\infty : t_n \geq 0, \sum_{n=0}^\infty t_n = 1, \sum_{n=0}^\infty nt_n < +\infty \text{ and } \sum_{n=0}^\infty nt_n = m(X)\}$ and the effective domain of $\tilde{\lambda}^*$ is contained in the set of all α -invariant measures M_α .

To generalize the formula (4) we had to solve two problems: to define the functional $\hat{\lambda}$ for infinite number of variables c_n and to consider the entropy function with infinite numbers of summands. It is known that not for all infinite sequences of probability weights (t_n) the entropy function $\sum_{n=0}^\infty t_n \ln t_n$ takes finite values (see Example 2.5). A solution of these problems there is in Theorem 2.9.

2. Convex conjugates

We begin by recalling Proposition 2.3 from [9].

Proposition 2.1. *For the sequence $(b_n)_{n=0}^\infty$ such that $b_n > 0$ and $\sum_{n=0}^\infty b_n < \infty$ the following holds*

$$\ln \sum_{n=0}^\infty b_n = \max_{t_n \geq 0, \sum t_n = 1} \limsup_{N \rightarrow \infty} \sum_{n=0}^N (t_n \ln b_n - t_n \ln t_n);$$

it is taken that $0 \ln 0 = 0$. This maximum is attained at $t_n = \frac{b_n}{\sum_{n=0}^\infty b_n}$.

Example 2.2. Let b_n equals r^n for $r \in (0, 1)$. Then the sum of the series $\sum_{n=0}^\infty r^n = 1/(1-r)$ and by the above Proposition we obtain formula

$$-\ln(1-r) = \max_{t_n \geq 0, \sum t_n = 1} \limsup_{N \rightarrow \infty} \sum_{n=0}^N (nt_n \ln r - t_n \ln t_n) \quad (5)$$

that can be rewritten as follows

$$\ln(1-r) = \min_{t_n \geq 0, \sum t_n = 1} \liminf_{N \rightarrow \infty} \sum_{n=0}^N t_n \ln \frac{t_n}{r^n}. \quad (6)$$

By Proposition 2.1, it is known that in this case the above maximum is attained for the geometric distribution, i.e. at $t_n = (1-r)r^n$. Let us emphasize that for the geometric distribution $((1-r)r^n)_{n=0}^\infty$ the series $\sum_{n=0}^\infty nt_n = \sum_{n=0}^\infty n(1-r)r^n = r/(1-r)$ is convergent. For this reason we can search for the maximum in (5) under the additional restricted condition $\sum_{n=0}^\infty nt_n < \infty$. So we can rewrite (5) in the form

$$-\ln(1-r) = \max_{\substack{t_n \geq 0, \sum t_n = 1 \\ \sum nt_n < \infty}} \left\{ \ln r \sum_{n=0}^\infty nt_n - \sum_{n=0}^\infty t_n \ln t_n \right\},$$

where the series $\sum_{n=0}^\infty t_n \ln t_n$ is either convergent or divergent to minus infinity. But the second opportunity is not possible because then the above maximum will be equal to $+\infty$ and it will not to be finite. It means that the restricted condition $\sum_{n=0}^\infty nt_n < +\infty$ ensures the convergence of the series $\sum_{n=0}^\infty t_n \ln t_n$.

Basing on this example we can formulate the following

Proposition 2.3. *If a probability distribution (t_n) satisfies the condition $\sum_{n=0}^{\infty} nt_n < \infty$ then the series $\sum_{n=0}^{\infty} t_n \ln t_n$ is convergent.*

The above statement is only a sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} t_n \ln t_n$.

Example 2.4. Using for instance the sum of series $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ and taking $t_n = 6/(\pi n)^2$ we see that the series $\sum_{n=1}^{\infty} nt_n = \sum_{n=1}^{\infty} 6/(\pi^2 n)$ is divergent but the series

$$\sum_{n=1}^{\infty} t_n \ln t_n = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\ln \frac{6}{\pi^2} - 2 \ln n \right) = \ln \frac{6}{\pi^2} - \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

is convergent.

Obviously not for all probability distributions (t_n) series of the form $\sum_{n=1}^{\infty} t_n \ln t_n$ are convergent.

Example 2.5. Using Cauchy condensation test one can check that the series $\sum_{n=2}^{\infty} 1/(n(\ln n)^2)$ is convergent and the series $\sum_{n=2}^{\infty} 1/(n \ln n)$ divergent. Denoting the sum $\sum_{n=2}^{\infty} 1/(n(\ln n)^2)$ by a and taking $t_n = 1/(n(\ln n)^2 a)$ one can check that

$$\begin{aligned} \sum_{n=2}^{\infty} t_n \ln t_n &= \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2 a} \ln \frac{1}{n(\ln n)^2 a} \\ &= -\ln a - \frac{1}{a} \sum_{n=2}^{\infty} \frac{1}{n \ln n} - \frac{2}{a} \sum_{n=2}^{\infty} \frac{\ln \ln n}{n(\ln n)^2} = -\infty. \end{aligned}$$

For each $(c_n) \in l_{\infty}$ the radius of convergence of an analytic function $f_{\mathbf{c}}(r) = \sum_{n=0}^{\infty} e^{c_n} r^n$ equals 1. Taking in Proposition 2.1 $b_n = e^{c_n} r^n$ for $r \in (0, 1)$ and $(c_n) \in l_{\infty}$ we obtain

$$\begin{aligned} \ln \sum_{n=0}^{\infty} e^{c_n} r^n &= \max_{t_n \geq 0, \sum t_n = 1} \limsup_{N \rightarrow \infty} \sum_{n=0}^N (c_n t_n + nt_n \ln r - t_n \ln t_n) \\ &= \max_{t_n \geq 0, \sum t_n = 1} \left\{ \sum_{n=0}^{\infty} c_n t_n - \liminf_{N \rightarrow \infty} \sum_{n=0}^N t_n \ln \frac{t_n}{r^n} \right\}. \end{aligned} \quad (7)$$

For each $(c_n) \in l_{\infty}$ the series $\sum_{n=0}^{\infty} c_n t_n$ is the bounded linear functional on the space l_1 . Let S denote the infinite standard simplex $\{\mathbf{t} = (t_n)_{n=0}^{\infty} : t_n \geq 0, \sum t_n = 1\}$. Notice that $S \subset l_1$ and $l_{\infty} \simeq (l_1)^*$.

Consider now the expression $\ln \sum_{n=0}^{\infty} e^{c_n} r^n$ as the functional $\lambda_r : l_{\infty} \mapsto \mathbb{R}$, i.e.

$$\lambda_r(\mathbf{c}) = \ln \sum_{n=0}^{\infty} e^{c_n} r^n \quad \text{for } \mathbf{c} = (c_n) \in l_{\infty} \quad (r \in (0, 1)).$$

The expression (7) means that λ_r is the convex conjugate of a functional $h_r : l_1 \mapsto \mathbb{R}$ defined as follows

$$h_r(\mathbf{t}) = \begin{cases} \liminf_{N \rightarrow \infty} \sum_{n=0}^N t_n \ln \frac{t_n}{r^n} & \text{if } \mathbf{t} \in S, \\ +\infty & \text{if } \mathbf{t} \in l_1 \setminus S. \end{cases}$$

The effective domain of h_r is contained in S . Moreover by (6) we have that $h_r(\mathbf{t}) \geq \ln(1-r)$ for $\mathbf{t} \in l_1$. Since the function $x \ln(x/a)$ ($a > 0$) is convex on $(0, +\infty)$ we have that $\sum_{n=0}^N t_n \ln \frac{t_n}{r^n}$ is convex on $(0, +\infty)^{N+1}$ but because in the definition of h_r appears the lower limit then one (we) can not prove convexity of it on S . For these reasons we only get that λ_r is the convex conjugate of h_r and λ_r^* is convex and lower semicontinuous regularization of h_r .

By $\tilde{S}$ we will denote the subset of sequences belonging to S and satisfying the condition $\sum n t_n < \infty$. By Proposition 2.1 the maximum in (7) is attained at the sequence $(e^{c_n} r^n / f_{\mathbf{c}}(r))$. Notice that for this sequence the value

$$\sum_{n=0}^{\infty} n t_n = \frac{r}{f_{\mathbf{c}}(r)} \sum_{n=1}^{\infty} n e^{c_n} r^{n-1} = \frac{r f'_{\mathbf{c}}(r)}{f_{\mathbf{c}}(r)}$$

is finite for $r \in (0, 1)$. Thus, using Proposition 2.3, we can express (7) as follows

$$\ln \sum_{n=0}^{\infty} e^{c_n} r^n = \max_{\mathbf{t} \in \tilde{S}} \left\{ \sum_{n=0}^{\infty} c_n t_n - \left(\sum_{n=0}^{\infty} t_n \ln t_n - \ln r \sum_{n=0}^{\infty} n t_n \right) \right\}. \quad (8)$$

Remark 2.6. Considering the logarithm of the finite sum $\sum_{n=0}^N e^{c_n} r^n$ we can assume that r is any nonnegative number. All series in the above formula become finite sums, $\tilde{S}$ will be the $(N+1)$ -dimensional standard simplex and for $r = 1$ we get the classical variational principle for the so-called log-exponential function (see Example 11.12 in [10]).

Define now the function g_r on l_1 in the following way

$$g_r(\mathbf{t}) = \begin{cases} \sum_{n=0}^{\infty} t_n \ln t_n - \ln r \sum_{n=0}^{\infty} n t_n & \text{if } \mathbf{t} \in \tilde{S}, \\ +\infty & \text{if } \mathbf{t} \in l_1 \setminus \tilde{S}. \end{cases}$$

The series $\sum_{n=0}^{\infty} n t_n$ is a linear but unbounded functional on l_1 . Because the space l_0 of sequences with finite supports is dense in l_1 and $\sum_{n=0}^{\infty} n t_n < \infty$ for $\mathbf{t} \in l_0$ then the set $\tilde{S}$ is dense in S ; besides it is a convex subset of S . The function g_r is convex on l_1 and we can defined λ_r^* as the lower semicontinuous regularization of g_r .

Let λ denote $\ln r$. If $r \in (0, 1)$ then $\lambda \in (-\infty, 0)$. Consider now the expression $\ln \sum_{n=0}^{\infty} e^{c_n} r^n = \ln \sum_{n=0}^{\infty} e^{c_n + n\lambda}$ as a functional $\tilde{\lambda} : l_{\infty} \times \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ defined in the following way

$$\tilde{\lambda}(\mathbf{c}, \lambda) = \begin{cases} \ln \sum_{n=0}^{\infty} e^{c_n + n\lambda} & \text{if } (\mathbf{c}, \lambda) \in l_{\infty} \times (-\infty, 0), \\ +\infty & \text{otherwise.} \end{cases} \quad (9)$$

We can rewrite (8) as

$$\tilde{\lambda}(\mathbf{c}, \lambda) = \max_{\mathbf{t} \in \tilde{S}} \left\{ \sum_{n=0}^{\infty} c_n t_n + \lambda \sum_{n=0}^{\infty} n t_n - \sum_{n=0}^{\infty} t_n \ln t_n \right\}. \quad (10)$$

By fixing $\mathbf{t} \in \tilde{S}$ the expression $\lambda \sum_{n=0}^{\infty} n t_n - \sum_{n=0}^{\infty} t_n \ln t_n$, as a linear function of the variable λ , is the convex conjugate of a function

$$f_{\mathbf{t}}(a) = \begin{cases} \sum_{n=0}^{\infty} t_n \ln t_n & \text{if } a = \sum_{n=0}^{\infty} n t_n, \\ +\infty & \text{otherwise,} \end{cases} \quad (11)$$

that is $\lambda \sum_{n=0}^{\infty} n t_n - \sum_{n=0}^{\infty} t_n \ln t_n = \max_{a \in \mathbb{R}} \{a\lambda - f_{\mathbf{t}}(a)\}$. Since $a = \sum_{n=0}^{\infty} n t_n$ is always nonnegative, we may search for this maximum only over the set $\mathbb{R}_+ = [0, +\infty)$.

Changing $\mathbf{t}$ and taking $f(\mathbf{t}, a) = f_{\mathbf{t}}(a)$ we can express $\tilde{\lambda}$ as follows

$$\begin{aligned} \tilde{\lambda}(\mathbf{c}, \lambda) &= \max_{\mathbf{t} \in \tilde{S}} \left\{ \sum_{n=0}^{\infty} c_n t_n + \max_{a \in \mathbb{R}_+} \{a\lambda - f(\mathbf{t}, a)\} \right\} \\ &= \max_{\mathbf{t} \in \tilde{S}} \max_{a \in \mathbb{R}_+} \left\{ \sum_{n=0}^{\infty} c_n t_n + \lambda a - f(\mathbf{t}, a) \right\}. \end{aligned} \quad (12)$$

Define now the set

$$D_{\tilde{\tau}} = \left\{ (\mathbf{t}, a) \in l_1 \times \mathbb{R} : \mathbf{t} \in \tilde{S}, a \geq 0 \text{ and } a = \sum_{n=0}^{\infty} n t_n \right\}$$

and the function

$$\tilde{\tau}(\mathbf{t}, a) = \begin{cases} \sum_{n=0}^{\infty} t_n \ln t_n & \text{if } (\mathbf{t}, a) \in D_{\tilde{\tau}}, \\ +\infty & \text{otherwise.} \end{cases} \quad (13)$$

Now we can rewrite (12) as follows

$$\tilde{\lambda}(\mathbf{c}, \lambda) = \max_{(\mathbf{t}, a) \in D_{\tilde{\tau}}} \left\{ \sum_{n=0}^{\infty} c_n t_n + \lambda a - \sum_{n=0}^{\infty} t_n \ln t_n \right\}.$$

Notice that for $(\mathbf{c}, a) \in l_{\infty} \times \mathbb{R}$ the expression $\sum_{n=0}^{\infty} c_n t_n + \lambda a$ is a linear and bounded functional on the space $l_1 \times \mathbb{R}$ and the above formula means that $\tilde{\lambda}$ is the convex conjugate of $\tilde{\tau}$.

Proposition 2.7. *The convex conjugate of $\tilde{\lambda}$ defined by (9) is the lower semicontinuous regularization of the functional $\tilde{\tau}$ defined by (13).*

Proof. We should prove that $D_{\tilde{\tau}}$ is a convex subset of $l_1 \times \mathbb{R}$ and $\tilde{\tau}$ is a convex function on it. Let $(\mathbf{t}^1, a_1)$, $(\mathbf{t}^2, a_2)$ belong to $D_{\tilde{\tau}}$ and s in the interval $(0, 1)$. Consider the element

$$s(\mathbf{t}^1, a_1) + (1-s)(\mathbf{t}^2, a_2) = (s\mathbf{t}^1 + (1-s)\mathbf{t}^2, sa_1 + (1-s)a_2).$$

Since $\tilde{S}$ is convex, $s\mathbf{t}^1 + (1-s)\mathbf{t}^2 \in \tilde{S}$ for $\mathbf{t}^1, \mathbf{t}^2 \in \tilde{S}$ and $s \in (0, 1)$. Moreover, if $\sum_{n=0}^{\infty} nt_n^1 = a_1$ and $\sum_{n=0}^{\infty} nt_n^2 = a_2$ then $sa_1 + (1-s)a_2 = \sum_{n=0}^{\infty} n(st_n^1 + (1-s)t_n^2)$. It follows the convexity of $D_{\tilde{\tau}}$ and the convexity of $\tilde{\tau}$ results from the convexity of the entropy function. $\square$

Let now $r : L \mapsto (0, +\infty]$ be a logarithmically convex and lower semicontinuous functional defined on a locally convex topological space L and $\lambda(\varphi) = \ln r(\varphi)$ for $\varphi \in L$. The functional λ is convex and lower semicontinuous. Let $D(\lambda^*)$ denote the effective domain of the convex conjugate of λ . For the functional $\lambda : L \mapsto \mathbb{R} \cup \{+\infty\}$ the following variational principle holds

$$\lambda(\varphi) = \sup_{\mu \in D(\lambda^*)} \{ \langle \mu, \varphi \rangle - \lambda^*(\mu) \}, \quad \varphi \in L. \quad (14)$$

Let $\mathcal{D}_\lambda$ denote a set $\{\varphi \in L : \lambda(\varphi) < 0\}$. We assume that $\mathcal{D}_\lambda$ is nonempty set. Define now the functional $\hat{\lambda} : l_\infty \times L \mapsto \mathbb{R} \cup \{+\infty\}$ as follows

$$\hat{\lambda}(\mathbf{c}, \varphi) = \begin{cases} \ln \sum_{n=0}^{\infty} e^{c_n + n\lambda(\varphi)} & \text{if } (\mathbf{c}, \varphi) \in l_\infty \times \mathcal{D}_\lambda, \\ +\infty & \text{otherwise.} \end{cases} \quad (15)$$

The effective domain $D(\hat{\lambda})$ is included in $l_\infty \times \mathcal{D}_\lambda$. Substituting (14) into (10) we obtain

$$\begin{aligned} \hat{\lambda}(\mathbf{c}, \varphi) &= \max_{\mathbf{t} \in \tilde{S}} \left\{ \sum_{n=0}^{\infty} c_n t_n + \sup_{\mu \in D(\lambda^*)} \{ \langle \mu, \varphi \rangle - \lambda^*(\mu) \} \sum_{n=0}^{\infty} n t_n - \sum_{n=0}^{\infty} t_n \ln t_n \right\} \\ &= \max_{\mathbf{t} \in \tilde{S}} \sup_{\mu \in D(\lambda^*)} \left\{ \sum_{n=0}^{\infty} c_n t_n + \left\langle \left(\sum_{n=0}^{\infty} n t_n \right) \mu, \varphi \right\rangle \right. \\ &\quad \left. - \left(\sum_{n=0}^{\infty} n t_n \right) \lambda^*(\mu) - \sum_{n=0}^{\infty} t_n \ln t_n \right\}. \end{aligned} \quad (16)$$

Notice that for $\mathbf{t} \in \tilde{S}$ the expression $\sum_{n=0}^{\infty} n t_n$ can be any nonnegative number, moreover $\sum_{n=0}^{\infty} n t_n = 0$ if and only if $\mathbf{t} = e_0$ ($e_0 = (1, 0, 0, \dots)$). The value of the above expression in the curly brackets at e_0 is equal to c_0 and is less than $\hat{\lambda}(\mathbf{c}, \varphi) = \ln \sum_{n=0}^{\infty} e^{c_n + n\lambda(\varphi)}$ for any $\mathbf{c} \in l_\infty$ and $\varphi \in \mathcal{D}_\lambda$. For this reason we can search for the above maximum without the point e_0 .

Let $\bar{\mu}$ denote $(\sum_{n=0}^{\infty} n t_n) \mu$ and

$$D_{\tilde{\tau}} = \left\{ (\mathbf{t}, \bar{\mu}) \in l_1 \times L^* : \mathbf{t} \in \tilde{S} \setminus \{e_0\} \text{ and } \frac{\bar{\mu}}{\sum_{n=0}^{\infty} n t_n} \in D(\lambda^*) \right\}.$$

Define the functional $\hat{\tau} : l_1 \times L^* \mapsto \mathbb{R} \cup \{+\infty\}$ in the following way

$$\hat{\tau}(\mathbf{t}, \bar{\mu}) = \begin{cases} (\sum_{n=0}^{\infty} nt_n)\lambda^* \left(\frac{\bar{\mu}}{\sum_{n=0}^{\infty} nt_n} \right) + \sum_{n=0}^{\infty} t_n \ln t_n & \text{if } (\mathbf{t}, \bar{\mu}) \in D_{\hat{\tau}}, \\ +\infty & \text{otherwise.} \end{cases} \quad (17)$$

Now we can rewrite (16) as follows

$$\hat{\lambda}(\mathbf{c}, \varphi) = \sup_{(\mathbf{t}, \bar{\mu}) \in D_{\hat{\tau}}} \left\{ \sum_{n=0}^{\infty} c_n t_n + \langle \bar{\mu}, \varphi \rangle - \hat{\tau}(\mathbf{t}, \bar{\mu}) \right\}.$$

It means that $\hat{\lambda}$ is the convex conjugate of $\hat{\tau}$ that is $\hat{\lambda} = \hat{\tau}^*$.

Proposition 2.8. *The convex conjugate of $\hat{\lambda}$ defined by (15) is the lower semi-continuous regularization of the functional $\hat{\tau}$ defined by (17).*

Proof. Let $(\mathbf{t}_1, \bar{\mu}_1)$ and $(\mathbf{t}_2, \bar{\mu}_2)$ belong to $D_{\hat{\tau}}$. Consider an element

$$s(\mathbf{t}_1, \bar{\mu}_1) + (1-s)(\mathbf{t}_2, \bar{\mu}_2) = (s\mathbf{t}_1 + (1-s)\mathbf{t}_2, s\bar{\mu}_1 + (1-s)\bar{\mu}_2)$$

for $s \in (0, 1)$. Since $\tilde{S} \setminus \{e_0\}$ is convex, $s\mathbf{t}_1 + (1-s)\mathbf{t}_2$ in $\tilde{S} \setminus \{e_0\}$. Let a_1 and a_2 denote $\sum_{n=0}^{\infty} nt_n^1$ and $\sum_{n=0}^{\infty} nt_n^2$, respectively. We should check that

$$\frac{s\bar{\mu}_1 + (1-s)\bar{\mu}_2}{sa_1 + (1-s)a_2} \in D(\lambda^*). \quad (18)$$

Notice that $\bar{\mu}_1/a_1, \bar{\mu}_2/a_2$ in $D(\lambda^*)$ and

$$\frac{s\bar{\mu}_1 + (1-s)\bar{\mu}_2}{sa_1 + (1-s)a_2} = \frac{sa_1}{sa_1 + (1-s)a_2} \frac{\bar{\mu}_1}{a_1} + \frac{(1-s)a_2}{sa_1 + (1-s)a_2} \frac{\bar{\mu}_2}{a_2}.$$

Because $D(\lambda^*)$ is convex and $sa_1/(sa_1 + (1-s)a_2), [(1-s)a_2]/(sa_1 + (1-s)a_2)$ are positive and their sum to be 1 then (18) is valid.

The entropy function $\sum_{n=0}^{\infty} t_n \ln t_n$ is convex on $\tilde{S}$ and the convexity of λ^* implies

$$(sa_1 + (1-s)a_2)\lambda^* \left(\frac{s\bar{\mu}_1 + (1-s)\bar{\mu}_2}{sa_1 + (1-s)a_2} \right) \leq sa_1\lambda^* \left(\frac{\bar{\mu}_1}{a_1} \right) + (1-s)a_2\lambda^* \left(\frac{\bar{\mu}_2}{a_2} \right).$$

For these reasons the functional $\hat{\tau}$ is convex on l_1 which completes the proof. $\square$

Return now to the spectral radius of the weighted composition operator $e^\varphi T_\alpha$. This operator, considered in L^p -spaces (Banach lattices), is an example of positive operators. The spectral radii of such operators belong to their spectrums (see Prop. 4.1 in Ch. V in [11]). Analytic functions with positive coefficients of positive operators are also positive ones and their spectral radii also belong to their spectrums. This fact yields that $r(f_{\mathbf{c}}(e^\varphi T_\alpha)) = f_{\mathbf{c}}(r(e^\varphi T_\alpha))$. This means that investigating the spectral radius $r(f_{\mathbf{c}}(e^\varphi T_\alpha))$ we can consider the functions of

the spectral radius $f_{\mathbf{c}}(r(e^{\varphi}T_{\alpha}))$. Since $r(e^{\varphi}T_{\alpha})$ depends logarithmically on φ , we have investigated $\ln f(e^{\lambda(\varphi)})$, where $\lambda(\varphi) = \ln r(e^{\varphi}T_{\alpha})$.

Proposition 2.8 gives us the general form of the Legendre-Fenchel transform of this functional. In our case we can precise some details. Because $D(\lambda^*)$ is included in the set of all probability and α -invariant measures M_{α}^1 then $\bar{\mu} = (\sum_{n=0}^{\infty} nt_n)\mu$ ($\mathbf{t} \in \tilde{S}$ and $\mu \in M_{\alpha}^1$) is any α -invariant measure. For $\bar{\mu} \equiv 0$, using the Legendre-Fenchel transform, in the same way as in the proof of the Theorem 3.1 form [7], one can calculate that $\lambda^*(e_0, \mathbf{0}) = 0$. In this way we can formulate a generalization of Theorem 3.1 form [7].

Theorem 2.9. *Let X be a Hausdorff compact space with Borel measure μ , $\alpha : X \mapsto X$ a continuous mapping preserving μ and φ be a continuous function on X . Let $\lambda : C(X) \mapsto \mathbb{R}$ denote a functional being the spectral exponent of weighted composition operators $e^{\varphi}T_{\alpha}$ acting in the space $L^p(X, \mu)$; $\lambda(\varphi) = \ln r(e^{\varphi}T_{\alpha})$. Then the Legendre-Fenchel transform of $\hat{\lambda}$ defined by (15) is the lower semicontinuous regularization of a functional defined by the following formula*

$$\hat{\lambda}^*(\mathbf{t}, \bar{\mu}) = \frac{1}{p} \left(\sum_{n=0}^{\infty} nt_n \right) \tau_{\alpha} \left(\frac{\bar{\mu}}{\sum_{n=0}^{\infty} nt_n} \right) + \sum_{n=0}^{\infty} t_n \ln t_n$$

on the set $\{(\mathbf{t}, \bar{\mu}) \in l_1 \times C(X)^* : \mathbf{t} \in \tilde{S} \setminus \{e_0\}, \bar{\mu} \in M_{\alpha}$ and $\bar{\mu}(X) = \sum_{n=0}^{\infty} nt_n\}$ and $+\infty$ otherwise. The functional τ_{α} is T -entropy. When $\bar{\mu}(X) = 0$ then $\hat{\lambda}^*$ takes value zero.

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