Journal of Convex Analysis
Volume 20 (2013), No. 1, 253–264

Best Constants in Poincaré Inequalities for Convex Domains

L. Esposito
*Dipartimento di Matematica, Università degli Studi di Salerno,
Via Ponte Don Melillo, 84084 Fisciano (SA), Italy
luesposi@unisa.it*

C. Nitsch
*Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università degli Studi di Napoli Federico II,
Complesso Monte S. Angelo, via Cintia, 80126 Napoli, Italy
c.nitsch@unina.it*

C. Trombetti
*Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università degli Studi di Napoli Federico II,
Complesso Monte S. Angelo, via Cintia, 80126 Napoli, Italy
cristina@unina.it*

Received: December 09, 2011
Revised manuscript received: March 29, 2012

We prove a Payne-Weinberger type inequality for the p-Laplacian Neumann eigenvalues ($p \geq 2$). The inequality provides the sharp upper bound on convex domains, in terms of the diameter alone, of the best constant in Poincaré inequality. The key point is the implementation of a refinement of the classical Pólya-Szegö inequality for the symmetric decreasing rearrangement which yields an optimal weighted Wirtinger inequality.

Keywords: Poincaré inequality, p-Laplacian eigenvalues, Neumann boundary conditions

1. Introduction

Let Ω be an open bounded Lipschitz connected subset in $\mathbb{R}^n$. The first nontrivial Neumann eigenvalue μ_p for the p-Laplacian equation ($p > 1$)

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = \mu_p |u|^{p-2}u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

can be characterized by

$$\mu_p = \min_{\substack{u \in W^{1,p}(\Omega) \\ \int_\Omega |u|^{p-2}u = 0}} \frac{\displaystyle\int_\Omega |Du|^p}{\displaystyle\int_\Omega |u|^p} \tag{2}$$

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

It is well known that (see for instance [15]), for any given open bounded Lipschitz connected Ω, a Poincaré inequality holds true, in the sense that there exists a constant $C_{\Omega,p}$ such that

$$\inf_{t\in\mathbb{R}} \|u-t\|_{L^p(\Omega)} \le C_{\Omega,p}\|Du\|_{L^p(\Omega)}, \tag{3}$$

for any $u \in W^{1,p}(\Omega)$.

The value of the best constant in (3) is

$$C_{\Omega,p} = \mu_p^{-1/p}.$$

In [16], and more recently in [2], it has been proved that, if $p = 2$, Ω is convex, in any dimension

$$\frac{1}{C_{\Omega,2}} = \sqrt{\mu_2} \ge \frac{\pi}{d}, \tag{4}$$

where d is the diameter of Ω. Observe that the right hand side of (4) is exactly the value achieved, in dimension $n = 1$ on any interval of length d, by the first nontrivial Laplacian eigenvalue (without distinction between the Neumann and the Dirichlet problems).

The proof of (4) in [16, 2] indeed relies on the reduction to a one dimensional problem. At this aim, for any given smooth admissible test function u in the right hand side of (2), the authors show that it is possible to perform a clever slicing of the domain Ω in convex sets which are as tiny as desired in at least $n-1$ orthogonal directions. On each one of such convex components of Ω, they are able to show that the Rayleigh quotient of u can be approximated by a 1-dimensional weighted Rayleigh quotient. This leads the authors to look for the best constants of a class of one dimensional weighted Poincaré-Wirtinger inequalities.

However the technique that they use strongly relies on the linearity of the Laplace operator (in particular the authors need the property that derivatives of solutions to homogeneous differential equation are still solutions to the same equation). For this reason the proof can not be generalized to $p \ne 2$ in a straightforward way as testified in [4], and to our knowledge the only other case where the best constant, in terms of the diameter of the domain, has been computed using these techniques is the limit case $p = 1$ (see [1]).

It is worth to mention that lower bounds for the p-Laplacian eigenvalues on compact manifolds have been widely studied (see for instance [13, 19, 24] and the references therein), and the very same estimate as (4) has been proved in [25].

The main result of this paper is actually to prove that

Theorem 1.1 (Main Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set having diameter d. For $p \ge 2$ and in any dimension we have*

$$\mu_p \ge \left(\frac{\pi_p}{d}\right)^p$$

where

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{dt}{(1-t^p/(p-1))^{1/p}} = 2\pi \frac{(p-1)^{1/p}}{p(\sin(\pi/p))}. \tag{5}$$

We observe that $\left(\frac{\pi_p}{d}\right)^p$ for $n = 1$ is the p-Laplacian eigenvalue on $[0, d]$, without distinction between the Neumann and the Dirichlet boundary conditions (see for instance [3], [17])

The formula (5) for π_p can be found for instance in [3, 14, 17, 20, 21]. The fact that $\pi_2 = \pi$ is consistent with the classical Wirtinger inequality (see [9]) and obviously also with (4).

We are grateful to the Referee for bringing to our attention a recent paper by D. Valtorta [23] where Theorem 1.1 has already been proved by means of techniques rather different in the general setting of compact manifolds with non negative Ricci curvature (and possibly with convex boundary). The main tools used in [23] are a gradient comparison method and a generalized p-Bochner formula.

In the spirit of the proof of (4) by Payne and Weinberger [16], our proof of Theorem 1.1 is completely different and it is based on the following estimate on the best constant in a class of weighted Wirtinger inequalities.

Proposition 1.2. *Let f be a nonnegative log-concave function defined on $[0, L]$ and $p \geq 2$ then*

$$\inf_{\substack{u\in W^{1,p}(0,L)\\ \int_0^L f|u|^{p-2}u=0}} \frac{\displaystyle\int_0^L f(x)|u'(x)|^p dx}{\displaystyle\int_0^L f(x)|u(x)|^p dx} \geq \min_{\substack{u\in W^{1,p}(0,L)\\ \int_0^L |u|^{p-2}u=0}} \frac{\displaystyle\int_0^L |u'(x)|^p dx}{\displaystyle\int_0^L |u(x)|^p dx} = \left(\frac{\pi_p}{L}\right)^p. \tag{6}$$

For an insight into generalized Wirtinger inequalities, and more generally into weighted Hardy inequalities, we refer to [12, 15, 22], other results can be found for instance in [5, 6, 7, 8, 18]

One of the main tools in the proof of Proposition 1.2 is a refinement of the Pólya-Szegö principle which, to our knowledge, is completely new. We provide such a proof in Section 2 after recalling the notion of symmetric decreasing rearrangement and a short list of its main properties. Thereafter we also explain, through a simple example, to which extent our result can not be deduced by the classical one.

Next, in Section 3, we prove the weighted Wirtinger inequality stated in Proposition 1.2 which is obtained piling up Lemma 3.1, Lemma 3.2 and Lemma 3.3. Finally, in Section 4, we show that Proposition 1.2 implies Theorem 1.1. This last proof is given for the sake of completeness but it can be considered an adaptation when $p > 2$ of the proof considered when $p = 2$ in [16] and later in [2] (see also [1] for the 1-Laplacian).

2. A Pólya-Szegö type inequality

The aim of this section is the proof of a one dimensional Pólya-Szegö type principle. Therefore, first of all, we recall the notion of symmetric decreasing rearrangement. Let $u : \mathbb{R} \to \mathbb{R}$ be a measurable function, denoted by $\mu_u(t) = \mathcal{L}^1\{x \in \mathbb{R} : |u(x)| > t\}$ the distribution function of u, the symmetric decreasing rearrangement of u is the function

$$u^\sharp(x) = \left[\begin{array}{ll} \sup\limits_{\mathbb{R}} u & x = 0 \\ \inf\{t \geq 0 : \mu_u(t) < 2|x|\} & x \neq 0 \end{array} \right.$$

As an immediate consequence of its definition, the symmetric decreasing rearrangement $u^\sharp$ and the function u are equimeasurable, hence, if $u \in L^p(\mathbb{R})$ then

$$\|u\|_p = \|u^\sharp\|_p.$$

Another useful and not so obvious property is the well known Pólya-Szegö principle, according to which if $u \in W^{1,p}(\mathbb{R})$ is compactly supported, then

$$\|u'\|_p \geq \|u^{\sharp\prime}\|_p.$$

Under the same hypothesis on the function u, it is possible to prove that if

$$G(s,t) : (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$$

is a measurable nonnegative function, convex and nondecreasing in t, then

$$\int_{\mathbb{R}} G(|u|, |u'|) \geq \int_{\mathbb{R}} G(u^\sharp, |u^{\sharp\prime}|)$$

For more details on rearrangements see [11].

In the following we shall prove an inequality that reminds the Pólya-Szegö principle but to our knowledge is not covered by any known result in the literature.

Proposition 2.1. *Let u be in $W^{1,p}(\mathbb{R})$ $(p \geq 2)$ a compactly supported function. Then for all real κ*

$$\int_{\mathbb{R}} |u'(x) + \kappa u(x)|^p dx \geq \int_{\mathbb{R}} |u^{\sharp\prime}(x)|^p dx.$$

Proof. We can always assume that u is nonnegative, if not we consider its modulus. Moreover we suppose that u is a simple function (i.e.: continuous and piecewise linear). The general case can be deduced by approximation arguments.

We decompose the set $[0, \max u]$ in the union of finitely many intervals (a_i, a_{i+1}), $i = 0, ..., N$ $(N \in \mathbb{N})$ such that $a_i < a_{i+1}$. Denoting by $D_i = \{x \in \mathbb{R} : a_i < u(x) < a_{i+1}\}$, then we chose a_i such that D_i is the union of an even number of open intervals, namely $X_{i,j}$ (with $j = 1, ..., 2M_i$ and $M_i \in \mathbb{N}$) in each of which u is linear and non constant.

We also denote by $u'_{i,j}$ the constant value that the derivative of u takes on $X_{i,j}$ and, without loss of generality, we can assume $u'_{i,j} = (-1)^{j+1}|u'_{i,j}|$

Taking into account the definition of symmetric rearrangement, $u^\sharp$ is simple when u is simple. Moreover, denoting by $D_i^\sharp = \{x \in \mathbb{R} : a_i < u^\sharp(x) < a_{i+1}\}$, equimeasurability of u and $u^\sharp$ implies

$$\frac{2}{|u^{\sharp\prime}(x)|} = -\mu_u'(t) = \sum_{j=1}^{2M_i} \frac{1}{|u'_{i,j}|} \qquad x \in D_i^\sharp,\ a_i < t < a_{i+1}.$$

Now we can use the coarea formula to get

$$\begin{aligned}
&\int_{\mathbb{R}} |u'(x) + \kappa u(x)|^p dx \\
&\geq \int_{\mathbb{R}\cap\{u'\neq 0\}} |u'(x) + \kappa u(x)|^p dx \\
&= \int_0^\infty \left\{ \int_{\{u=t\}} \frac{|u' + \kappa t|^p}{|u'|} d\mathcal{H}^0 \right\} dt = \sum_{i=0}^N \int_{a_i}^{a_{i+1}} \left(\sum_{j=1}^{2M_i} \frac{|u'_{i,j} + \kappa t|^p}{|u'_{i,j}|} \right) dt \\
&= \sum_{i=0}^N \int_{a_i}^{a_{i+1}} \left(\sum_{j=1}^{M_i} \frac{\big||u'_{i,2j-1}| + \kappa t\big|^p}{|u'_{i,2j-1}|} + \frac{\big||u'_{i,2j}| - \kappa t\big|^p}{|u'_{i,2j}|} \right) dt \\
&\geq \sum_{i=0}^N \left[2^p (a_{i+1} - a_i) \sum_{j=1}^{M_i} \left(\frac{|u'_{i,2j-1} u'_{i,2j}|}{|u'_{i,2j-1}| + |u'_{i,2j}|} \right)^{p-1} \right] \\
&\geq \sum_{i=0}^N \left[2^p (a_{i+1} - a_i) M_i \left(\frac{1}{M_i} \sum_{j=1}^{M_i} \frac{|u'_{i,2j-1}| + |u'_{i,2j}|}{|u'_{i,2j-1} u'_{i,2j}|} \right)^{1-p} \right] \\
&\geq \sum_{i=0}^N \left[2^p (a_{i+1} - a_i) \left(\sum_{j=1}^{2M_i} |u'_{i,j}|^{-1} \right)^{1-p} \right] \\
&= \int_0^\infty \left\{ \int_{\{u^\sharp = t\}} |u^{\sharp\prime}|^{p-1} d\mathcal{H}^0 \right\} dt = \int_{\mathbb{R}} |u^{\sharp\prime}(x)|^p dx
\end{aligned}$$

Here we have used the fact that for $p \geq 2$

$$\frac{\big||u'_{i,2j-1}| + \kappa t\big|^p}{|u'_{i,2j-1}|} + \frac{\big||u'_{i,2j}| - \kappa t\big|^p}{|u'_{i,2j}|} \geq 2^p \left(\frac{|u'_{i,2j-1} u'_{i,2j}|}{|u'_{i,2j-1}| + |u'_{i,2j}|} \right)^{p-1}. \tag{7}$$

Infact, differentiating with respect to κ, the left hand side of (7) achieves the minimum when

$$\kappa t = \frac{|u'_{i,2j-1}|^{1/(p-1)} |u'_{i,2j}| - |u'_{i,2j-1}| |u'_{i,2j}|^{1/(p-1)}}{|u'_{i,2j-1}|^{1/(p-1)} + |u'_{i,2j}|^{1/(p-1)}}$$

hence

$$\frac{\left||u'_{i,2j-1}|+\kappa t\right|^p}{|u'_{i,2j-1}|}+\frac{\left||u'_{i,2j}|-\kappa t\right|^p}{|u'_{i,2j}|}\geq\frac{(|u'_{i,2j-1}|+|u'_{i,2j}|)^p}{\left(|u'_{i,2j-1}|^{1/(p-1)}+|u'_{i,2j}|^{1/(p-1)}\right)^{p-1}}.$$

Thereafter inequality (7) becomes a trivial consequence of the convexity of the power $p-1$ and of the arithmetic-geometric mean inequality. □

Arguing exactly as in the proof of Proposition 2.1, we can easily generalize the result and state the following Corollary.

Corollary 2.2. *Let u be in $W^{1,p}(\mathbb{R})$ ($p\geq 2$) a nonnegative compactly supported function. Given any measurable function f*

$$\int_{\mathbb{R}}|u'(x)+f(u(x))|^p dx\geq\int_{\mathbb{R}}|u^{\sharp\prime}(x)|^p dx.$$

Remark 2.3. We emphasize that the previous inequality is by no means a straightforward consequence of the classical Pólya-Szegö principle. Indeed the reader may wonder whether

$$\int_{\mathbb{R}}|u'(x)+f(u(x))|^p dx\geq\int_{\mathbb{R}}|u^{\sharp\prime}(x)+f(u^\sharp(x))|^p dx, \tag{8}$$

since it is trivially true when $p=2$. In fact the last inequality would imply Proposition 2.1 since, using the symmetry of $u^\sharp$

$$\begin{aligned}&\int_{\mathbb{R}}|u^{\sharp\prime}(x)+f(u^\sharp(x))|^p dx\\ =&\int_{\mathbb{R}^+}|u^{\sharp\prime}(x)+f(u^\sharp(x))|^p+|u^{\sharp\prime}(x)-f(u^\sharp(x))|^p dx\geq\int_{\mathbb{R}}|u^{\sharp\prime}(x)|^p.\end{aligned}$$

However, even if (8) holds true for $p=2$, it fails for $p>2$.

For instance we fix f identically equal to one and $p>2$. For any $0<\varepsilon<1$ we can define a continuous piecewise linear function u, which is compactly supported in $[-1/(1-\varepsilon),1]$ such that

$$u'(x)=\left[\begin{array}{ll}1-\varepsilon & x\in[-1/(1-\varepsilon),0]\\ -1 & x\in[0,1]\end{array}\right.$$

A simple calculation yields

$$\int_{\mathbb{R}}|u'(x)+f(u(x))|^p dx=2^p(1-\varepsilon(p/2-1))+o(\varepsilon)$$

and

$$\int_{\mathbb{R}}|u^{\sharp\prime}(x)+f(u^\sharp(x))|^p dx=2^p(1-(\varepsilon/2)(p/2-1))+o(\varepsilon).$$

Therefore for some ε small enough (8) is false.

3. Weighted Wirtinger inequality (proof of Proposition 1.2)

This section is devoted to the proof of Proposition 1.2. For the reader convenience we have split the claim in three Lemmata.

Lemma 3.1. *Let f be a nonnegative log-concave function defined on $[0,L]$ and $p>1$. Then there exists $\kappa\in\mathbb{R}$ such that*

$$\inf_{\substack{u\in W^{1,p}(0,L)\\ \int_0^L f|u|^{p-2}u=0}} \frac{\displaystyle\int_0^L f(x)|u'(x)|^p dx}{\displaystyle\int_0^L f(x)|u(x)|^p dx} \geq \min_{\substack{u\in W^{1,p}(0,L)\\ \int_0^L e^{\kappa x}|u|^{p-2}u=0}} \frac{\displaystyle\int_0^L e^{\kappa x}|u'(x)|^p dx}{\displaystyle\int_0^L e^{\kappa x}|u(x)|^p dx} \tag{9}$$

Proof. We assume that the function f is smooth and bounded away from 0 in $[0,L]$. The general case can be deduced by approximation arguments.
Under these assumptions the infimum on the left had side of (9) is achieved by a function u_λ belonging to $\{u\in W^{1,p}(0,L), \int_0^L f|u|^{p-2}u=0\}$, which is a solution to the following Neumann eigenvalue problem

$$\begin{cases}(-u'|u'|^{p-2})' = \lambda u|u|^{p-2} + h'(x)u'|u'|^{p-2} & x\in(0,L)\\ u'(0)=u'(L)=0.\end{cases}$$

Here $h(x)=\log f(x)$ is a smooth bounded concave function and λ is the left hand side of (9).

Standard arguments ensure that u_λ vanishes in one and only one point namely $x_\lambda\in(0,L)$ and without loss of generality we may assume that $u_\lambda(L)<0<u_\lambda(0)$.

We claim that if $\kappa=h'(x_\lambda)$ then

$$\lambda \geq \min_{\substack{u\in W^{1,p}(0,L)\\ \int_0^L e^{\kappa x}|u|^{p-2}u=0}} \frac{\displaystyle\int_0^L e^{\kappa x}|u'(x)|^p dx}{\displaystyle\int_0^L e^{\kappa x}|u(x)|^p dx} \equiv \bar\lambda.$$

Arguing by contradiction we assume that $\lambda<\bar\lambda$. Therefore there exists a function $u_{\bar\lambda}$ solution to

$$\begin{cases}(-u'|u'|^{p-2})' = \bar\lambda u|u|^{p-2} + h'(x_\lambda)u'|u'|^{p-2} & x\in(0,L)\\ u'(0)=u'(L)=0\end{cases}$$

Standard arguments ensures that $u_{\bar\lambda}$ vanishes in one and only one point namely $x_{\bar\lambda}\in(0,L)$, and we may always assume that $u_{\bar\lambda}(L)<0<u_{\bar\lambda}(0)$. Since h' is non increasing in $[0,L]$, a strightforward consequence of the comparison principle applied to u_λ and $u_{\bar\lambda}$ on the interval $[0,x_\lambda]$ enforces $x_{\bar\lambda}<x_\lambda$. On the other hand the comparison principle applied to u_λ and $u_{\bar\lambda}$ on the interval $[x_\lambda,L]$ enforces $x_{\bar\lambda}>x_\lambda$ and eventually a contradiction arises. □

In view of Lemma 3.1 we restrict our investigation to (log-concave functions) f of exponetial type (i.e. $f(x) = e^{\kappa x}$ for some $\kappa \in \mathbb{R}$). For such a class of functional we are able to prove that the first nontrivial Neumann eigenvalue equals the first Dirichlet eigenvalue, namely we prove

Lemma 3.2. *For all $\kappa \in \mathbb{R}$ and $p > 1$*

$$\min_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L e^{\kappa x}|u|^{p-2}u=0}} \frac{\displaystyle\int_0^L e^{\kappa x}|u'(x)|^p dx}{\displaystyle\int_0^L e^{\kappa x}|u(x)|^p dx} = \min_{u \in W_0^{1,p}(0,L)} \frac{\displaystyle\int_0^L e^{\kappa x}|u'(x)|^p dx}{\displaystyle\int_0^L e^{\kappa x}|u(x)|^p dx} \tag{10}$$

Proof. If v minimizes the left hand side of (10) then it solves

$$\begin{cases} (-u'|u'|^{p-2})' = \mu u|u|^{p-2} + \kappa u'|u'|^{p-2} & x \in (0,L) \\ u'(0) = u'(L) = 0, \end{cases}$$

where

$$\mu = \min_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L e^{\kappa x}|u|^{p-2}u=0}} \frac{\displaystyle\int_0^L e^{\kappa x}|u'(x)|^p dx}{\displaystyle\int_0^L e^{\kappa x}|u(x)|^p dx}.$$

We denote by $x_0 \in [0, L]$ the unique zero of v and by

$$w(x) = \left[\begin{array}{ll} \left|\dfrac{v(x+x_0)}{v(L)}\right| & x \in [0, L-x_0] \\ \left|\dfrac{v(x-L-x_0)}{v(0)}\right| & x \in [L-x_0, L]. \end{array}\right.$$

We observe that any minimizer of the right hand side of (10) is a solution to

$$\begin{cases} (-u'|u'|^{p-2})' = \nu u|u|^{p-2} + \kappa u'|u'|^{p-2} & x \in (0,L) \\ u(0) = u(L) = 0, \end{cases}$$

where

$$\nu = \min_{u \in W_0^{1,p}(0,L)} \frac{\displaystyle\int_0^L e^{\kappa x}|u'(x)|^p dx}{\displaystyle\int_0^L e^{\kappa x}|u(x)|^p dx}.$$

It is trivial to check that w is a solution of

$$\begin{cases} (-u'|u'|^{p-2})' = \mu u|u|^{p-2} + \kappa u'|u'|^{p-2} & x \in (0,L) \\ u(0) = u(L) = 0. \end{cases}$$

Hence $\mu \geq \nu$. On the other hand w is an eigenfunction having constant sign, and we can easily deduce by standard argument (for instance by using the maximum principle) that $\mu \leq \nu$. Therefore we have $\mu = \nu$. $\square$

Lemma 3.3. *Let $p \geq 2$ and let u be in $W_0^{1,p}(0, L)$. Then there exists a nonnegative function $\tilde{u} \in W_0^{1,p}(0, L)$ such that*

$$\int_0^L e^{\kappa x} |u'(x)|^p dx \geq \int_0^L |\tilde{u}'(x)|^p dx$$

$$\int_0^L e^{\kappa x} |u(x)|^p dx = \int_0^L |\tilde{u}(x)|^p dx$$

for all $\kappa \in \mathbb{R}$.

Proof. Setting $v(x) = u(x) e^{\frac{\kappa}{p} x}$ we have

$$\int_0^L e^{\kappa x} |u'(x)|^p dx = \int_0^L \left| v(x)' - \frac{k}{p} v(x) \right|^p dx$$

and

$$\int_0^L e^{\kappa x} |u(x)|^p dx = \int_0^L |v(x)|^p dx.$$

The claim can be easily deduced from Proposition 2.1 once we chose $\tilde{u}(x + L/2) = v^\sharp(x) \equiv (e^{\frac{\kappa}{p} x} |u(x)|)^\sharp$. $\square$

4. Reduction to one dimensional case (proof of Theorem 1.1)

The aim of this section is to prove that Theorem 1.1 can be deduced from Proposition 1.2. As we already mentioned the idea is based on a slicing method worked out in [16] and proved in a slightly different way also in [2, 1]. We outline the technique for the sake of completeness.

Lemma 4.1. *Let Ω be a convex set in $\mathbb{R}^n$ having diameter d, and let u be any function such that $\int_\Omega |u(x)|^{p-2} u(x) dx = 0$. Then, for all positive ε, there exists a decomposition of the set Ω in mutually disjoint convex sets Ω_i $(i = 1, ..., k)$ such that*

$$\bigcup_{i=1}^k \bar{\Omega}_i = \bar{\Omega}$$

$$\int_{\Omega_i} |u(x)|^{p-2} u(x) dx = 0$$

and for each i there exists a rectangular system of coordinates such that

$$\Omega_i \subseteq \{(x_1, ..., x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_\ell| \leq \varepsilon, \ell = 2, ..., n\} \quad (d_i \leq d, i = 1, ..., k)$$

Proof. Among all the $n-1$ hyperplanes of the form $ax_1 + bx_2 = c$, orthogonal to the 2-plane $\Pi_{1,2}$ generated by the directions x_1 and x_2, by continuity there exists certainly one that divides Ω into two nonempty subsets on each of which the integral of $u|u|^{p-2}$ is zero and their projections on $\Pi_{1,2}$ have the same area. We go on subdividing recursively in the same way both subsets and eventually we stop when all the subdomains $\Omega_j^{(1)}$ $(j = 1, ..., 2^{N_1})$ have projections with area smaller then $\varepsilon^2/2$. Since the width w of a planar set of area A is bounded by the trivial inequality $w \le \sqrt{2A}$, each subdomain $\Omega_j^{(1)}$ can be bounded by two parallel $n-1$ hyperplanes of the form $ax_1 + bx_2 = c$ whose distance is less than ε. If $n = 2$ the proof is complete, provided that we understand $\Pi_{1,2}$ as $\mathbb{R}^2$, the projection of Ω on $\Pi_{1,2}$ as Ω itself, and the $n-1$ orthogonal hyperplanes as lines. If $n > 2$ for any given $\Omega_j^{(1)}$ we can consider a rectangular system of coordinates such that the normal to the above $n-1$ hyperplanes which bound the set, points in the direction x_n. Then we can repeat the previous arguments and subdivide the set $\Omega_i^{(1)}$ in subsets $\Omega_j^{(2)}$ $(j = 1, ..., 2^{N_2})$ on each of which the integral of $u|u|^{p-2}$ is zero and their projections on $\Pi_{1,2}$ have the same area which is less then $\varepsilon^2/2$. Therefore, any given $\Omega_j^{(2)}$, can be bounded by two $n-1$ hyperplanes of the form $ax_1 + bx_2 = c$ whose distance is less than ε. If $n = 3$ the proof is over. If $n > 3$ we can go on considering $\Omega_j^{(2)}$ and rotating the coordinate system such that the normal to the above $n-1$ hyperplanes which bound $\Omega_j^{(2)}$, points in the direction x_{n-1} and such that the rotation keeps the x_n direction unchanged. The procedure ends after $n-1$ iterations, at that point we have perfomed $n-1$ rotations of the coordinate system and all the directions have been fixed. Up to a translation, in the resulting coordinate system

$$\Omega_j^{(n-1)} \subseteq \{(x_1, ..., x_n) \in \mathbb{R}^n : 0 \le x_1 \le d_i, |x_\ell| \le \varepsilon, \ell = 2, ..., n\} \qquad \square$$

Proof of Theorem 1.1. From (2), using the density of smooth functions in Sobolev spaces it will be enough to prove that

$$\frac{\displaystyle\int_\Omega |Du|^p}{\displaystyle\int_\Omega |u|^p} \ge \left(\frac{\pi_p}{d}\right)^p$$

when u is a smooth function with bounded second derivatives and $\int_\Omega |u(x)|^{p-2}u(x)dx = 0$.

Let u be any such function. According to Lemma 4.1 we fix ε and we decompose the set Ω in convex domains Ω_i $(i = 1, ..., k)$. We use the notation of Lemma 4.1 and we focus on one of the subdomains Ω_i and fix the reference system such that

$$\Omega_i \subseteq \{(x_1, ..., x_n) \in \mathbb{R}^n : 0 \le x_1 \le d_i, |x_\ell| \le \varepsilon, \ell = 2, ..., n\}.$$

For $t \in [0, d_i]$ we denote by $f_i(t)$ the $n-1$ volume of the intersection of Ω_i with the $n-1$ hyperplane $x_1 = t$. Since Ω_i is convex, then by Brunn-Minkowski inequality (see [10]) f is a log concave function in $[0, d_i]$.

Therefore, if for any $t \in [0, d_i]$ we denote by $v(t) = u(t, 0, ..., 0)$, and by C any constant depending just on $||u||_{C^2}$, we have

$$\left| \int_{\Omega_i} \left| \frac{\partial u}{\partial x_1} \right|^p dx - \int_0^{d_i} f_i(t) |v'(t)|^p \, dt \right| \leq C|\Omega_i|\varepsilon, \tag{11}$$

$$\left| \int_{\Omega_i} |u|^p \, dx - \int_0^{d_i} f_i(t) |v(t)|^p \, dt \right| \leq C|\Omega_i|\varepsilon \tag{12}$$

and

$$\left| \int_0^{d_i} f_i(t) |v(t)|^{p-2} v(t) \, dt \right| \leq C|\Omega_i|\varepsilon. \tag{13}$$

Since $d_i \leq d$, applying Proposition 1.2 we have

$$\int_{\Omega_i} |Du|^p \, dx \geq \int_{\Omega_i} \left| \frac{\partial u}{\partial x_1} \right|^p dx \geq \left(\frac{\pi_p}{d} \right)^p \int_{\Omega_i} |u(x)|^p \, dx + C\varepsilon|\Omega_i|.$$

We sum up the last inequality over $i = 1, ..., k$

$$\int_{\Omega} |Du|^p \, dx \geq \left(\frac{\pi_p}{d} \right)^p \int_{\Omega} |u(x)|^p \, dx + C\varepsilon|\Omega|.$$

and we let $\varepsilon \to 0$ to obtain the desired inequality. □

References

[1] G. Acosta, R. G. Durán: An optimal Poincaré inequality in L^1 for convex domains, Proc. Amer. Math. Soc. 132(1) (2004) 195–202.

[2] M. Bebendorf: A note on the Poincaré inequality for convex domains, Z. Anal. Anwend. 22(4) (2003) 751–756.

[3] P. Binding, L. Boulton, J. Cepicka, P. Drabek, P. Girg: Basis properties of eigenfunctions of the p-Laplacian, Proc. Amer. Math. Soc. 134 (2006) 3487–3494.

[4] S. K. Chua, R. Wheeden: Estimates of best constants for weighted Poincaré inequalities on convex domains, Proc. Lond. Math. Soc. 93 (2006) 197–226.

[5] G. Croce, B. Dacorogna: On a generalized Wirtinger inequality, Discrete Contin. Dyn. Syst. A 5 (2003) 1329–1341.

[6] B. Dacorogna, W. Gangbo, N. Subia: Sur une généralisation de l'inégalité de Wirtinger, Ann. Inst. H. Poincaré, Anal. Non Linéaire 9 (1992) 29–50.

[7] F. Farroni, R. Giova, T. Ricciardi: Best constants and extremals for a vector Poincaré inequality with weights, Sci. Math. Jpn. 71 (2010) 111–126.

[8] R. Giova, T. Ricciardi: A sharp Wirtinger inequality and some related functional spaces, Bull. Belg. Math. Soc. – Simon Stevin 17 (2010) 1–10.

[9] G. H. Hardy, J. E. Littlewood, G. Pólya: Inequalities. Reprint of the 1952 Ed.. Cambridge Mathematical Library. Cambridge University Press, Cambridge (1988).

[10] R. J. Gardner: The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. 39(2) (2002) 355–405.

[11] B. Kawohl: Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Mathematics 1150, Springer, Berlin (1985).

[12] A. Kufner, L. E. Persson: Weighted Inequalities of Hardy Type, World Scientific, River Edge (2003).

[13] P. Li, S. T. Yau: Estimates of eigenvalues of a compact Riemannian manifold, in: Geometry of the Laplace Operator (Honolulu, 1979), R. Osserman, A. Weinstein (eds.), Proc. Sympos. Pure Math. 36, Amer. Math. Soc., Providence (1980) 205–239.

[14] P. Lindqvist: Some remarkable sine and cosine functions, Ric. Mat. 44(2) (1995) 269–290.

[15] V. Maz'ya: Sobolev Spaces, Springer, Berlin (2011).

[16] L. E. Payne, H. F. Weinberger: An optimal Poincaré inequality for convex domains, Arch. Ration. Mech. Anal. 5 (1960) 286–292.

[17] W. Reichel, W. Walter: Sturm-Liouville type problems for the p-Laplacian under asymptotic non-resonance conditions, J. Differ. Equations 156(1) (1999) 50–70.

[18] T. Ricciardi: A sharp weighted Wirtinger inequality, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8) 8 (2005) 259–267.

[19] R. Schoen, S. T. Yau: Lectures on Differential Geometry, Conference Proceedings and Lecture Notes in Geometry and Topology 1, International Press, Cambridge (1994).

[20] A. Stanoyevitch: Geometry of Poincaré Domains, Doctoral Dissertation, Univ. of Michigan (1990).

[21] A. Stanoyevitch: Products of Poincaré Domains, Proc. Amer. Math. Soc. 117(1) (1993) 79–87.

[22] B. O. Turesson: Nonlinear Potential Theory and Weighted Sobolev Spaces, Lecture Notes in Mathematics 1736, Springer, Berlin (2000).

[23] D. Valtorta: Sharp estimate on the first eigenvalue of the p-Laplacian, Nonlinear Anal. 75(13) (2012) 4974–4994.

[24] H. Zhang: Lower bounds for the first eigenvalue of the p-Laplace operator on compact manifolds with nonnegative Ricci curvature, Adv. Geom. 7 (2007) 145–155.

[25] J. Q. Zhong, H. C. Yang: On the estimate of the first eigenvalue of a compact Riemannian manifold, Sci. Sin., Ser. A 27 (1984) 1265–1273.