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Convexity on Complex Hyperbolic Space*

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In a Riemannian manifold a regular convex domain is said to be λ-convex if its normal curvature at each point is greater than or equal to $\lambda > 0$. In a Hadamard manifold, the asymptotic behaviour of the quotient $\operatorname{vol}(\Omega_t)/\operatorname{vol}(\partial\Omega_t)$ for a family of λ-convex domains Ω_t expanding over the whole space has been studied and general bounds for this quotient are known. In this paper we improve this general result in the complex hyperbolic space $\mathbb{C}H^n(-4k^2)$, a Hadamard manifold with constant holomorphic curvature equal to $-4k^2$. Furthermore, we give some specific properties of convex domains in $\mathbb{C}H^n(-4k^2)$ and we prove that λ-convex domains of arbitrary diameter exists if $\lambda \leq k$.

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1. Introduction

Given a family of convex domains expanding over the whole Euclidean plane, the quotient between the area and the perimeter tends to infinity. This behaviour does not hold in hyperbolic plane $\mathbb{H}^2(-1)$ where the quotient tends to a value less or equal than 1 depending on the shape of the domain.

A convex domain (i.e. a convex set with interior points) in $\mathbb{H}^2(-1)$ is said to be h-convex if for every pair of points, each horocyclic segment joining them also belongs to the convex domain. The first result about the asymptotic behaviour of the quotient area/perimeter of convex domains in $\mathbb{H}^2(-1)$ was stated for a family of h-convex domains by Santaló and Yañez [14] in 1972. They proved that if $\{\Omega_t\}_{t\in\mathbb{R}^+}$ is a family of compact h-convex domains in $\mathbb{H}^2(-1)$ expanding over the

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whole plane, then

$$\lim_{t\to\infty} \frac{\operatorname{area}(\Omega_t)}{\operatorname{length}(\partial\Omega_t)} = 1. \tag{1}$$

In the hyperbolic plane, the set of equidistant points to a geodesic line are two curves called equidistant lines. These curves have constant geodesic curvature λ with value between 0 and 1. It is said that a convex domain is λ-convex if for every pair of points, each segment of curve with constant geodesic curvature λ joining them also belongs to the convex domain. A convex domain in $\mathbb{H}^2(-1)$ is λ-convex if and only if the geodesic curvature of the boundary is greater than or equal to λ (see for instance [7]). It is known that that the quotient in (1) can take any value between $[\lambda, 1]$ for a family of λ-convex sets ([7]). The notion of λ-convexity is also defined for arbitrary Riemannian manifolds (see Definition 2.6).

Recall that a Hadamard manifold is a complete simply connected Riemannian manifold with non-positive sectional curvature. For Hadamard manifolds it was stated in [3] the following result concerning λ-convexity.

Theorem ([3]). *Let M be a Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$. If a family of λ-convex domains expands over the whole space M we must have $\lambda \leq k_2$.*

Complex hyperbolic space $\mathbb{C}H^n(-4k^2)$ is a Hadamard manifold of real dimension $2n$ with real sectional curvature K such that $-4k^2 \leq K \leq -k^2$.

In this paper we prove that the estimate of the preceding theorem can be sharpened from $\lambda \leq 2k$ to $\lambda \leq k$.

Theorem 1.1. *In the complex hyperbolic space $\mathbb{C}H^n(-4k^2)$, if a family of compact λ-convex domains expands over the whole $\mathbb{C}H^n(-4k^2)$ then $\lambda \leq k$.*

The result about the asymptotic behaviour of $\operatorname{vol}(\Omega_t)/\operatorname{vol}(\partial\Omega_t)$ was studied for families of λ-convex domains $\{\Omega_t\}$ expanding over the whole $\mathbb{H}^n$ in [4] and [5]. In [3] it was stated for Hadamard manifolds as follows:

Theorem ([3]). *Let $\{\Omega_t\}_{t\in\mathbb{R}^+}$ be a piecewise $\mathcal{C}^2$ family of λ-convex compact domains, $0 \leq \lambda \leq k_2$, in a Hadamard manifold M of dimension n with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ and suppose that it expands over the whole M. Then*

$$\frac{\lambda}{(n-1)k_2^2} \leq \liminf_{t\to\infty} \frac{\operatorname{vol}(\Omega_t)}{\operatorname{vol}(\partial\Omega_t)} \leq \limsup_{t\to\infty} \frac{\operatorname{vol}(\Omega_t)}{\operatorname{vol}(\partial\Omega_t)} \leq \frac{1}{(n-1)k_1}.$$

When we apply this result to $\mathbb{H}^n$ we obtain the same bounds as in [5]. Moreover, in this case these bounds cannot be improved (see [15]). The same theorem applied to the complex hyperbolic space gives

$$\frac{\lambda}{4(2n-1)k^2} \leq \liminf_{t\to\infty} \frac{\operatorname{vol}(\Omega_t)}{\operatorname{vol}(\partial\Omega_t)} \leq \limsup_{t\to\infty} \frac{\operatorname{vol}(\Omega_t)}{\operatorname{vol}(\partial\Omega_t)} \leq \frac{1}{(2n-1)k}.$$

We show that these inequalities can be improved. We obtain new bounds for the limits of the quotient and we prove that the new estimate for the upper bound is sharp. It is attained, among others, for a family of geodesic spheres. More precisely we have

Theorem 1.2. *Let $\{\Omega_t\}_{t\in\mathbb{R}^+}$ be a piecewise $\mathcal{C}^2$ family of λ-convex compact domains, $0 \leq \lambda \leq k$, expanding over the whole space $\mathbb{C}H^n(-4k^2)$, $n \geq 2$. Then,*

$$\frac{\lambda}{4nk^2} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \frac{1}{2nk}. \tag{2}$$

Moreover, the upper bound is sharp.

2. Preliminaries

2.1. Complex hyperbolic space

Definition 2.1. The *Complex hyperbolic space* is the only (up to holomorphic isometries) complete simply connected Kähler manifold of constant holomorphic curvature $-4k^2$. We denote it by $\mathbb{C}H^n(-4k^2)$.

Consider on $\mathbb{C}^{n+1}$ the Hermitian product

$$\langle z, w\rangle = -z_0\overline{w_0} + \sum_{j=1}^{n} z_j\overline{w_j} \tag{3}$$

and the subset

$$M = \{z \in \mathbb{C}^{n+1} \ : \ \langle z, z\rangle < 0\}.$$

The projection of M into $\mathbb{CP}^n$ corresponds to complex hyperbolic space $\mathbb{C}H^n(-4k^2)$. This space is equipped with the Bergman metric (see [8, 68-74]) rescaled in a way such that sectional curvature lies in the interval $[-4k^2, -k^2]$. The projective line through two points in $\mathbb{C}H^n(-4k^2)$ is a real hyperbolic plane. Geodesics in $\mathbb{C}H^n(-4k^2)$ are the geodesics in these real hyperbolic planes. The isometry group is the projection unitary group $PU(1, n)$ with respect to the product (3).

The *holomorphic sectional curvature* $K_{\mathrm{hol}}(v)$ for a unit tangent vector v in a Kähler manifold with complex structure J is the sectional curvature of the plane generated by $\{v, Jv\}$. Kähler manifolds with constant holomorphic curvature are locally isometric to complex Euclidean space $\mathbb{C}^n$, complex projective space $\mathbb{C}P^n(4k^2)$, or complex hyperbolic space $\mathbb{C}H^n(-4k^2)$. Their real sectional curvature of planes spanned by orthonormal vectors u, v is given by

$$g(R(u, v)v, u) = \frac{K_{\mathrm{hol}}}{4}(1 + 3(g(u, Jv))^2)$$

(see [11, p. 167]). From this we have that $\mathbb{C}H^n(-4k^2)$ has sectional curvature between $-4k^2$ and $-k^2$.

In $\mathbb{C}H^n(-4k^2)$ geodesic spheres are not umbilical hypersurfaces as they are in real space forms. The explicit value of its principal curvatures is essential in the proof of Theorem 1.1.

Proposition 2.2 ([12]). *In $\mathbb{C}H^n(-4k^2)$ the principal curvatures of a geodesic sphere of radius r are:*

a) $2k\coth(2kr)$ *with multiplicity 1 and principal direction $-JN$ (where N is the inward unit normal vector).*

b) $k\coth(kr)$ *with multiplicity $2n-2$.*

We shall use the following result:

Corollary 2.3. *Let z be a point in a geodesic sphere in $\mathbb{C}H^n(-4k^2)$, let N be the inward unit normal vector and let v be a principal direction. Then the submanifold generated by the exponential map on* $\operatorname{span}\{N, v\}$ *at z is totally geodesic in $\mathbb{C}H^n(-4k^2)$.*

Proof. If $v = -JN$, then the submanifold generated by the exponential map on $\operatorname{span}\{N, JN\}$ is isometric to $\mathbb{H}^2(-4k^2)$ which is totally geodesic in $\mathbb{C}H^n(-4k^2)$.

If $v \neq -JN$, then $\langle v, JN\rangle = 0$ since v and JN are principal directions of different principal curvatures, but this is the condition for vectors $\{N, v\}$ to generate a totally real plane. So, the submanifold generated by the exponential map is isometric to $\mathbb{H}^2(-k^2)$ and also totally geodesic in $\mathbb{C}H^n(-4k^2)$. $\square$

2.2. $\lambda-$convexity in Hadamard manifolds

Let us recall the definition of λ-convexity in a Hadamard manifold (see [3] for more details).

Definition 2.4. A $\mathcal{C}^2$ *hypersurface* of a Riemannian manifold is said to be *regular λ-convex* if at every point all the normal curvatures (with respect to the inward unit normal vector) are greater than or equal to $\lambda \geq 0$.

In the non-regular case we consider:

Definition 2.5. A *λ-convex hypersurface* is a hypersurface such that for every point p there is a regular λ-convex hypersurface S leaving a neighborhood of p in the hypersurface in the convex side of S.

Definition 2.6. A domain (set with interior points) is said to be *(regular) λ-convex* if its boundary is a (regular) λ-convex hypersurface.

Remark 2.7. Note that every λ-convex hypersurface is also λ'-convex for any $\lambda' \leq \lambda$.

Remark 2.8. In spaces of constant sectional curvature and in complete simply connected manifolds with non-positive sectional curvature the notion of 0-convexity is equivalent to the convexity with respect to geodesics (cf. [1]).

3. $\lambda-$convexity in complex hyperbolic space

Definition 3.1. Let $\Omega \subset \mathbb{C}H^n(-4k^2)$ be a compact convex domain and let $p \in \partial\Omega$ be a point of class $\mathcal{C}^1$. The largest (smallest) sphere tangent to $\partial\Omega$ at p contained in (containing) the domain is called *inscribed (circumscribed) sphere at p.*

Lemma 3.2. *Let $\Omega \subset \mathbb{C}H^n(-4k^2)$ be a compact convex domain with piecewise $\mathcal{C}^2$ boundary. If there exists an inscribed sphere S_i and a circumscribed sphere S_c at $p \in \partial\Omega$, then*

$$K_{n,S_c}(v) \leq K_{n,\partial\Omega}(v) \leq K_{n,S_i}(v), \tag{4}$$

where $K_{n,\cdot}(v)$ denotes the normal curvature in the principal direction v *of the inscribed sphere at p.*

Remark 3.3. If at a point p there exists only circumscribed or inscribed sphere, then the corresponding inequality remains true.

Proof. Let N be the inward unit normal vector to $\partial\Omega$ at p, and let $\{e_1, ..., e_{2n-1} = -JN\}$ be a basis of principal directions of the inscribed sphere at p. Notice that it is also a basis of principal directions of the circumscribed sphere at p (cf. Proposition 2.2).

From Corollary 2.3, the submanifold generated by the exponential map on $\mathrm{span}\{N, e_i\}$ at p is isometric to $\mathbb{H}^2(-k^2)$ if $i \neq 2n-1$, and to $\mathbb{H}^2(-4k^2)$ otherwise.

In the real hyperbolic plane the inequality (4) is satisfied. That is, suppose p is a point in the boundary of a convex domain in a real hyperbolic plane. If there exists circumscribed and inscribed sphere at p, then the geodesic curvature of the convex domain at p lies between the geodesic curvature of the spheres at p. Thus, inequality (4) holds. $\square$

Proposition 3.4. *Let $\Omega \subset \mathbb{C}H^n(-4k^2)$ be a compact convex domain with piecewise $\mathcal{C}^2$ boundary and N the inward normal vector of $\partial\Omega$. If in a regular point $p \in \partial\Omega$ there exists inscribed and circumscribed sphere of radius r and R respectively, then for directions v such that $\langle v, JN\rangle = 0$ we have*

$$k\coth(kR) \leq K_{n,\partial\Omega}(v) \leq k\coth(kr).$$

In the direction $-JN$ the inequality becomes

$$2k\coth(2kR) \leq K_{n,\partial\Omega}(-JN) \leq 2k\coth(2kr).$$

Proof. If $v \in T_p\partial\Omega$ such that $\langle v, JN\rangle = 0$, then v is a principal direction of the inscribed and circumscribed sphere at p. Using Proposition 2.2 and Lemma 3.2 we deduce the inequalities. Similarly we prove the other case. $\square$

Using this proposition we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{\Omega_t\}$ be a family of λ-convex domains expanding over the whole space and let $p \in \partial\Omega_t$ be a regular point with inscribed sphere $S_i(r)$ with radius r. It follows from the definition of λ-convexity and Proposition 3.4 that for some direction v tangent to $\partial\Omega_t$ at p it is satisfied

$$\lambda \leq K_{n,\partial\Omega_t}(v) \leq k\coth(kr). \tag{5}$$

Since the radius of the inscribed ball tends to infinity when the convex domain grows, for a convex domain big enough, $k\coth(kr)$ is close to k. From inequalities in (5) it is necessary that $\lambda \leq k$. □

Remark 3.5. The proof uses the fact that every geodesic sphere of radius r has a direction with normal curvature equal to $k\coth(kr)$. In general this may not be true: given a Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$ the normal curvature K_n of a sphere of radius r satisfies $k_1\coth(k_1 r) \leq K_n \leq k_2\coth(k_2 r)$ but the equality in (5) may not be taken (cf. [13]).

Example 3.6. *Geodesic spheres.* In any Hadamard manifold geodesic spheres are compact convex domains. In $\mathbb{C}H^n(-4k^2)$ the normal curvature of geodesic spheres of radius r varies from $k\coth(kr)$ to $2k\coth(2kr)$ (cf. Proposition 2.2). Then, they are $k\coth(kr)$-convex domains.

Horospheres. If we fix a point in a sphere and let the radius tend to infinity we get a non-compact convex domain called *horosphere*. It is a k-convex domain.

Equidistant hypersurfaces. Let $\mathbb{C}H^p(-4k^2)$, $1 \leq p < n$, be an isometrically embedded complex hyperbolic space in $\mathbb{C}H^n(-4k^2)$ and let $\mathbb{R}H^n(-k^2)$ be an isometrically embedded real hyperbolic space in the complex hyperbolic space $\mathbb{C}H^n(-4k^2)$. The *equidistant hypersurface* from a $\mathbb{C}H^p(-4k^2)$, $1 \leq p < n$, or from a $\mathbb{R}H^n(-k^2)$ consists of all points at distance r from the submanifold. Equidistant hypersurfaces are also convex hypersurfaces bounding a non-compact domain. It is $k\tanh(kr)$-convex (see [12] for more details).

Bisectors. It is known that in $\mathbb{C}H^n(-4k^2)$ do not exist totally geodesic (real) hypersurfaces (see [8, p. 84]). In some cases bisectors are considered as a substitute of totally geodesic hypersurfaces in $\mathbb{C}H^n(-4k^2)$. Bisectors are the image of the exponential map on a $(2n-1)$-dimensional tangent space at a point. They are not convex hypersurfaces and do not bound convex domains (cf. [8, p. 193]). So, we cannot construct any convex domain with part of its boundary contained in a bisector.

Tube along a geodesic. Other examples of compact convex domains in $\mathbb{C}H^n(-4k^2)$ can be constructed from the tube of radius r about a geodesic.

Definition 3.7. Let (M, g) be a Hadamard manifold and γ a complete geodesic. The *tube* $\tau(\gamma, r)$ of radius $r > 0$ about γ is defined as (see for instance [9, p. 32])

$$\tau(\gamma, r) = \{x \in M \ : \ \exists \text{ a geodesic segment } \xi \text{ of length } L(\xi) \leq r \\ \text{from } x \text{ meeting } \gamma \text{ orthogonally}\}.$$

Remark 3.8. As γ is complete, the definition of $\tau(\gamma, r)$ is equivalent to

$$\bigcup_{y\in\gamma}\{\exp_y(v) \ : \ v \in (\gamma_y)^{\perp} \text{ and } ||v|| \leq r\}$$

and also to

$$\bigcup_{y\in\gamma}\{\exp_y(v) \ : \ v \in T_yM \text{ and } ||v|| \leq r\}.$$

Proposition 3.9. *Let (M, g) be a Hadamard manifold and let γ be a complete geodesic in M. The tube of radius r about γ is a convex domain.*

Proof. For every Hadamard manifold M and a convex subset $Y \subset M$, it is known that the function $x \mapsto d(x, Y)$ is convex (in the sense that for every constant-speed geodesic $\gamma(t)$ the function $t \mapsto d(\gamma(t), Y)$ is convex) (cf. [6, 331]). This implies that the tube $\tau(\gamma, r)$ is convex. Indeed, if $p, q \in \tau(\gamma, r)$ and $\beta(t)$ is the unit geodesic joining p and q, the function $d(\beta(t), \gamma)$ is convex, therefore the geodesic segment joining p and q lies in $\tau(\gamma, r)$. □

In order to obtain compact convex domains we can, for example, intersect a tube with a ball centered in the geodesic with radius greater than the radius of the tube. We call this domain a *round tube* about γ.

In complex hyperbolic space the tube $\tau(\gamma, r)$ about a geodesic segment is not a convex domain since a part of its boundary is contained in a bisector. If γ_L denotes a geodesic segment of length L, we define the *modified tube* $\tau_L(\gamma, r)$ of radius $r > 0$ about γ_L as the set of points at distance r of γ_L.

Corollary 3.10. *Let (M, g) be a Hadamard manifold and let γ_L be a geodesic segment of length L in M. The modified tube $\tau_L(\gamma, r)$ is a convex domain.*

Proof. Clearly, every point of the boundary is supported by a convex hypersurface. □

4. Asymptotic behaviour

Let p be a fixed point in the interior of Ω and let $T_p'\mathbb{C}H^n(-4k^2)$ be the unit tangent space at p. If $u \in T_p'\mathbb{C}H^n(-4k^2)$ and $l(u)$ is the distance from p to $\partial\Omega$ in the direction u, then the exponential map at p on the set

$$A = \{(u, t) \in T_p'\mathbb{C}H^n(-4k^2) \times \mathbb{R} \mid 0 < t \leq l(u)\}$$

parametrizes $\Omega \setminus \{p\}$ and

$$\mathrm{vol}(\Omega) = \int_{\Omega} \tau = \int_A \exp_p^* \tau,$$

where τ is the volume element of $\mathbb{C}H^n(-4k^2)$.

If $\mathcal{J}(t, u)$ is the Jacobian of the exponential map, then $\exp_p^*\tau = \mathcal{J}(t, u)t^{2n-1}dtdS_{2n-1}$. Using an orthonormal basis and the corresponding Jacobi fields along the geodesic given by the direction u we have (see, for instance, [9, 123])

$$\mathcal{J}(t, u) = \frac{\sinh^{2n-1}(kt)\cosh(kt)}{(kt)^{2n-1}}.$$

Thus, the volume of a compact convex domain Ω is given by

$$\mathrm{vol}(\Omega) = \frac{1}{2nk^{2n}} \int_{S^{2n-1}} \sinh^{2n}(kl(u))dS_{2n-1}, \tag{6}$$

and the $(2n-1)$-dimensional volume of $\partial\Omega$ by

$$\mathrm{vol}(\partial\Omega) = \int_{S^{2n-1}} \frac{\sinh^{2n-1}(kl(u))\cosh(kl(u))}{k^{2n-1}\langle \partial_t, N\rangle} dS_{2n-1} \tag{7}$$

where N is the outward unit normal vector of $\partial\Omega$ and ∂_t the radial field from p.

We shall need the following proposition:

Proposition 4.1 ([3]). *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that $-k_2^2 \leq K \leq -k_1^2$. Let Ω be a λ-convex domain with $\mathcal{C}^2$ boundary, $\lambda < k_2$ and p an interior point of Ω. If φ denotes the angle of the normal to $\partial\Omega$ and the exterior radial direction and $d(p, \partial\Omega) \geq (1/k_2)\mathrm{arctanh}(\lambda/k_2)$, then*

$$\cos\varphi \geq \frac{\lambda}{k_2}.$$

Proof of Theorem 1.2. Let φ be the angle between the inward unit normal vector N and the radial direction ∂_t from p. From expressions (6) and (7) we find

$$\frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \frac{\displaystyle\int_{S^{2n-1}} \sinh^{2n}(kl(u))dS_{2n-1}}{2nk\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))dS_{2n-1}} \leq \frac{1}{2nk}.$$

From Proposition 4.1 we have that for a sufficiently large convex domain $\cos\varphi \geq \frac{\lambda}{2k}$ and

$$\begin{aligned}
\frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} &= \frac{\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))\tanh(kl(u))dS_{2n-1}}{2nk\displaystyle\int_{S^{2n-1}} \frac{\sinh^{2n-1}(kl(u))\cosh(kl(u))}{\cos\varphi}dS_{2n-1}} \\
&\geq \tanh(kr)\frac{\lambda\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))dS_{2n-1}}{4nk^2\displaystyle\int_{S^{2n-1}} \sinh^{2n-1}(kl(u))\cosh(kl(u))dS_{2n-1}} \\
&= \frac{\lambda}{4nk^2}\tanh(kr),
\end{aligned}$$

where r is the distance between p and $\partial\Omega_t$. As $\tanh(kr)$ tends to 1 when Ω_t expands to the whole space we obtain

$$\liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \geq \frac{1}{4nk^2}.$$

It remains to show the sharpness of the upper bound. A ball B_r of radius r is λ-convex for every $\lambda \in [0, k\coth(kr)]$. Hence a family of balls expanding over the whole space is λ-convex for every $\lambda \in [0,k]$. From the expressions

$$\mathrm{vol}(B_r) = \frac{\sinh^{2n}(kr)\pi^n}{2nk^{2n}n!}, \qquad \mathrm{vol}(\partial B_r) = \frac{\sinh^{2n-1}(kr)\cosh(kr)\pi^n}{k^{2n-1}n!},$$

we obtain

$$\begin{aligned}\limsup_{r\to\infty} \frac{\mathrm{vol}(B_r)}{\mathrm{vol}(\partial B_r)} &= \limsup_{r\to\infty} \frac{\sinh^{2n}(kr)\pi^n n!}{2nk\sinh^{2n-1}(kr)\cosh(kr)\pi^n n!} \\ &= \limsup_{r\to\infty} \frac{\tanh(kr)}{2nk} = \frac{1}{2nk},\end{aligned}$$

which gives the upper bound on (2). □

Notice that modified and round tubes give also the upper bound when expanding over $\mathbb{C}H^n(-4k^2)$.

Remark 4.2. We denote the non-compact rank one symmetric spaces different from $\mathbb{R}H^n(-k^2)$ by $\mathbb{K}H^n(-4k^2)$ with $\mathbb{K} = \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ the associated real division algebra with real dimension d. When $\mathbb{K} = \mathbb{O}$, we assume $n = 2$. Theorem 1.1 and Theorem 1.2 can be generalized to the quaternionic hyperbolic space and the Cayley hyperbolic plane.

Theorem A. *In $\mathbb{K}H^n(-4k^2)$ it can only exist families of compact λ-convex domains expanding over the whole $\mathbb{K}H^n(-4k^2)$ if $\lambda \leq k$.*

Theorem B. *Let $\{\Omega_t\}_{t\in\mathbb{R}^+}$ be a piecewise $\mathcal{C}^2$ family of λ-convex compact domains, $0 \leq \lambda \leq k$, expanding over the whole space $\mathbb{K}H^n(-4k^2)$, $n \geq 2$. Then*

$$\frac{\lambda}{2dnk} \leq \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega_t)}{\mathrm{vol}(\partial\Omega_t)} \leq \frac{1}{(dn+d-2)k},$$

where d is the dimension of the associated real division algebra to $\mathbb{K}H^n(-4k^2)$.

Moreover, the upper bound is sharp. The upper bound is attained, for instance, for the geodesic balls.

For the proof of Theorem B, the main ingredient is, as in Theorem 1.2, the expression for the Jacobian of the exponential map in $\mathbb{K}H^n(-4k^2)$. In [10] this expression is explicitly given for the other non-compact rank one symmetric spaces $\mathbb{K}H^n(-4k^2)$. If d denotes the dimension of the associated real division algebra to $\mathbb{K}H^n(-4k^2)$, then the Jacobian of the exponential map is given by

$$\mathcal{J}(t,u) = \frac{\sinh^{dn-1}(kt)\cosh^{d-1}(kt)}{(kt)^{dn-1}}.$$

Therefore the volume of a compact convex domain Ω is given by

$$\begin{aligned}\mathrm{vol}(\Omega) &= \int_{S^{dn-1}} \int_0^{l(u)} \frac{\sinh^{dn-1}(kt)\cosh^{d-1}(kt)}{k^{dn-1}} dt dS_{dn-1} \\ &= \sum_{j=0}^{d/2-1} \binom{d/2-1}{j} \int_{S^{dn-1}} \frac{\sinh^{dn+2j}(kl(u))}{(dn+2j)k^{dn}} dS_{dn-1}.\end{aligned}$$

Remark 4.3. In Theorem 1.2 we have seen that the upper bound is the best possible one. It is attained mainly for families of convex sets that are "quite round" at infinity, as spheres, round tubes and modifies tubes. It seems reasonable to expect that sequences of convex sets whose quotient V/A can tend to the value of the lower bound must have some kind of sharpened ends.

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