

A Non-Type (D) Linear Isometry

O. Bueno*

*IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
obueno@impa.br*

B. F. Svaiter†

*IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
benar@impa.br*

Received: April 4, 2012

Revised manuscript received: April 17, 2012

Previous constructions of non-type (D) maximal monotone operators were based on the non-type (D) operators introduced by Gossez, and the construction of such operators or the proof that they were non-type (D) were not straightforward. The aim of this paper is to present a very simple non-type (D) linear isometry.

Keywords: Monotone operators, non-reflexive Banach spaces, type (D)

2010 Mathematics Subject Classification: 47H05, 49J52, 47N10

1. Introduction

Maximal Monotone operators were defined and used in the early sixties as a theoretical framework for the study of electrical networks, and, later on, for the study of non-linear partial differential equations. The first works on monotone operators were due to Zarantonello [15], Minty [12], Kato [11], Browder [4], Rockafellar [13], Brézis [3], among others. Since then, monotone operators were object of intense study. See [2] for a survey on the subject.

Fitzpatrick proved that any maximal monotone operator can be represented by convex functions by providing the explicit formula for one of these functions. Therefore, it is quite natural to ask the inverse question: under which conditions a convex function represents a maximal monotone operator. Characterizations of the convex functions which represent such operators in reflexive Banach spaces were presented in [6]. In non-reflexive Banach spaces, a characterization of convex functions which represents a sub-class of maximal monotone operators (those of type (D)) were presented in [14]. In the non-reflexive space ℓ^1 , non-type (D) maximal monotone operators were presented by Gossez in [8, 9]. Since then, examples of non-type (D) (maximal monotone) operators in c_0 were presented in

*Partially supported by CAPES.

†Partially supported by CNPq grants 300755/2005-8, 475647/2006-8 and by PRONEX-Optimization.

[5], and such examples in James spaces where presented in [1]. So far, the examples of non-type (D) maximal monotone operators were constructed using always Gossez's original examples and ideas in their construction.

The technicalities presented in the construction of previous non-type (D) operator may lead one to believe that operators of this type are inherently complex and/or pathological. Our aim is to present a very simple linear isometry which happens to be a non-type (D) maximal monotone operator.

2. Notation and basics definitions

Let A and B be arbitrary sets. A point-to-set operator $T : A \rightrightarrows B$ of A in B , is a triple (A, B, G) where $G \subset A \times B$. The set G , called the *graph* of T , is denoted as

$$\text{gra}(T) = G$$

and, for $a \in A$

$$T(a) = \{b \mid (a, b) \in \text{gra}(T)\}$$

Hence, $T : A \rightrightarrows B$ may also be regarded as a map of A in to $\wp(B)$, the power set of B , and be denoted as $T : A \rightarrow \wp(B)$. From now on we identify a map $F : A \rightarrow B$ with the operator $F : A \rightrightarrows B$ with the same graph.

Let X be a real Banach space, with topological dual X^* . We will use the notation

$$\langle x, x^* \rangle = \langle x^*, x \rangle = x^*(x), \quad \forall x \in X, x^* \in X^*.$$

The weak-* topology of X^* is the smallest topology (in X^*), in which the maps $X^* \ni x^* \mapsto \langle x, x^* \rangle$ are continuous for each $x \in X$. The bidual of X is $X^{**} = (X^*)^*$ and the canonical injection of X in to X^{**} is

$$J : X \rightarrow X^{**}, \quad \langle x^*, J(x) \rangle = \langle x^*, x \rangle, \quad \forall x^* \in X^*.$$

Note that this map is a linear isometry. From now on, we identify X with its image under the canonical injection of X into X^{**} . The space X is *non-reflexive* if J is not onto, which under the above convention means $X \subsetneq X^{**}$.

A point-to-set operator $T : X \rightrightarrows X^*$ (respectively, $T : X^* \rightrightarrows X$) is monotone if for any $(x, x^*), (y, y^*) \in \text{gra}(T)$ (respectively, for any $(x^*, x), (y^*, y) \in \text{gra}(T)$)

$$\langle x - y, x^* - y^* \rangle \geq 0$$

and is *maximal monotone* if it is monotone and its graph is maximal, with respect to the partial order of inclusion, in the family of graphs of maximal monotone operators from X to X^* (respectively, from X^* to X)

Define, for $T : X \rightrightarrows X^*$ the operator $\bar{T} : X^{**} \rightrightarrows X^*$ which graph is given by the limits, in the weak-* $\times$ strong topology of $X^{**} \times X^*$, of *bounded* nets of elements in the graph of T .

A maximal monotone operator $T : X \rightrightarrows X^*$ is of type (D) [7] if every point $(x^{**}, x^*) \in X^{**} \times X^*$ such that

$$\langle x^* - y^*, x^{**} - y \rangle \geq 0, \quad \forall (y, y^*) \in \text{gra}(T),$$

is contained in the graph of $\overline{T}$. This is equivalent to the fact of $\overline{T} : X^{**} \rightrightarrows X^*$ being maximal monotone.

3. A self-canceling non-type (D) maximal monotone linear isometry

The next theorem is our main result.

Theorem 3.1. *Let X be a non-reflexive real Banach space. The operator*

$$T : X \times X^* \rightarrow X^* \times X^{**}, \quad T(x, x^*) = (-x^*, x)$$

*is a non-type (D) maximal monotone linear isometry. Furthermore, the operator $\overline{T} : X^{**} \times X^{***} \rightrightarrows X^* \times X^{**}$ has infinitely many maximal monotone extensions in $(X^{**} \times X^{***}) \times (X^* \times X^{**})$.*

Proof. It follows trivially from its definition that T is a linear monotone isometry. Since T is a continuous monotone map, it is maximal monotone.

Take $(p, p^*) \in \text{gra } \overline{T}$. This means that there exists a bounded net of elements in the graph of T

$$\{((x_i, x_i^*), (y_i^*, y_i^{**}))\}_{i \in I}$$

which converges in the weak-* $\times$ strong topology to (p, p^*) . Using the definition of T we have

$$x_i^* = -y_i^*, \quad y_i^{**} = x_i.$$

Since $\{(y_i^*, y_i^{**})\}$ converges in the norm topology, $\{(x_i, x_i^*)\}$ also converges in the strong topology and its limit belongs to $X \times X^*$. Therefore for some $x \in X$, $x^* \in X^*$,

$$p = (x, x^*) \in X \times X^*, \quad p^* = (-x^*, x).$$

Using again the definition of T we have $(p, p^*) \in \text{gra}(T)$. Altogether we proved that

$$\text{gra}(\overline{T}) = \text{gra}(T).$$

Now we will prove that $\overline{T} : X^{**} \times X^{***} \rightrightarrows X^* \times X^{**}$ has infinitely many maximal monotone extensions. Since X is non-reflexive and a closed subspace of X^{**} , there exist $x_0^{**} \in X^{**} \setminus X$ and $w_0 \in X^{***}$ such that

$$\langle x_0^{**}, w_0 \rangle = 1, \quad w_0(x) = 0, \quad \forall x \in X. \quad (1)$$

Define, for each $t \in (0, \infty)$,

$$p_t = (tx_0^{**}, (1/t)w_0), \quad q_t = (0, tx_0^{**}).$$

Take $(x, x^*) \in X \times X^*$. Direct calculation yields:

$$\begin{aligned} \langle (x, x^*) - p_t, (-x^*, x) - q_t \rangle &= \langle (x - tx_0^{**}, x^* - (1/t)w_0), (-x^*, x - tx_0^{**}) \rangle \\ &= \langle x - tx_0^{**}, -x^* \rangle + \langle x^* - (1/t)w_0, x - tx_0^{**} \rangle = 1 \end{aligned}$$

where the last equality follows from (1). Therefore, the operator $A_t : X^{**} \times X^{***} \rightrightarrows X^* \times X^{**}$ with graph

$$\text{gra}(A_t) = \text{gra } T \cup \{(p_t, q_t)\}.$$

is monotone. Hence, for each $t > 0$, there exists a maximal monotone extension of A_t , say B_t . If $t, s > 0$, $t \neq s$, then

$$\begin{aligned} \langle p_t - p_s, q_t - q_s \rangle &= \left\langle \left((t-s)x_0^{**}, (1/t - 1/s)w_0 \right), \left(0, (t-s)x_0^{**} \right) \right\rangle \\ &= (t-s)(1/t - 1/s) = -\frac{(t-s)^2}{ts} < 0 \end{aligned}$$

which proves that $(p_t, q_t) \in B_t \setminus B_s$ and, in particular $B_t \neq B_s$. $\square$

Observe that if X is a James' space [10], then $X \times X^*$ is a non-reflexive Banach space isometrically isomorphic to its dual. Indeed, if $A : X \rightarrow X^{**}$ is an isometric isomorphism, then

$$\mathbb{A} : X \times X^* \rightarrow X^* \times X^{**}, \quad \mathbb{A}(x, x^*) = (x^*, A(x))$$

is also an isometric isomorphism.

References

- [1] H. H. Bauschke, J. M. Borwein, X. Wang, L. Yao: Construction of pathological maximally monotone operators on non-reflexive Banach spaces, *Set-Valued Var. Analysis* 20(3) (2012) 387–415.
- [2] J. M. Borwein: Fifty years of maximal monotonicity, *Optim. Lett.* 4(4) (2010) 473–490.
- [3] H. Brézis: *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland Mathematics Studies 5, Notas de Matemática 50, North-Holland, Amsterdam (1973).
- [4] F. E. Browder: Nonlinear maximal monotone operators in Banach space, *Math. Ann.* 175 (1968) 89–113.
- [5] O. Bueno, B. F. Svaiter: A non-type (D) operator in c_0 , *Math. Program.*, to appear (Mar. 2011).
- [6] R. S. Burachik, B. F. Svaiter: Maximal monotone operators, convex functions and a special family of enlargements, *Set-Valued Anal.* 10(4) (2002) 297–316.
- [7] J.-P. Gossez: Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs, *J. Math. Anal. Appl.* 34 (1971) 371–395.
- [8] J.-P. Gossez: On a convexity property of the range of a maximal monotone operator, *Proc. Amer. Math. Soc.* 55(2) (1976) 359–360.
- [9] J.-P. Gossez: On the extensions to the bidual of a maximal monotone operator, *Proc. Amer. Math. Soc.* 62(1) (1977) 67–71.
- [10] R. C. James: A non-reflexive Banach space isometric with its second conjugate space, *Proc. Natl. Acad. Sci. USA* 37 (1951) 174–177.

- [11] T. Kato: Demicontinuity, hemicontinuity and monotonicity, *Bull. Amer. Math. Soc.* 70 (1964) 548–550.
- [12] G. J. Minty: Monotone networks, *Proc. R. Soc. Lond., Ser. A* 257 (1960) 194–212.
- [13] R. T. Rockafellar: Local boundedness of nonlinear, monotone operators, *Michigan Math. J.* 16 (1969) 397–407.
- [14] B. F. Svaiter, M. Marques Alves: Brønsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces, *J. Convex Analysis* 15 (2008) 693–706.
- [15] E. Zarantonello: Solving functional equations by contractive averaging, MRC Technical Summary Report 160, Mathematics Research Center, United States Army, Univ. of Wisconsin (1960).