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Attractive Point Theorems for Generalized Nonspreading Mappings in Banach Spaces

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In this paper, we introduce the concept of attractive points of a nonlinear mapping in a Banach space and obtain some fundamental properties for the points. Using these results, we prove attractive point theorems for generalized nonspreading mappings in a Banach space. Using these results, we also obtain some results for skew-generalized nonspreading mappings in a Banach space. Finally, we prove nonlinear ergodic theorems without convexity for generalized nonspreading mappings in a Banach space. These results extend attractive point theorems which were proved by Takahashi and Takeuchi [42] in Hilbert spaces to Banach spaces.

Keywords: Attractive point, Banach space, fixed point, generalized nonspreading mapping, skew-generalized nonspreading mapping

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of fixed points of T and by $A(T)$ the set of attractive points [42] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = Z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

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A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. Let C be a nonempty subset of H. A mapping $F : C \to H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [6] and Goebel and Kirk [9]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [4] and [7]. Recently, Kohsaka and Takahashi [31], and Takahashi [41] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [31] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [41] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [30] and Iemoto and Takahashi [20]. Recently, Kocourek, Takahashi and Yao [26] defined a broad class of nonlinear mappings containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is called generalized hybrid [26] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. Then, Kocourek, Takahashi and Yao [26] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem which generalizes Baillon's nonlinear ergodic theorem [3] in a Hilbert space. Very recently, Takahashi and Takeuchi [42] proved the following fixed point and mean convergence theorem without convexity in a Hilbert space.

Theorem 1.1. *Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$v_1 \in C, \qquad v_{n+1} = Tv_n, \qquad b_n = \frac{1}{n}\sum_{k=1}^{n} v_k$$

for all $n \in \mathbb{N}$. If $\{v_n\}$ is bounded, then the following hold:

(1) *$A(T)$ is nonempty, closed and convex;*

(2) *$\{b_n\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)}v_n$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.*

In this paper, we introduce the concept of attractive points of a nonlinear mapping in a Banach space and obtain some fundamental properties for the points.

Using these results, we prove attractive point theorems for generalized nonspreading mappings in a Banach space. Using these results, we also obtain some results for skew-generalized nonspreading mappings in a Banach space. Finally, we prove nonlinear ergodic theorems without convexity for generalized nonspreading mappings in a Banach space. These results extend attractive point theorems which were proved by Takahashi and Takeuchi [42] in Hilbert spaces to Banach spaces.

2. Preliminaries

Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{z \in C : Tz = z\}$. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [21]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If

E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [37, 38]. The following result is also well known; see [37].

Lemma 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [25]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x-z, Jz-Jy\rangle \tag{3}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x-y, Jz-Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{4}$$

for $x, y, z, w \in E$. Let $\phi_*\colon E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{5}$$

for $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{6}$$

The following theorems are in Xu [46] and Kamimura and Takahashi [25].

Lemma 2.2 (Xu [46]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 (Kamimura and Takahashi [25]). *Let E be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* of T, i.e., $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x),\ \forall x \in C\}$.

Lemma 2.4. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

Proof. We show that $A(T)$ is closed. Let $\{z_n\} \subset A(T)$ be a sequence which converges strongly to $z \in E$. Then we have from $z_n \in A(T)$ that

$$\phi(z_n, Tx) \leq \phi(z_n, x), \quad \forall x \in C.$$

From the definition of ϕ, we have

$$\|z_n\|^2 - 2\langle z_n, JTx\rangle + \|Tx\|^2 \leq \|z_n\|^2 - 2\langle z_n, Jx\rangle + \|x\|^2, \quad \forall x \in C.$$

Since $z_n \to z$, we have that

$$\phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C.$$

This implies $z \in A(T)$. Next, we prove that $A(T)$ is convex. Let $z_1, z_2 \in A(T)$, $\alpha \in [0, 1]$ and $z = \alpha z_1 + (1-\alpha)z_2$. Then we have that for any $x \in C$,

$$\begin{aligned}
\phi(z, Tx) &= \|z\|^2 - 2\alpha\langle z_1, JTx\rangle - 2(1-\alpha)\langle z_2, JTx\rangle + \|Tx\|^2 \\
&= \|z\|^2 + \alpha\phi(z_1, Tx) + (1-\alpha)\phi(z_2, Tx) - \alpha\|z_1\|^2 - (1-\alpha)\|z_2\|^2 \\
&\leq \|z\|^2 + \alpha\phi(z_1, x) + (1-\alpha)\phi(z_2, x) - \alpha\|z_1\|^2 - (1-\alpha)\|z_2\|^2 \\
&= \|z\|^2 - 2\alpha\langle z_1, Jx\rangle - 2(1-\alpha)\langle z_2, Jx\rangle + \|x\|^2 \\
&= \phi(z, x).
\end{aligned}$$

This implies $z \in A(T)$. $\square$

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized nonexpansive [16] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [13, 15, 16] for more details. The following results are in Ibaraki and Takahashi [16].

Lemma 2.5 (Ibaraki and Takahashi [16]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.6 (Ibaraki and Takahashi [16]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(*i*) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz \rangle \leq 0$ *for all* $y \in C$;

(*ii*) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [29] proved the following results:

Lemma 2.7 (Kohsaka and Takahashi [29]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(*a*) *C is a sunny generalized nonexpansive retract of E;*

(*b*) *C is a generalized nonexpansive retract of E;*

(*c*) *JC is closed and convex.*

Lemma 2.8 (Kohsaka and Takahashi [29]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(*i*) $z = Rx$;

(*ii*) $\phi(x, z) = min_{y \in C} \phi(x, y)$.

Very recently, Ibaraki and Takahashi [19] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.9 (Ibaraki and Takahashi [19]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.7 and 2.9.

Lemma 2.10 (Ibaraki and Takahashi [19]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

3. Attractive Point Theorems

In this section, we try to extend Takahashi and Takeuchi's attractive point theorem [42] in a Hilbert space to Banach spaces. Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. Then, a mapping $T : C \to E$ is called generalized nonspreading [27] if there exist

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{7}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. For example, a (1,1,1,0)-generalized nonspreading mapping is a nonspreading mapping in the sense of Kohsaka and Takahashi [31], i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x), \quad \forall x, y \in C.$$

Let T be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (7), we obtain

$$\phi(u, Ty) + \gamma\{\phi(Ty, u) - \phi(Ty, u)\} \leq \phi(u, y) + \delta\{\phi(y, u) - \phi(y, u)\}.$$

So, we have that

$$\phi(u, Ty) \leq \phi(u, y) \tag{8}$$

for all $u \in F(T)$ and $y \in C$. Thus for such a mapping, we have $F(T) \subset A(T)$. Furthermore, if E is a Hilbert space, then we have $\phi(x, y) = \|x - y\|^2$ for $x, y \in E$. So, from (7) we obtain the following:

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \\ &\leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\} \end{aligned} \tag{9}$$

for all $x, y \in C$. This implies that

$$\begin{aligned} &(\alpha+\gamma)\|Tx - Ty\|^2 + \{1 - (\alpha+\gamma)\}\|x - Ty\|^2 \\ &\leq (\beta+\delta)\|Tx - y\|^2 + \{1 - (\beta+\delta)\}\|x - y\|^2 \end{aligned} \tag{10}$$

for all $x, y \in C$. That is, T is a generalized hybrid mapping [26] in a Hilbert space. Now, using the technique developed by [36], we prove an attractive point theorem for generalized nonspreading mappings in a Banach space.

Theorem 3.1. *Let E be a smooth and reflexive Banach space. Let C be a nonempty subset of E and let T be a generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(a) $A(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Let T be a generalized nonspreading mapping of C into itselt. Then, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\}\\ \le\ &\beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}\end{aligned}$$

for all $x, y \in C$. If $A(T) \neq \emptyset$, then $\phi(u, Ty) \le \phi(u, y)$ for all $u \in A(T)$ and $y \in C$. Thus we have $\phi(u, T^n x) \le \phi(u, x)$ for all $n \in \mathbb{N}$ and $x \in C$. This means $(a) \Longrightarrow (b)$. Let us show $(b) \Longrightarrow (a)$. Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Then for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}&\alpha\phi(T^{k+1}x, Ty) + (1-\alpha)\phi(T^k x, Ty) + \gamma\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\}\\ \le\ &\beta\phi(T^{k+1}x, y) + (1-\beta)\phi(T^k x, y) + \delta\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}\\ =\ &\beta\{\phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle\}\\ &\quad + (1-\beta)\{\phi(T^k x, Ty) + \phi(Ty, y) + 2\langle T^k x - Ty, JTy - Jy\rangle\}\\ &\quad + \delta\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.\end{aligned} \tag{11}$$

This implies that

$$\begin{aligned}0 \le\ &(\beta-\alpha)\{\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\} + \phi(Ty, y)\\ &+ 2\langle \beta T^{k+1}x + (1-\beta)T^k x - Ty, JTy - Jy\rangle\\ &- \gamma\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} + \delta\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.\end{aligned} \tag{12}$$

Summing up these inequalities (12) with respect to $k = 0, 1, \dots, n-1$, we have

$$\begin{aligned}0 \le\ &(\beta-\alpha)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + n\phi(Ty, y)\\ &+ 2\langle x + Tx + \dots + T^{n-1}x + \beta(T^n x - x) - nTy, JTy - Jy\rangle\\ &- \gamma\{\phi(Ty, T^n x) - \phi(Ty, x)\} + \delta\{\phi(y, T^n x) - \phi(y, x)\}.\end{aligned} \tag{13}$$

Dividing by n in (13), we have

$$\begin{aligned}0 \le\ &\frac{1}{n}(\beta-\alpha)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + \phi(Ty, y)\\ &+ 2\left\langle S_n x + \frac{1}{n}\beta(T^n x - x) - Ty, JTy - Jy\right\rangle\\ &- \frac{1}{n}\gamma\{\phi(Ty, T^n x) - \phi(Ty, x)\} + \frac{1}{n}\delta\{\phi(y, T^n x) - \phi(y, x)\},\end{aligned} \tag{14}$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Since $\{T^n x\}$ is bounded by assumption, $\{S_n x\}$ is bounded. Thus we have a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ such that $\{S_{n_i} x\}$ converges weakly to a point $u \in E$. Letting $n_i \to \infty$ in (14), we obtain

$$0 \le \phi(Ty, y) + 2\langle u - Ty, JTy - Jy\rangle. \tag{15}$$

Using (4), we obtain

$$
\begin{aligned}
0 &\le \phi(Ty,y) + 2\langle u - Ty, JTy - Jy\rangle \\
&= \phi(Ty,y) + \phi(u,y) + \phi(Ty,Ty) - \phi(u,Ty) - \phi(Ty,y) \\
&= \phi(u,y) - \phi(u,Ty).
\end{aligned} \tag{16}
$$

Hence we have $\phi(u,Ty) \le \phi(u,y)$ and then $u \in A(T)$. Therefore $A(T)$ is nonempty. Additionally, assume that E is strictly convex and C is closed and convex. Then if a subsequence $\{S_{n_i}x\}$ of $\{S_n x\}$ converges weakly to a point $u \in E$, then $u \in C$ because C is weakly closed. Putting $y = u$ in (15), we obtain

$$
\begin{aligned}
0 &\le \phi(Tu,u) + 2\langle u - Tu, JTu - Ju\rangle \\
&= \phi(Tu,u) + \phi(u,u) + \phi(Tu,Tu) - \phi(u,Tu) - \phi(Tu,u) \\
&= -\phi(u,Tu).
\end{aligned} \tag{17}
$$

Hence $\phi(u,Tu) = 0$. Since E is strictly convex, we have $u \in F(T)$. Therefore $F(T)$ is nonempty. It is obvious that if $F(T) \neq \emptyset$, then $\{T^n x\}$ is bounded for some $x \in C$. This completes the proof. □

Using Theorem 3.1, we have the following attractive point theorems in a Banach space. The first result is a generalization of Kohsaka and Takahashi [31].

Theorem 3.2. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a nonspreading mapping* [31]*, i.e.,*

$$\phi(Tx,Ty) + \phi(Ty,Tx) \le \phi(Tx,y) + \phi(Ty,x)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $A(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (7), we obtain that

$$\phi(Tx,Ty) + \phi(Ty,Tx) \le \phi(Tx,y) + \phi(Ty,x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 3.1. □

Theorem 3.3. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a hybrid mapping* [41]*, i.e.,*

$$2\phi(Tx,Ty) + \phi(Ty,Tx) \le \phi(Tx,y) + \phi(Ty,x) + \phi(x,y)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $A(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = 1$, $\beta = \gamma = \frac{1}{2}$ and $\delta = 0$ in (7), we obtain that

$$2\phi(Tx,Ty) + \phi(Ty,Tx) \le \phi(Tx,y) + \phi(Ty,x) + \phi(x,y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 3.1. □

Theorem 3.4. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping such that*

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $A(T) \neq \emptyset$;
(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\gamma = \delta = 0$ in (7), we obtain that

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$. So, we have the desired result from Theorem 3.1. □

As a direct consequence of Theorem 3.4, we have Takahashi and Takeuchi's attractive point theorem [42] in a Hilbert space.

Theorem 3.5 (Takahashi and Takeuchi [42]). *Let C be a nonempty subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \qquad (18)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $A(T) \neq \emptyset$;
(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. We know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in C$ in Theorem 3.4. So, we have the desired result from Theorem 3.4. □

4. Skew-Attractive Point Theorems

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $T : C \to E$ be a generalized nonspreading mapping; see (7). This mapping has the property that if $u \in F(T)$ and $x \in C$, then $\phi(u, Tx) \leq \phi(u, x)$. This property can be revealed by putting $x = u \in F(T)$ in (7). Similarly, putting $y = u \in F(T)$ in (7), we obtain that for $x \in C$,

$$\begin{aligned} &\alpha\phi(Tx, u) + (1 - \alpha)\phi(x, u) + \gamma\{\phi(u, Tx) - \phi(u, x)\} \\ &\leq \beta\phi(Tx, u) + (1 - \beta)\phi(x, u) + \delta\{\phi(u, Tx) - \phi(u, x)\} \end{aligned} \qquad (19)$$

and hence

$$(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0. \qquad (20)$$

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u).$$

Motivated by this property of T and $F(T)$, we give the following defintition. Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $B(T)$ the set of *skew-attractive points* of T, i.e., $B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z),\ \forall x \in C\}$.

Lemma 4.1. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $B(T)$ is closed.*

Proof. We show that $B(T)$ is closed. Let $\{z_n\} \subset B(T)$ with $z_n \to z$. Then we have that

$$\phi(Tx, z_n) \leq \phi(x, z_n), \quad \forall x \in C,\ n \in \mathbb{N}.$$

This implies that

$$\|Tx\|^2 - 2\langle Tx, Jz_n\rangle + \|z_n\|^2 \leq \|x\|^2 - 2\langle x, Jz_n\rangle + \|z_n\|^2.$$

Since E is smooth, J is norm-to-weak* continuous. So, we have from $z_n \to z$ that

$$\|Tx\|^2 - 2\langle Tx, Jz\rangle + \|z\|^2 \leq \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2.$$

Thus we have $\phi(Tx, z) \leq \phi(x, z)$ for all $x \in C$. This implies $z \in B(T)$. Therefore $B(T)$ is closed. $\square$

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the duality mapping of T; see also [44] and [11]. It is easy to show that if T is a mapping of C into itselt, then T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then, T^* is a mapping of JC into itself. We can prove the following result in a Banach space.

Lemma 4.2. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E and let T^* be the duality mapping of T. Then, the following hold:*

(1) $JB(T) = A(T^*)$;

(2) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Proof. *(1)* Let $z^* \in E^*$. We have from the definition of T^* that

$$\begin{aligned}
z^* \in JB(T) &\Longleftrightarrow J^{-1}z^* \in B(T) \\
&\Longleftrightarrow \phi(Tx, J^{-1}z^*) \leq \phi(x, J^{-1}z^*), \quad \forall x \in C \\
&\Longleftrightarrow \phi_*(z^*, JTx) \leq \phi_*(z^*, Jx), \quad \forall x \in C \\
&\Longleftrightarrow \phi_*(z^*, JTJ^{-1}Jx) \leq \phi_*(z^*, Jx), \quad \forall x \in C \\
&\Longleftrightarrow \phi_*(z^*, T^*Jx) \leq \phi_*(z^*, Jx), \quad \forall x \in C \\
&\Longleftrightarrow z^* \in A(T^*).
\end{aligned}$$

This implies that $JB(T) = A(T^*)$.

(2) Similarly, we have from the definition of T^* that

$$\begin{aligned}
z^* \in JA(T) &\Longleftrightarrow J^{-1}z^* \in A(T) \\
&\Longleftrightarrow \phi(J^{-1}z^*, Tx) \leq \phi(J^{-1}z^*, x), \quad \forall x \in C \\
&\Longleftrightarrow \phi_*(JTx, z^*) \leq \phi_*(Jx, z^*), \quad \forall x \in C \\
&\Longleftrightarrow \phi_*(JTJ^{-1}Jx, z^*) \leq \phi_*(Jx, z^*), \quad \forall x \in C \\
&\Longleftrightarrow \phi_*(T^*Jx, z^*) \leq \phi_*(Jx, z^*), \quad \forall x \in C \\
&\Longleftrightarrow z^* \in B(T^*).
\end{aligned}$$

This implies that $JA(T) = B(T^*)$. We know from Lemma 2.4 that $A(T^*)$ is closed and convex. So, $JB(T)$ is closed and convex. □

Let E be a smooth Banach space, let J be the duality mapping from E into E^* and let C be a nonempty subset of E. A mapping $T : C \to E$ is called skew-generalized nonspreading [12] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\
&\leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\} \qquad (21)
\end{aligned}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspreading mapping. For example, a (1,1,1,0)-skew-generalized nonspreading mapping is a skew-nonspreading mapping in the sense of Ibaraki and Takahashi [18], i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx), \quad \forall x, y \in C.$$

Theorem 4.3. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a skew-generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(a) $B(T) \neq \emptyset$;

(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Additionally, if C is closed and JC is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Let T be a skew-generalized nonspreading mapping of C into itselt. Then, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ &\leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\} \end{aligned}$$

for all $x, y \in C$. If $B(T) \neq \emptyset$, then $\phi(Ty, u) \leq \phi(y, u)$ for all $u \in B(T)$ and $y \in C$. Thus we have that $\phi(T^n x, u) \leq \phi(x, u)$ for all $n \in \mathbb{N}$ and $x \in C$. This implies $(a) \Longrightarrow (b)$. Let us show $(b) \Longrightarrow (a)$. Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Then for any $x^*, y^* \in JC$ with $x^* = Jx$ and $y^* = Jy$ and $T^* = JTJ^{-1}$, we have from (5) that

$$\begin{aligned} &\alpha\phi_*(T^*x^*, T^*y^*) + (1-\alpha)\phi_*(x^*, T^*y^*) + \gamma\{\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)\} \\ &= \alpha\phi_*(JTx, JTy) + (1-\alpha)\phi_*(Jx, JTy) + \gamma\{\phi_*(JTy, JTx) - \phi_*(JTy, Jx)\} \\ &= \alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\beta\phi_*(T^*x^*, y^*) + (1-\beta)\phi_*(x^*, y^*) + \delta\{\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)\} \\ &= \beta\phi_*(JTx, Jy) + (1-\beta)\phi_*(Jx, Jy) + \delta\{\phi_*(Jy, JTx) - \phi_*(Jy, Jx)\} \\ &= \beta\phi(JTx, Jy) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\}. \end{aligned}$$

Since T is skew-generalized nonspreading, we have that

$$\begin{aligned} &\alpha\phi_*(T^*x^*, T^*y^*) + (1-\alpha)\phi_*(x^*, T^*y^*) + \gamma\{\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)\} \\ &\leq \beta\phi_*(T^*x^*, y^*) + (1-\beta)\phi_*(x^*, y^*) + \delta\{\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)\}. \end{aligned}$$

This implies that T^* is a generalized nonspreading mapping of JC into itself. Furthermore, we have that

$$(JTJ^{-1})^n Jx = JT^n x$$

for each $x \in C$ and $n \in \mathbb{N}$. In fact, for any $x \in C$, we have $JTx = JTJ^{-1}Jx$. So, the equality is true in the case of $k = 1$. Suppose that

$$(JTJ^{-1})^k Tx = JT^k x$$

for some $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} (JTJ^{-1})^{k+1}Tx &= (JTJ^{-1})(JTJ^{-1})^k Tx \\ &= JTJ^{-1}JT^k x \\ &= JTT^k x \\ &= JTT^{k+1} x. \end{aligned}$$

So, the equality is true in the case of $k+1$. Hence

$$\|T^n x\| = \|JT^n x\| = \|(JTJ^{-1})^n Jx\|$$

for each $x \in C$ and $n \in \mathbb{N}$. Thus if $\{T^n x\}$ is bounded for some $x \in C$, then $\{(T^*)^n Jx\}$ is bounded. We have from Theorem 3.1 that $A(T^*)$ is nonempty. We also know from Lemma 4.2 that $A(T^*) = JB(T)$. Therefore $B(T)$ is nonempty. Additionally, assume that C is closed and JC is closed and convex. If $\{T^n x\}$ is bounded for some $x \in C$, then $\{(T^*)^n Jx\}$ is bounded. So, we have from Theorem that 3.1 that $F(T^*)$ is nonempty. As in the proof of Lemma 4.2, we have that $JF(T) = F(T^*)$. So, $F(T)$ is nonempty. The converse is obvious. This completes the proof. $\square$

Using Theorem 4.3, we have the following skew-attractive point theorems in a Banach space. The first result is a generalization of Dhompongsa, Fupinwong, Takahashi and Yao [8].

Theorem 4.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a skew-nonspreading mapping, i.e.,*

$$\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $B(T) \neq \emptyset$;
(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (21), we obtain that

$$\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.3. $\square$

Theorem 4.5. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping such that*

$$2\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty) + \phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $B(T) \neq \emptyset$;
(*b*) $\{T^n x\}$ *is bounded for some* $x \in C$.

Proof. Putting $\alpha = 1$, $\beta = \gamma = \frac{1}{2}$ and $\delta = 0$ in (21), we obtain that

$$2\phi(Ty, Tx) + \phi(Tx, Ty) \leq \phi(y, Tx) + \phi(x, Ty) + \phi(y, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.3. $\square$

Theorem 4.6. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping such that*

$$\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) \le \beta\phi(y, Tx) + (1-\beta)\phi(y, x)$$

for all $x, y \in C$. Then, the following are equivalent:

(*a*) $B(T) \neq \emptyset$;

(*b*) $\{T^n x\}$ *is bounded for some $x \in C$.*

Proof. Putting $\gamma = \delta = 0$ in (21), we obtain that

$$\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) \le \beta\phi(y, Tx) + (1-\beta)\phi(y, x)$$

for all $x, y \in C$. So, we have the desired result from Theorem 4.3. □

5. Nonlinear Ergodic Theorems

In this section, we can prove the following nonlinear ergodic theorem without convexity for generalized nonspreading mappings in a Banach space.

Theorem 5.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $A(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. We know from Lemmas 4.1 and 4.2 that $B(T)$ is closed, and $JB(T)$ is closed and convex. So, from Lemma 2.7 there exists the sunny generalized nonexpansive retraction R of E onto $B(T)$. From Lemma 2.8, this retraction R is characterized by

$$Rx = \arg\min_{u \in B(T)} \phi(x, u).$$

We also know from Lemma 2.6 that

$$0 \le \langle v - Rv, JRv - Ju \rangle, \quad \forall u \in B(T),\ v \in C.$$

Adding up $\phi(Rv, u)$ to both sides of this inequality, we have

$$\begin{aligned} \phi(Rv, u) &\le \phi(Rv, u) + 2\langle v - Rv, JRv - Ju \rangle \\ &= \phi(Rv, u) + \phi(v, u) + \phi(Rv, Rv) - \phi(v, Rv) - \phi(Rv, u) \\ &= \phi(v, u) - \phi(v, Rv). \end{aligned} \tag{22}$$

Since $\phi(Tz, u) \leq \phi(z, u)$ for any $u \in B(T)$ and $z \in C$, it follows that

$$\begin{aligned}\phi(T^n x, RT^n x) &\leq \phi(T^n x, RT^{n-1}x) \\ &\leq \phi(T^{n-1}x, RT^{n-1}x).\end{aligned}$$

Hence the sequence $\phi(T^n x, RT^n x)$ is nonincreasing. Putting $u = RT^n x$ and $v = T^m x$ with $n \leq m$ in (22), we have from Lemma 2.3 that

$$\begin{aligned}g(\|RT^m x - RT^n x\|) &\leq \phi(RT^m x, RT^n x) \\ &\leq \phi(T^m x, RT^n x) - \phi(T^m x, RT^m x) \\ &\leq \phi(T^n x, RT^n x) - \phi(T^m x, RT^m x),\end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RT^n x\}$ is a Cauchy sequence. Therefore $\{RT^n x\}$ converges strongly to a point $q \in B(T)$ since $B(T)$ is closed from Lemma 4.1. Next, consider a fixed $x \in C$ and an arbitrary subsequence $\{S_{n_i}x\}$ of $\{S_n x\}$ convergent weakly to a point v. From the proof of the attractive point theorem (Theorem 3.1) we know that $v \in A(T)$. Rewriting the characterization of the retraction R, we have that for any $u \in B(T)$,

$$0 \leq \left\langle T^k x - RT^k x, JRT^k x - Ju \right\rangle$$

and hence

$$\begin{aligned}\left\langle T^k x - RT^k x, Ju - Jq \right\rangle &\leq \left\langle T^k x - RT^k x, JRT^k x - Jq \right\rangle \\ &\leq \|T^k x - RT^k x\| \cdot \|JRT^k x - Jq\| \\ &\leq K\|JRT^k x - Jq\|,\end{aligned}$$

where K is an upper bound for $\|T^k x - RT^k x\|$. Summing up these inequalities for $k = 0, 1, \ldots, n-1$, we arrive to

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} RT^k x, Ju - Jq \right\rangle \leq K\frac{1}{n}\sum_{k=0}^{n-1} \|JRT^k x - Jq\|,$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Letting $n_i \to \infty$ and remembering that J is continuous, we get

$$\langle v - q, Ju - Jq \rangle \leq 0.$$

This holds for any $u \in B(T)$. Therefore $Rv = q$. But because $v \in A(T) = B(T)$, we have $v = q$. Thus the sequence $\{S_n x\}$ converges weakly to the point q. □

Using Theorem 5.1, we obtain the following theorems.

Theorem 5.2 (Kocourek, Takahashi and Yao [27]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume* *that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. We also know that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. We also note that $A(T) = F(T)$ and $B(T) = F(T)$. So, we have the desired result from Theorem 5.1. $\square$

Theorem 5.3 (Takahashi and Takeuchi [42]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $A(T) \neq \emptyset$ and let P be the mertic projection of H onto $A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $A(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof. We first note that $A(T) = B(T)$. Since $A(T) = B(T)$ is a nonempty closed convex subset of H, there exists the metric projection of H onto $A(T)$. In a Hilbert space, the metric projection of H onto $A(T)$ is equivalent to the sunny generalized nonexpansive retraction of E onto $A(T)$. So, we have the desired result from Theorem 5.1. $\square$

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