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Uniformly Metrizable Bornologies

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This paper is devoted to study the following question: given a bornology $\mathbf{B}$ on a metric space (X, d), when is there a metric d' on X, uniformly equivalent to d, for which the bornology of d'-bounded sets coincides with $\mathbf{B}$? If that happens, we will say that $\mathbf{B}$ is uniformly metrizable. We characterize these kinds of bornologies on X, and we obtain necessary and sufficient conditions in order to some of the most common bornologies enjoy this property. More precisely, we will consider the compact bornology, the totally bounded bornology and the bornology of the Bourbaki-bounded sets.

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1. Introduction

Bounded sets play an important role in the frame of metric spaces and normed vector spaces. For that reason this concept has been extended to some different settings, like topological vector spaces, uniform spaces or even the general context of topological spaces. In spite of the fact that there are different generalizations of the notion of bounded set, we can observe that all of them define what is called a *bornological universe*. Recall that a bornological universe is a pair $(X, \mathbf{B})$ where X is a topological space and $\mathbf{B}$ is a bornology on X, that is, a family of subsets of X that satisfies the following conditions:

- For every $x \in X$, the set $\{x\} \in \mathbf{B}$.
- If $B \in \mathbf{B}$ and $A \subset B$, then $A \in \mathbf{B}$.
- If $A, B \in \mathbf{B}$ then $A \cup B \in \mathbf{B}$.

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The elements in $\mathbf{B}$ are called the *bounded sets* in the bornological universe $(X, \mathbf{B})$.

Bornological universes have shown to be very useful in the study of some convergence structures on hyperspaces as well as in optimization theory (see for instance [5] and references therein). The most typical example of bornological universe is obtained when we consider a metric space (X, d) together with the family $\mathbf{B}_d(X)$ of its metric bounded sets, i.e., the sets having finite diameter. In fact, a natural question in this frame consists of characterizing which bornological universes $(X, \mathbf{B})$ have this form, i.e., there is a metric d, compatible with the topology on X, such that $\mathbf{B} = \mathbf{B}_d(X)$. The problem of metrizability of bornologies was studied and solved in a nice way by Hu in [12]. An interesting study about the uniformizability of bornological universes has been recently done by Vroegrijk in [15].

In this paper, we are mainly interested in the following related problem: given a metric space (X, d) and a bornology $\mathbf{B}$ on X, we want to know if there is some metric d' on X, uniformly equivalent to d, such that $\mathbf{B} = \mathbf{B}_{d'}(X)$. In that case we will say that the bornology $\mathbf{B}$ is uniformly metrizable (Definition 2.3). This problem has been also considered by Beer in [3] but in a different approach. Namely, he uses the so-called *metric modes of convergence to infinity*, which provides a kind of dual point of view to boundedness.

We present here, in Section 2, a characterization of uniformly metrizable bornologies. This characterization allows us to obtain, in a unified way, different results about the uniform metrizability of some of the most common bornologies. We refer to the compact bornology (Section 3), the totally bounded bornology (Section 4) and the bornology of the Bourbaki-bounded sets (Section 5). In particular, we can derive as corollaries some of the already known results in this context.

2. Uniform metrizability for bornologies

Let (X, d) be a metric space. We denote by $B(x_0, r)$ the open ball of center $x_0 \in X$ and radius $r > 0$, i.e., $B(x_0, r) = \{x : d(x_0, x) < r\}$. Analogously, we denote the closed ball $B[x_0, r] = \{x : d(x_0, x) \leq r\}$. For a nonempty subset A of X, its *ε-enlargement* with respect to d, is defined by

$$A^\varepsilon = \{x \in X : d(x, A) < \varepsilon\}$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$. That is, $A^\varepsilon = \bigcup_{a \in A} B(a, \varepsilon)$.

Given a bornology $\mathbf{B}$ on X, a family $\mathbf{C} \subset \mathbf{B}$ is said to be a *base* for $\mathbf{B}$ whenever for every $B \in \mathbf{B}$ there is $C \in \mathbf{C}$ such that $B \subset C$, i.e., every set in $\mathbf{B}$ is contained in some set of $\mathbf{C}$. For instance, the (countable) family $\mathbf{C} = \{B(x_0, n) : n \in \mathbb{N}\}$, of the open balls of center $x_0 \in X$ and radius $n \in \mathbb{N}$, is a base for the bornology $\mathbf{B}_d(X)$ of the metric bounded sets in the metric space (X, d).

For further notation and basic results on topological and uniform spaces we refer to the books of Engelking [6] and Kelley [8].

Now we recall the notion of *characteristic function* that Hu gives in [12] in order to obtain his metrization theorem for bornological universes. That notion was also considered by Beer in [3] under the name of *forcing function*.

Definition 2.1. A *characteristic function* of a bornological universe $(X, \mathbf{B})$ is any continuous function $\chi : X \to [0, \infty)$ such that a subset $E \subset X$ is bounded if, and only if, $\chi(E)$ is bounded in $[0, \infty)$, that is,

$$\mathbf{B} = \{E \subset X : \chi(E) \subset [0, K], \text{ for some } K > 0\}.$$

Proposition 2.2. *Let $\mathbf{B}$ be a bornology on the metric space (X, d). Then, $(X, \mathbf{B})$ admits a uniformly continuous characteristic function if, and only if, $\mathbf{B}$ has a countable base $\{B_n : n \in \mathbb{N}\}$, such that for some $\delta > 0$,*

$$B_n{}^{\delta} \subset B_{n+1}, \quad \textit{for every } n \in \mathbb{N}.$$

Proof. First, suppose that $\mathbf{B}$ has a countable base satisfying the above hypotheses. Then, for each $n \in \mathbb{N}$, we define the function $\phi_n : X \to [0, 1]$ by

$$\phi_n(x) = \min \left\{1, \frac{1}{\delta}\, d(x, B_n)\right\}.$$

Clearly, $\phi_n(B_n) = 0$, $\phi_n(X - B_{n+1}) = 1$ and ϕ_n is uniformly continuous, in fact it is a $1/\delta$-Lipschitz function. Now if we put $B_0 = \emptyset$ and $\phi_0(x) = 1$, for all $x \in X$, we can consider $\chi : X \to [0, \infty)$ defined by,

$$\chi(x) = n - 2 + \phi_{n-1}(x)$$

when $x \in B_n - B_{n-1}$, $n \in \mathbb{N}$. In this way, χ is a uniformly continuous characteristic function of $(X, \mathbf{B})$. Indeed, if $d(x, y) < \delta$, there is $n \in \mathbb{N}$ such that both x, y are in $B_n - B_{n-1}$, or $x \in B_n - B_{n-1}$ and $y \in B_{n+1} - B_n$. In any case, it is easy to check that

$$|\chi(x) - \chi(y)| \leq |\phi_{n-1}(x) - \phi_{n-1}(y) + \phi_n(x) - \phi_n(y)| \leq \frac{2}{\delta}\, d(x, y).$$

And so, χ is not only uniformly continuous but it is in fact $\frac{2}{\delta}-$Lipschitz on each $\delta-$ball. In particular, χ is what is called a *Lipschitz in the small* function (see Garrido and Jaramillo, [7]).

Finally, if $E \in \mathbf{B}$ there exists $n \in \mathbb{N}$ such that $E \subset B_n$. So $\chi(E) \subset [0, n-1]$. On the other hand, if $\chi(E)$ is bounded then $E \subset B_n$ for some $n \in \mathbb{N}$, because otherwise for every $n \in \mathbb{N}$ there will be $x \in E$ such that $\chi(x) \geq n$.

Conversely, suppose that $(X, \mathbf{B})$ admits a uniformly continuous characteristic function χ. Then from the uniform continuity of χ, there exists some $\delta > 0$ such that $d(x, y) < \delta$ implies $|\chi(x) - \chi(y)| \leq 1$. Now if we take, $B_n = \chi^{-1}([0, n])$, $n \in \mathbb{N}$, then it is easy to see that the family $\{B_n\}_n$ is a countable base for $\mathbf{B}$, satisfying the required property for this δ. □

Definition 2.3. Let $\mathbf{B}$ be a bornology on the metric space (X, d). We say that $\mathbf{B}$ is *uniformly metrizable* if there exists a metric d' on X, uniformly equivalent to d, such that $\mathbf{B} = \mathbf{B}_{d'}(X)$.

Next we give a characterization of uniform metrizability of bornologies, which is a natural extension of the metrization theorem by Hu, which we spoke of in the introduction. An analogous result was given by Beer in [3] in the setting of metric spaces having a metric mode of convergence to infinity.

Theorem 2.4. *Let $\mathbf{B}$ be a bornology on the metric space (X, d). Then, the following conditions are equivalent:*

(*1*) $\mathbf{B}$ *is uniformly metrizable.*

(*2*) $\mathbf{B}$ *has a countable base $\{B_n : n \in \mathbb{N}\}$, such that for some $\delta > 0$ we have*

$$B_n{}^{\delta} \subset B_{n+1}, \quad \text{for every } n \in \mathbb{N}.$$

(*3*) *There exists a uniformly continuous characteristic function for $(X, \mathbf{B})$.*

Proof. According to Proposition 2.2, conditions (*2*) and (*3*) are equivalent.

(*1*) $\Rightarrow$ (*2*). If $\mathbf{B} = \mathbf{B}_{d'}(X)$, for some metric d' uniformly equivalent to d, then for a fixed $x_0 \in X$ the family,

$$\big\{B_n = \{x \in X : d'(x, x_0) < n\},\, n \in \mathbb{N}\big\}$$

is clearly a countable base for $\mathbf{B}$. On the other hand, from the uniform equivalence between d and d', and for $\varepsilon = 1$ there exists some $\delta > 0$ such that $d'(x, y) < 1$ whenever $d(x, y) < \delta$. And therefore condition (*2*) follows for this δ.

(*2*) $\Rightarrow$ (*1*). If (*2*) holds, we can define a new metric $d' : X \times X \to \mathbb{R}$, by:

$$d'(x, y) = d^*(x, y) \vee |\chi(x) - \chi(y)|$$

where $d^* = \min\{1, d\}$, $\vee$ denotes the maximum, and χ is the uniformly continuous characteristic function associated to $\mathbf{B}$ that appears in the proof of Proposition 2.2.

To show that d' and d are uniformly equivalent it is enough to prove that d' and d^* are uniformly equivalent. Indeed, since $d^* \le d'$ the uniformity $\mathbf{U}_{d^*}$ is coarser than $\mathbf{U}_{d'}$. Conversely, using the property seen in the proof of Proposition 2.2 that $d(x, y) < \delta$ imply $|\chi(x) - \chi(y)| < \frac{2}{\delta} d(x, y)$, we have that

$$d'(x, y) = d(x, y) \vee |\chi(x) - \chi(y)| \le \max\{1, 2/\delta\} \cdot d(x, y)$$

whenever $d(x, y) < \min\{1, \delta\}$, and this gives the equivalence between both metrics.

Finally, the equality $\mathbf{B} = \mathbf{B}_{d'}(X)$ is again an easy consequence of Proposition 2.2. $\square$

Remark 2.5. Note that with a little modification in the above proof of (*2*) $\Rightarrow$ (*1*), we can get a metric d' which is not only uniformly equivalent to d but in fact uniformly locally identical to d. Recall that two metrics on X are *uniformly locally identical* when they are uniformly equivalent and coincide on some entourage U of the diagonal of $X \times X$ (see Williamson and Janos, [16]). Indeed, if we consider

$$d'(x, y) = d^*(x, y) \vee (\delta/2)\, |\chi(x) - \chi(y)|$$

then it is clear that the metric d' also satisfies condition (*1*) in Theorem 2.4, and in addition $d'(x, y) = d(x, y)$ whenever $d(x, y) < \min\{1, \delta\}$.

The remainder of the paper will be devoted to apply Theorem 2.4 to some of the most usual bornologies on a metric space, in order to know when they are uniformly metrizable. Thus, we start by considering the simplest bornology on any space X, that is the bornology $\mathbf{FB}(X)$ of the finite subsets of X. Recall that a metric space (X, d) is said to be *locally finite* if every ball in it has finitely many elements, i.e, whenever $\mathbf{FB}(X) = \mathbf{B}_d(X)$. This class of metric spaces contains in particular the metric graphs and also the so-called metric spaces with bounded geometry.

In this line, we define a metric space (X, d) to be *uniformly locally finite* whenever there exists some $\delta > 0$ such that every open ball of radius δ is finite. It is clear that locally finite implies uniformly locally finite, but the converse is not true. For instance, if we consider the discrete metric on $\mathbb{N}$, then every 1/2-ball is finite but not every ball is finite.

Theorem 2.6. *Let (X, d) be a metric space. The following conditions are equivalent:*

(*1*) $\mathbf{FB}(X)$ *is uniformly metrizable.*

(*2*) *There exists d', uniformly equivalent to d, such that (X, d') is locally finite.*

(*3*) *(X, d) is countable and uniformly locally finite.*

Proof. It is clear that conditions (*1*) and (*2*) are equivalent.

(*2*) $\Rightarrow$ (*3*). Let d' be a metric uniformly equivalent to d, such that (X, d') is locally finite. Since $X = \bigcup_n B_{d'}(x_0, n)$, for any fixed $x_0 \in X$, we have that X is a countable union of finite sets and then X is countable. On the other hand, from the uniform equivalence between d and d', and for $\varepsilon = 1$ there exists some $\delta > 0$ such that $d'(x, y) < 1$ whenever $d(x, y) < \delta$. Therefore, for every $x \in X$, we have that $B_d(x, \delta) \subset B_{d'}(x, 1)$, and then every open δ-ball in (X, d) is finite. Hence, (X, d) is uniformly locally finite.

(*3*) $\Rightarrow$ (*1*). Suppose $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ is countable and choose $\delta > 0$ such that every open ball of radius δ in (X, d) is finite. Now, if we consider $B_1 = \{x_1\}$ and for every $n \geq 2$,

$$B_n = \{x_n\} \cup (B_{n-1})^\delta$$

then the family $\mathbf{B} = \{B_n : n \in \mathbb{N}\}$ is a base for the bornology $\mathbf{FB}(X)$ satisfying condition (*2*) in Theorem 2.4, and then $\mathbf{FB}(X)$ is uniformly metrizable. $\square$

As a easy consequence of above result we can deduce that the usual metric on the subset $X = \{1/n : n \in \mathbb{N}\}$ of the real line does not admit any uniformly equivalent metric making it locally finite.

3. The compact bornology and the Heine-Borel Property

This section is devoted to study the so-called *compact bornology* $\mathbf{CB}(X)$, which is the bornology generated by the compact subsets of X. Thus, a set $B \subset X$ belongs to $\mathbf{CB}(X)$ if it is contained in some compact subset of X, or equivalently, when $\mathrm{cl}_X B$ (the closure of B in X) is compact. Recall that a topological space is said to be *hemicompact* whenever the family of all compact subspaces ordered by the inclusion admits a countable cofinal subfamily, that is whenever $\mathbf{CB}(X)$ has a countable base.

We denote by $\mathbf{CB}_d(X)$, when we consider the compact bornology on the metric space (X, d). It is clear that this bornology $\mathbf{CB}_d(X)$ is uniformly metrizable when there exists a metric d' on X, uniformly equivalent to d, in such a way that (X, d') satisfies the Heine-Borel property. We say that the metric space (X, d') satisfies the *Heine-Borel property* when every closed and bounded subset is compact, that is, when $\mathbf{CB}_{d'}(X) = \mathbf{B}_{d'}(X)$.

The problem of the metrizability of the compact bornology was studied by Vaughan in [14], where he proved that a metric space (X, d) admits a topologically equivalent metric with the Heine-Borel property if, and only if, it is locally compact and separable (or equivalently, Lindelöf). Note that for metric spaces, or more generally for first countable spaces, to be locally compact and Lindelöf is equivalent to be hemicompact (see e.g. Engelking [6]). And therefore, it follows that the compact bornology is metrizable if, and only if, it has a countable base.

Now we are going to establish, from our Theorem 2.4, the uniform version of the Vaughan's result. First, recall that a uniform space $(X, \mathbf{U})$ is *uniformly locally compact* if there is an entourage $U \in \mathbf{U}$ such that $U[x]$ is a compact subspace of X, for every $x \in X$. In particular, a metric space (X, d) is uniformly locally compact whenever there exists some $\delta > 0$ such that, for every $x \in X$, the closed ball $B[x, r]$ of center x and radius δ is compact.

Theorem 3.1. *Let (X, d) be a metric space. The following conditions are equivalent:*

(*1*) $\mathbf{CB}_d(X)$ *is uniformly metrizable.*

(*2*) *There exists d', uniformly equivalent to d, with the Heine-Borel property.*

(*3*) (X, d) *is Lindelöf and uniformly locally compact.*

Proof. (*1*) $\Rightarrow$ (*2*). Let d' be a metric on X, uniformly equivalent to d, such that $\mathbf{CB}_d(X) = \mathbf{B}_{d'}(X)$. Since compact sets coincide for both equivalent metrics, then $\mathbf{CB}_{d'}(X) = \mathbf{B}_{d'}(X)$, and this means that d' satisfies the Heine-Borel property.

(*2*) $\Rightarrow$ (*3*). If d and d' are uniformly equivalent, then both spaces (X, d) and (X, d') have the same uniform and topological properties. On the other hand, it is clear that every metric space with the Heine-Borel property is uniformly locally compact, for whatever $\delta > 0$. Finally, (X, d') is Lindelöf since X can be written as a countable union of compact sets, namely $X = \bigcup_n B[x_0, n]$, $x_0 \in X$.

(3) $\Rightarrow$ *(1)*. Let $\delta > 0$ such that every closed ball of radius δ is compact. Then if $C \subset X$ is compact, then $C^{\delta/2} \in \mathbf{CB}_d(X)$. Indeed, from the open cover of $C \subset \bigcup_{x \in C} B(x, \delta/2)$, we can take a finite subcover $C \subset \bigcup_{i=1}^n B(x_i, \delta/2)$. Since

$$C^{\delta/2} \subset \bigcup_{i=1}^{n} B(x_i, \delta/2)^{\delta/2} \subset \bigcup_{i=1}^{n} B[x_i, \delta]$$

and $\bigcup_{i=1}^n B[x_i, \delta]$ is compact, it follows that $C^{\delta/2} \in \mathbf{CB}_d(X)$.

On the other hand, it is not difficult to prove that every locally compact and Lindelöf space is hemicompact (see Engelking [6]), that is, there exists a countable family of compact sets $\{C_n\}_n$ which is a base for $\mathbf{CB}_d(X)$. Now we are going to construct another countable base $\{B_n\}_n$ for this bornology satisfying condition *(2)* in Theorem 2.4. First, take $B_1 = C_1$. Since ${B_1}^{\delta/2}$ is in $\mathbf{CB}_d(X)$, then there exists C_{n_1} containing ${B_1}^{\delta/2}$. If we define $B_2 = C_{n_1} \cup C_2$, then B_2 is a compact set with ${B_1}^{\delta/2} \subset B_2$ and $C_2 \subset B_2$. Now repeating this argument, we get $B_1, B_2, \dots B_n \dots$ such that, for every $n \in \mathbb{N}$,

$$ {B_n}^{\delta/2} \subset B_{n+1} \quad \text{and} \quad C_{n+1} \subset B_{n+1}. $$

And we finish, since $\{B_n\}_n$ is a desired base for $\mathbf{CB}_d(X)$, and then $\mathbf{CB}_d(X)$ is uniformly metrizable. □

Remark 3.2. The equivalence between *(2)* and *(3)* was directly proved by Potter in [13], without using bornologies.

Now we are going to apply Theorem 3.1 above in order to derive easily a result given by Williamson and Janos in [16]. In this way, we illustrate how it is possible to obtain topological and/or uniform results from bornological results.

Corollary 3.3 (Williamson and Janos, [16]). *A metric space (X, d) admits a metric d', with the Heine-Borel property, which is uniformly locally identical to d if, and only if, (X, d) is uniformly locally compact and σ-compact.*

Proof. First note that in the context of locally compactness, σ-compactness is equivalent to Lindelöfness. Moreover, from Remark 2.5, it is the same for a bornology on (X, d) to be uniformly metrizable with a metric uniformly equivalent to d, and to be uniformly metrizable with a metric uniformly locally identical to d. And therefore the result follows at once from Theorem 3.1. □

Since every uniformly locally compact metric space (X, d) is in particular complete, then Theorem 3.1 says that completeness is a necessary condition in order to $\mathbf{CB}_d(X)$ be uniformly metrizable. For instance, if we consider the usual metric on the real open interval $(0, 1)$, then its compact bornology is metrizable (according to the mentioned Vaughan's result) but not uniformly metrizable.

The next example provides a complete Lindelöf locally compact metric space (X, d) that is not uniformly locally compact. Therefore $\mathbf{CB}_d(X)$ is metrizable but not uniformly metrizable.

Example 3.4. Let ℓ_2 be the classical Banach space of square summable real sequences endowed with its usual norm $\|\cdot\|$, and let $B = \{e_n : n \in \mathbb{N}\}$ be the canonical basis. Consider $X = \bigcup_n A_n$ the countable subspace of ℓ_2, where

$$A_n = \{e_n\} \cup \left\{e_n + \frac{1}{2^n} e_k : k \in \mathbb{N}\right\}.$$

Then $(X, d_{\|\cdot\|})$ is locally compact. Indeed, if $x \in X$ then x belongs to some A_n and in this case the open ball $B(x, 1/2^n) = \{x\}$. On the other hand, X is not uniformly locally compact. Note that for every $n \in \mathbb{N}$, the closed ball of center e_n and radius $1/2^n$ coincides with the whole set A_n, that is infinite and discrete, and therefore noncompact. Finally X is complete since the only Cauchy sequences are those being eventually constant.

4. The totally bounded bornology

Next, we are going to study the bornology $\mathbf{TB}(X)$ of all totally bounded subsets of X. A subset B in a uniform space $(X, \mathbf{U})$ is said to be *totally bounded* or *precompact* if for every $U \in \mathbf{U}$ there exists a finite set $F \subset X$ such that $B \subset U[F]$. In particular, in a metric space (X, d), a subset B is precompact if for every $\varepsilon > 0$ there is a finite set $F = \{x_1, \ldots, x_n\}$ such that $B \subset F^\varepsilon = \bigcup_{i=1}^n B(x_i, \varepsilon)$. We will denote by $\mathbf{TB}_d(X)$ the totally bounded bornology on the metric space (X, d).

It is easy to check that if B is precompact then its closure in X, $\mathrm{cl}_X B$, is also precompact. On the other hand, since a subset in a uniform space X is compact whenever it is precompact and complete, we can deduce immediately that the compact bornology and the totally bounded bornology coincide if (and only if) X is complete. Thus, if we consider the usual metric d_u on $\mathbb{Q}$, the set of rational numbers, then we have,

$$\mathbf{CB}_{d_u}(\mathbb{Q}) \neq \mathbf{TB}_{d_u}(\mathbb{Q}) = \mathbf{B}_{d_u}(\mathbb{Q}).$$

And clearly $\mathbf{TB}_{d_u}(\mathbb{Q})$ is uniformly metrizable, but $\mathbf{CB}_{d_u}(\mathbb{Q})$ is not uniformly metrizable according to Theorem 3.1.

In order to characterize the uniform metrizability of $\mathbf{TB}_d(X)$ in the non complete case, we need the following lemma that is essentially contained in [4]. In particular, this result gives the relationship between the bornology $\mathbf{TB}_d(X)$ on the metric space (X, d) and the corresponding bornology $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$ on its completion $(\widetilde{X}, \tilde{d})$.

Lemma 4.1. *Let (X, d) be a metric space and let $(\widetilde{X}, \tilde{d})$ be its completion. Then,*

(i) For each $C \in \mathbf{TB}_{\tilde{d}}(\widetilde{X})$, there is some $B \in \mathbf{TB}_d(X)$ such that $C \subset \mathrm{cl}_{\widetilde{X}} B$.

(ii) If $\{B_i\}_{i \in I}$ is a base for $\mathbf{TB}_d(X)$, then $\{\mathrm{cl}_{\widetilde{X}} B_i\}_{i \in I}$ is a base for $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$.

Proof. (i) Let $C \in \mathbf{TB}_{\tilde{d}}(\widetilde{X})$ be. Then, for every $k \in \mathbb{N}$, we can take a finite set F_k in $\widetilde{X}$ such that $C \subset F_k{}^{1/(2k)}$. Without loss of generality, we can suppose that $F_k \subset C$, for all k.

Now, from the density of X in $\widetilde{X}$, we choose for each point $y \in F_k$ some point $x \in X$, with $\tilde{d}(x, y) < 1/(2k)$. In that way, we get a finite set B_k in X, such that

$$F_k \subset B_k{}^{1/(2k)} \quad \text{and} \quad B_k \subset F_k{}^{1/(2k)}$$

for every $k \in \mathbb{N}$. If we define $B = \bigcup B_k$, then it is not difficult to see that $C \subset \mathrm{cl}_{\widetilde{X}} B$. Indeed, let $y \in C$, $\varepsilon > 0$ and take $k \in \mathbb{N}$ with $1/k < \varepsilon$. Since $C \subset F_k{}^{1/(2k)} \subset B_k{}^{1/k}$, it follows that

$$\tilde{d}(y, B) \leq \tilde{d}(y, B_k) < 1/k < \varepsilon.$$

Finally, we are going to check that B belongs to $\mathbf{TB}_d(X)$. Let $\varepsilon > 0$ and take $n_0 \in \mathbb{N}$ with $1/n_0 < \varepsilon/2$. Since $B_k \subset F_k{}^{1/(2k)} \subset C^{1/(2k)}$, for every $k \in \mathbb{N}$, then if $k \geq n_0$, we have $B_k \subset C^{1/(2k)} \subset C^{1/(2n_0)} \subset (B_{n_0})^{3/(2n_0)} \subset (B_{n_0})^{\varepsilon}$. And we finish since $B_1 \cup \cdots \cup B_{n_0}$ is a finite set and,

$$B = \bigcup B_k \subset (B_1 \cup \cdots \cup B_{n_0})^{\varepsilon}.$$

(ii) It is clear that $\mathrm{cl}_{\widetilde{X}} B_i \in \mathbf{TB}_{\tilde{d}}(\widetilde{X})$, for every $B_i \in \mathbf{TB}_d(X)$. On the other hand, if C is a precompact in $\widetilde{X}$, then for (i) there exists some $B \in \mathbf{TB}_d(X)$ with $C \subset \mathrm{cl}_{\widetilde{X}} B$. Now, if B_i is an element of the given base containing B, then $C \subset \mathrm{cl}_{\widetilde{X}} B_i$, as we wanted. □

From the above lemma it follows at once that the bornology $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$ has a countable base if, and only if, $\mathbf{TB}_d(X)$ has a countable base. Since $\mathbf{TB}_{\tilde{d}}(\widetilde{X}) = \mathbf{CB}_{\tilde{d}}(\widetilde{X})$, the above mentioned Vaughan's metrization result says that $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$ is metrizable if, and only, it has a countable base. On the other hand, it is easy to see that if $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$ is metrizable then $\mathbf{TB}_d(X)$ is also metrizable. Therefore, we obtain that $\mathbf{TB}_d(X)$ is metrizable if, and only if, it has a countable base. This result was given by Beer, Constantini and Levi in [4].

Now, in order to obtain the uniform metrizability of $\mathbf{TB}_d(X)$, we recall that a uniform space $(X, \mathbf{U})$ is said to be *uniformly locally totally bounded* if there is an entourage $U \in \mathbf{U}$ such that $U[x]$ is a totally bounded subspace of X, for every $x \in X$. In particular, a metric space (X, d) is uniformly locally totally bounded whenever there exists some $\delta > 0$ such that every ball of radius δ is totally bounded.

Theorem 4.2. *Let (X, d) be a metric space and let $(\widetilde{X}, \tilde{d})$ be its completion. Then, the following are equivalent:*

(1) $\mathbf{TB}_d(X)$ *is uniformly metrizable.*

(2) $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$ *is uniformly metrizable.*

(3) $(\widetilde{X}, \tilde{d})$ *is Lindelöf and uniformly locally compact.*

(4) (X, d) *is Lindelöf and uniformly locally totally bounded.*

Proof. (1) $\Rightarrow$ (2). From Theorem 2.4, condition (1) implies that $\mathbf{TB}_d(X)$ admits a countable base $\{B_n\}_n$ such that, for some $\delta > 0$, ${B_n}^{\delta} \subset B_{n+1}$, for all n. Now, according to Lemma 4.1, we have that $\{\mathrm{cl}_{\widetilde{X}} B_n\}_n$ is a countable base for $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$. Moreover, this base satisfies that,

$$(\mathrm{cl}_{\widetilde{X}} B_n)^{\delta/2} \subset \mathrm{cl}_{\widetilde{X}} B_{n+1},$$

for every n. Indeed, it is not difficult to prove that in fact when $A, B \subset X$ with $A^{\delta} \subset B$, then $(\mathrm{cl}_{\widetilde{X}} A)^{\delta/2} \subset \mathrm{cl}_{\widetilde{X}} B$. And finally, again from Theorem 2.4, we deduce that $\mathbf{TB}_{\tilde{d}}(\widetilde{X})$ is uniformly metrizable.

(2) $\Rightarrow$ (1). If $\hat{d}$ is a metric on $\widetilde{X}$ uniformly equivalent to $\tilde{d}$ such that $\mathbf{TB}_{\tilde{d}}(\widetilde{X}) = \mathbf{B}_{\hat{d}}(\widetilde{X})$, then it is clear that the restriction to X, $d' = \hat{d}_{|X \times X}$ is a metric uniformly equivalent to d, such that $\mathbf{TB}_d(X) = \mathbf{B}_{d'}(X)$. Hence (1) follows.

(2) $\Leftrightarrow$ (3). This equivalence follows directly from Theorem 3.1, since for $(\widetilde{X}, \tilde{d})$ we have that $\mathbf{TB}_{\tilde{d}}(\widetilde{X}) = \mathbf{CB}_{\tilde{d}}(\widetilde{X})$.

(3) $\Rightarrow$ (4). It is clear.

(4) $\Rightarrow$ (3). Firstly, note that X is Lindelöf if, and only if, $\widetilde{X}$ is Lindelöf. On the other hand, let $\delta > 0$ such that every δ-ball in X is totally bounded. Then every closed $\delta/2$-ball in $\widetilde{X}$ is compact. Indeed, let $y \in \widetilde{X}$, then there is $x \in X$ with $\tilde{d}(x, y) < \delta/2$. Then we have,

$$B_{\tilde{d}}[y, \delta/2] \subset B_{\tilde{d}}(x, \delta) \subset \mathrm{cl}_{\widetilde{X}}[B_{\tilde{d}}(x, \delta) \cap X] = \mathrm{cl}_{\widetilde{X}} B_d(x, \delta).$$

And we finish since $\mathrm{cl}_{\widetilde{X}} B_d(x, \delta)$ is totally bounded in $\widetilde{X}$ and then it is compact. $\square$

5. The Bourbaki-bounded bornology

Our last section will be devoted to the bornology $\mathbf{BB}(X)$ of the Bourbaki-bounded sets. This kind of sets were introduced by Atsuji [1] in the frame of metric spaces, under the name of *finitely chainable* sets. Later on, in order to get a notion of boundedness stable under uniformly continuous functions, Hejcman [9] defines these sets in the general setting of uniform spaces. The name of Bourbaki-bounded set cames from the book of Bourbaki [2], where the study of certain properties of these sets is proposed as an exercise.

Recall that a subset B in a uniform space $(X, \mathbf{U})$ is said to be *Bourbaki-bounded* if for every $U \in \mathbf{U}$ there exist a finite set $F \subset X$ and some $m \in \mathbb{N}$ such that $B \subset U^m[F]$, where $U^1 = U$, $U^2 = U \circ U$, and in general $U^m = U \circ (U^{m-1})$. In a metric space (X, d), B is Bourbaki-bounded if for every $\varepsilon > 0$ there are a finite set $F = \{x_1, \ldots, x_n\} \subset X$ and some $m \in \mathbb{N}$ such that

$$B \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)^m.$$

Here a point y belongs to $B(x,\varepsilon)^m$ means that there exists an *ε-chain of length m* joining the points x and y, i.e., there exist points $a_0 = x, a_1, \ldots, a_m = y$ in X with $d(a_i, a_{i+1}) < \varepsilon$, for $i = 0, \ldots (m-1)$. We will denote by $\mathbf{BB}_d(X)$ the bornology of Bourbaki-bounded sets in the metric space (X, d).

It is easy to see that if in the definition of Bourbaki-bounded set, the number m is always 1 (or even $m \le m_0$, for some m_0), then we have precompactness. On the other hand, every Bourbaki-bounded set is in particular bounded since $B(x,\varepsilon)^m \subset B(x, m\cdot\varepsilon)$. Thus, in any metric space (X, d) we have the following,

$$\mathbf{TB}_d(X) \subset \mathbf{BB}_d(X) \subset \mathbf{B}_d(X).$$

Next examples show that these bornologies can be different.

Example 5.1. Let $(X, d_{\|\cdot\|})$ be an infinite dimensional normed space. It is easy to check that every ball of center $x = 0$ is Bourbaki-bounded, and therefore the same is true for every bounded set. On the other hand, it is well known that any ball in an infinite dimensional normed space cannot be precompact. Moreover, since in this case $\mathbf{BB}_d(X) = \mathbf{B}_d(X)$, it is clear that $\mathbf{BB}_d(X)$ is in fact uniformly metrizable.

Example 5.2. Consider the real line $\mathbb{R}$ with the metric $d^* = \min\{1, d_u\}$, where d_u is the usual metric. Then every subset in $(\mathbb{R}, d^*)$ is bounded but not Bourbaki-bounded. Indeed, since uniform equivalent metrics have the same Bourbaki-bounded sets, we have that $\mathbf{BB}_{d^*}(\mathbb{R}) = \mathbf{BB}_{d_u}(\mathbb{R}) = \mathbf{B}_{d_u}(\mathbb{R})$. And, in particular, it follows that $\mathbf{BB}_{d^*}(\mathbb{R})$ is also uniformly metrizable.

Example 5.3. Combining last two examples we can find metric spaces (X, d) having all these bornologies different at the same time. For instance, consider $X = \ell_2 \times \mathbb{R}$ endowed with the metric $d = \sup\{d_{\|\cdot\|}, d^*\}$. Now it is easy to check that each of those bornologies on X is generated by the product of the corresponding bornologies on each factor, and therefore all of them are different. In addition, we have that $\mathbf{BB}_d(X)$ can be uniformly metrizable by means of the metric $d' = \sup\{d_{\|\cdot\|}, d_u\}$.

Note that in the above examples we have determined in an easy way the Bourbaki-bounded sets using the fact that in a normed space these sets coincide with bounded sets. Then a natural question arises here.

Question. For which metric spaces the Bourbaki-bounded sets and the bounded sets coincide?

A partial answer can be obtained when we use a nice characterization of the Bourbaki-bounded sets given by Atsuji [1] in the setting of metric spaces (for uniform spaces see Hejcman [9]). It is proved that a set B is Bourbaki-bounded in X if, and only if, every real uniformly continuous function on X is bounded on B. Then, according to this result it is easy to check that every normed space, every length space, every quasi-convex space or more generally every *small-determined* metric space enjoys that property. Small-determined metric spaces were introduced by Garrido and Jaramillo in [7], where it is proved that these spaces can be characterized by the fact that every real-valued uniformly continuous function can be uniformly approximated by Lipschitz functions. Then it is clear that in these spaces uniformly continuous functions must be bounded on the bounded sets. But note that the converse is not true. For instance, if we take the set of natural numbers $\mathbb{N}$ endowed with the usual metric, we have that Bourbaki-bounded sets and bounded sets coincide, but $\mathbb{N}$ it is not a small-determined space.

On the other hand, the related problem of determining when the Bourbaki-bounded sets in a uniform space $(X, \mathbf{U})$ coincide with the bounded sets for some uniform (pseudo)metric on X, was studied by Hejcman in [11]. When this happens, it is said that the uniform space is *B-simple*. In that paper, it is proved that a uniform space $(X, \mathbf{U})$ is B-simple if, and only if, X is *σ-Bourbaki-bounded* (i.e., X is a countable union of Bourbaki-bounded sets) and *B-conservative*. There $(X, \mathbf{U})$ is said to be B-conservative if the Bourbaki-bounded sets are stable under uniform enlargement, that is, there exists some entourage $U \in \mathbf{U}$ such that $U[B] \in \mathbf{BB}(X)$, for every $B \in \mathbf{BB}(X)$. Thus a metric space (X, d) is B-conservative if for some $\delta > 0$, we have $B^\delta \in \mathbf{BB}_d(X)$, whenever $B \in \mathbf{BB}_d(X)$.

Remark 5.4. It is interesting to note here that a metric space (X, d) is B-simple if, and only if, its bornology $\mathbf{BB}_d(X)$ is uniformly metrizable. Indeed, if $\mathbf{BB}_d(X)$ is uniformly metrizable then there is a metric d' uniformly equivalent to d, and hence uniformly continuous for (X, d), such that $\mathbf{BB}_d(X) = \mathbf{B}_{d'}(X)$. And therefore (X, d) is B-simple. Conversely, if $\hat{d}$ is a uniformly continuous metric in (X, d), such that $\mathbf{BB}_d(X) = \mathbf{B}_{\hat{d}}(X)$, then taking some $x_0 \in X$, the function on X defined by $\chi(x) = \hat{d}(x, x_0)$ is a uniformly continuous characteristic function for the bornology $\mathbf{BB}_d(X)$, and then from Theorem 2.4 this bornology is uniformly metrizable.

According to the above remark, we can obtain as a consequence of Theorem 2.4, the metric version of the mentioned result by Hejcman.

Corollary 5.5 (Hejcman, [11]). *Let (X, d) be a metric space. The following conditions are equivalent:*

(*1*) $\mathbf{BB}_d(X)$ *is uniformly metrizable.*
(*2*) (X, d) *is B-simple.*
(*3*) (X, d) *is σ-Bourbaki-bounded and B-conservative.*

Proof. From the above Remark 5.4, we have the equivalence between (*1*) and (*2*). Moreover it is clear that (*1*) $\Rightarrow$ (*3*). So, in order to see (*3*) $\Rightarrow$ (*1*), suppose that $X = \bigcup_n B_n$ where $B_n \in \mathbf{BB}_d(X)$, for every $n \in \mathbb{N}$. And let $\delta > 0$ such that every δ-enlargement of a Bourbaki-bounded set is also Bourbaki-bounded. Consider $\widetilde{B}_1 = B_1$ and, for every $n \geq 2$, let

$$\widetilde{B}_n = (\widetilde{B}_{n-1})^\delta \cup B_n.$$

Then we finish since $\{\widetilde{B}_n\}_n$ is a base for $\mathbf{BB}_d(X)$ satisfying condition (*2*) in Theorem 2.4. Indeed, it is clear that $\widetilde{B}_n$ is Bourbaki-bounded and that ${\widetilde{B}_n}^{\delta} \subset \widetilde{B}_{n+1}$, for every $n \in \mathbb{N}$. On the other hand, if $B \in \mathbf{BB}_d(X)$, let $x_1, \dots, x_k \in X$, and $m \in \mathbb{N}$ such that, $B \subset \bigcup_{i=1}^{k}(B(x_i, \delta))^m$. Then, there exists some $n_0 \in \mathbb{N}$ such that $\{x_1, \dots, x_k\} \subset \widetilde{B}_{n_0}$, and therefore $\{x_1, \dots, x_k\}^{\delta} \subset \widetilde{B}_{n_0+1}$. Finally, we have that $B \subset {\widetilde{B}_{n_0+1}}^{m} \subset \widetilde{B}_{n_0+1+m}$. Hence, $\mathbf{BB}_d(X)$ is uniformly metrizable. □

In the preceding sections, we have proved that if the metric space (X, d) is uniformly locally $\mathbf{B}$, where $\mathbf{B}$ is any of the bornologies $\mathbf{FB}_d(X)$, $\mathbf{CB}_d(X)$ or $\mathbf{TB}_d(X)$, then $\mathbf{B}$ is in particular stable under uniform enlargement. And this means that, in all these cases, the uniform metrizability of these bornologies follows if, in addition, the metric space can be written as a countable union of sets in $\mathbf{B}$ (we will say that X is σ-$\mathbf{B}$). This means that we have obtained a general result of uniform metrizability for all these bornologies. Namely, $\mathbf{B}$ is uniformly metrizable if, and only if, the metric space is σ-$\mathbf{B}$ and uniformly locally $\mathbf{B}$ (see the corresponding condition *(3)* in Theorems 2.6 and 3.1, and condition *(4)* in Theorem 4.2, where to be σ-$\mathbf{B}$ is equivalent there to be countable or Lindelöf, respectively).

Then a natural question here is if an analogous result is true for the bornology $\mathbf{BB}_d(X)$. And the answer is no, in general. In [10], Hejcman gives an example of a σ-Bourbaki-bounded metric space that is uniformly locally Bourbaki-bounded but $\mathbf{BB}_d(X)$ is not stable under uniform enlargement, and then it cannot be uniformly metrizable.

Then, in order to get that $\mathbf{BB}_d(X)$ is stable under uniform enlargement, it is not enough with the existence of some $\delta > 0$ such that $\{x\}^{\delta} = B(x, \delta)$ is Bourbaki-bounded, for all $x \in X$. It would be necessary for all the consecutive δ-enlargement of singletons to be Bourbaki-bounded. In fact, this condition is also sufficient as the next result proves.

Theorem 5.6. *Let (X, d) be a metric space. The following conditions are equivalent:*

(1) $\mathbf{BB}_d(X)$ *is uniformly metrizable.*
(2) *(X, d) is σ-Bourbaki-bounded and there exists some $\delta > 0$ such that $B(x, \delta)^m$ is Bourbaki-bounded, for all $m \in \mathbb{N}$ and for all $x \in X$.*

Proof. According to Corollary 5.5, we need only to prove that condition *(2)* implies that (X, d) is conservative. Then, if $B \in \mathbf{BB}_d(X)$, there exist a finite set $F = \{x_1, \dots, x_n\} \subset X$, and some $m \in \mathbb{N}$, such that $B \subset \bigcup_{i=1}^{n} B(x_i, \delta)^m$. And therefore the δ-enlargement

$$B^{\delta} \subset B(x_1, \delta)^{m+1} \cup \dots \cup B(x_n, \delta)^{m+1}$$

is in $\mathbf{BB}_d(X)$ since it is contained in a finite union of Bourbaki-bounded sets. □

Now, we wonder if the condition of being σ-Bourbaki-bounded in the above theorem can be replaced by Lindelöfness (or equivalently separability), in an analogous way as in the cases of the compact bornology and the totally bounded bornology.

And the answer is clearly negative. In order to see this, it is enough to consider any non separable Banach space. Since separability seems to be a strong property in this context, we introduce the following notion.

Definition 5.7. Let (X, d) be a metric space and let $D \subset X$. We say that D is *Bourbaki-dense* in X when, for every $x \in X$ and $\varepsilon > 0$, there exist some $y \in D$ and $m \in \mathbb{N}$, such that $x \in B(y, \varepsilon)^m$. We say that (X, d) is *Bourbaki-separable* if there exists a countable Bourbaki-dense subset of X.

It is clear that separability implies Bourbaki-separability. On the other hand, every normed space X (separable or not) also enjoys this property, since the set $D = \{0\}$ is Bourbaki-dense in X. More generally, in every chainable metric space each singleton set is Bourbaki-dense. Recall that a metric space is said to be *chainable* when every pair of points in the space can be joined by an ε-chain, for every $\varepsilon > 0$. Note that chainability is a property related with connectedness. In fact, a metric space is chainable if, and only if, it is *uniformly connected*, that is, it is not the union of two non-empty subsets which are a positive distance apart. So every connected metric space is uniformly connected. But clearly the converse is not true. For instance, the subspace of $\mathbb{R}^2$ with the usual metric, $X = \{(x, y) : x \cdot y = \pm 1\}$ is uniformly connected but not connected.

To finish we are going to see that Bourbaki separability is a suitable condition for the uniform metrizability of Bourbaki-bounded sets.

Theorem 5.8. *Let (X, d) be a metric space. The following conditions are equivalent:*

(*1*) $\mathbf{BB}_d(X)$ *is uniformly metrizable.*

(*2*) (X, d) *is Bourbaki-separable and there exists some $\delta > 0$ such that $B(x, \delta)^m$ is Bourbaki-bounded, for all $m \in \mathbb{N}$ and for all $x \in X$.*

Proof. (*1*) $\Rightarrow$ (*2*). It follows from Theorem 5.6, since every σ-Bourbaki-bounded metric space is also Bourbaki-separable. Indeed, suppose $X = \bigcup B_n$ where $B_n \in \mathbf{BB}_d(X)$, for all n. We can suppose without loss of generality that $B_1 \subset B_2 \subset \cdots \subset B_n \ldots$, is a increasing sequence. Now, for every $n \in \mathbb{N}$, there exist a finite set $F_n = \{x_1^n, \ldots x_{k_n}^n\} \subset X$ and some $m_n \in \mathbb{N}$ such that

$$B_n \subset B(x_1^n, 1/n)^{m_n} \cup \cdots \cup B(x_{k_n}^n, 1/n)^{m_n}.$$

If we take $D = \bigcup F_n$, it is easy to check that D is a countable Bourbaki-dense in X, and therefore X is Bourbaki-separable.

(*2*) $\Rightarrow$ (*1*). Now, let $D = \{x_n : n \in \mathbb{N}\}$ a countable Bourbaki-dense subset in X, and let $\delta > 0$ such that every consecutive δ-enlargement of any singleton is Bourbaki-bounded. Then, it is easy too see that

$$X = \bigcup_{n,m \in \mathbb{N}} B(x_n, \delta)^m.$$

We finish since X is a countable union of Bourbaki-bounded sets, and then (*1*) follows. $\square$

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