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Bartle–Dunford–Schwartz Integral versus Bochner, Pettis and Dunford Integrals*

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We study some relationships between the Bartle–Dunford–Schwartz integral of a scalar valued function f, with respect to a vector measure m, and the Dunford, Pettis or Bochner integrals of its (vector valued) distribution function m_f. The Dunford (or Pettis) integrability of m_f is strongly related to the weak integrability (or the integrability) of f in the sense of Bartle–Dunford–Schwartz. In the case of the Bochner integrability of m_f, a new function space appears. It is defined through the Choquet integrability of f with respect to the semivariation $\|m\|$ of the measure m. We also study this space and present its main properties.

Keywords: Bartle–Dunford–Schwartz integral, Dunford, Pettis and Bochner integrals, Choquet integral, Vector measures

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1. Introduction

In general, *integration theory* deals with *measures* whose values are in some set E, integrating certain *functions* with values in a second set F, and resulting in *integrals* in a third set G. Furthermore, one needs to consider some linear and topological structure on the sets E, F and G, and a bilinear form from $E \times F$ into G that determines how measures and functions interact with each other. The classical Bochner, Pettis and Dunford integrals fit into this context when we consider a positive scalar measure and functions with values in a Banach space.

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In the opposite situation is the Bartle–Dunford–Schwartz integral which appears by considering vector measures with values in a Banach space and scalar valued functions. In both cases, the bilinear form is the product of scalars and vectors.

In this paper we study some relationships between these two kind of integrals. Let us recall and describe briefly the spaces of integrable functions that we are going to consider in the paper.

For a σ–finite complete measure space $(\Lambda, \mathcal{A}, \mu)$ and a Banach space X, following the definitions of integrability of [7, Chapter III, §1], we denote the spaces of all Dunford or Pettis integrable functions, respectively, by $\mathcal{D}(\mu, X)$ and $\mathcal{P}(\mu, X)$. These spaces are normed spaces (in general non complete), with the usual identification of functions, endowed with the norm

$$\|G\|_{\mathcal{P}} := \sup\left\{\int_{\Lambda} |\langle G, x'\rangle| \, d\mu : \|x'\| \leq 1\right\}, \quad G \in \mathcal{D}(\mu, X),$$

where x' are elements of the dual X' of the Banach space X. The space of all Bochner integrable functions is denoted by $\mathcal{B}(\mu, X)$, and (when we identify functions) it is a Banach space when is endowed with the norm

$$\|G\|_{\mathcal{B}} := \int_{\Lambda} \|G\| \, d\mu, \quad G \in \mathcal{B}(\mu, X).$$

Every Bochner integrable function is Pettis integrable, and the inclusion $\mathcal{B}(\mu, X) \subseteq \mathcal{P}(\mu, X)$ is continuous. It is also really well–known that for every function $G \in \mathcal{P}(\mu, X)$ the set function m_G, given by

$$m_G(A) := \int_A G d\mu \in X, \quad A \in \mathcal{A}$$

is a μ–continuous countably additive vector measure [7, Theorem 3.7.2].

For a general countably additive vector measure m with values in a real Banach space X defined on a measurable space (Ω, Σ) we can consider the spaces $L^1(m)$ (or $L^1_w(m)$) of $\mathbb{R}$–valued measurable functions which are *integrable* (or *weakly integrable*) with respect to m in the sense of Bartle–Dunford–Schwartz. The space $L^1_w(m)$ becomes a Banach lattice when it is endowed with the natural order and the norm

$$\|f\|_{L^1_w(m)} := \sup\left\{\int_{\Omega} |f| \, d\,|\langle m, x'\rangle| : \|x'\| \leq 1\right\}, \quad f \in L^1_w(m).$$

The space $L^1(m)$ of integrable functions becomes an order continuous closed lattice ideal of $L^1_w(m)$. In these spaces we identify functions which are equal almost everywhere with respect to the semivariation of m, that is, the set function defined by

$$\|m\|(A) := \sup\{|\langle m, x'\rangle|(A) : \|x'\| \leq 1\}, \quad A \in \Sigma,$$

where $|\langle m, x'\rangle|$ is the *variation measure* of the scalar measure $\langle m, x'\rangle$ given by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$, for all $A \in \Sigma$. Let us mention that a continuous integral operator $f \mapsto \int_\Omega f dm$ can be defined in both spaces with values in the bidual, X'', of the space X for weakly integrable functions f, and with values in X in the case of integrable functions f. For details about this kind of integral see [6], [8], [9], [10] and [11].

We find a first connection between the Dunford, Pettis or Bochner integrals and the Bartle–Dunford–Schwartz integral by considering a function $G \in \mathcal{P}(\mu, X)$ and the corresponding vector measure m_G given by $m_G(A) = \int_A G d\mu$, for all $A \in \mathcal{A}$. Taking into account that it is a μ–continuous countably additive vector measure it is not difficult to see that $L^1_w(m_G)$ is the space of measurable functions $f : \Lambda \longrightarrow \mathbb{R}$ such that $f \cdot G \in \mathcal{D}(\mu, X)$. Moreover

$$\|f\|_{L^1_w(m_G)} = \|f \cdot G\|_{\mathcal{P}}, \quad f \in L^1_w(m_G).$$

The integration map is given by

$$f \in L^1_w(m_G) \longrightarrow \int_\Lambda f dm_G = \int_\Lambda f \cdot G d\mu \in X''.$$

In this way, the Banach space $L^1_w(m_G)$ can be isometrically embedded in the space $\mathcal{D}(\mu, X)$. Analogously, the space $L^1(m_G)$ can be isometrically embedded in the space $\mathcal{P}(\mu, X)$, since $\int_\Lambda f dm_G = \int_\Lambda f \cdot G \, d\mu \in X$, for all $f \in L^1(m_G)$. This situation has been considered by Stefansson in [11] for the case of a finite measure μ. Furthermore, when the function $G \in \mathcal{B}(\mu, X)$ it is well–known that the variation measure of m_G is given by $|m_G|(A) = \int_A \|G\| \, d\mu$, for all $A \in \mathcal{A}$, and consequently is a finite measure. Moreover, the Lebesgue space $L^1(|m_G|)$ consists of all measurable functions $f : \Lambda \longrightarrow \mathbb{R}$ such that $f \cdot G \in \mathcal{B}(\mu, X)$. For the norms we have, in this case, the following equality

$$\|f\|_{L^1(|m_G|)} = \|f \cdot G\|_{\mathcal{B}}, \quad f \in L^1(|m_G|).$$

In this paper we are interested in another connection between Dunford, Pettis or Bochner integrals and the Bartle–Dunford–Schwartz integral. Starting from a Banach–valued countably additive vector measure $m : \Sigma \to X$, defined on a σ–algebra Σ of subsets of a non–empty set Ω, we want to relate the integral $\int_\Omega f dm$ of a function f belongs to $L^1_w(m)$ or $L^1(m)$ with the integral of a vector valued function (related to f and m) in the sense of Dunford, Pettis or Bochner. For scalar measures this situation is really well–known. For example, if (Ω, Σ, μ) is a positive measure space, then (see [1, Proposition II.1.8])

$$\int_\Omega |f| \, d\mu = \int_0^\infty \mu_f d\lambda, \quad f \in L^1(\mu), \tag{1}$$

where λ denote the Lebesgue measure and μ_f is the distribution function of f with respect to μ. For a vector measure m this is achieved via the *distribution function* with respect to m that we are going to introduce right now. Given a measurable function $f : \Omega \longrightarrow \mathbb{R}$, the distribution function of f with respect to the vector measure m is the vector valued function m_f given by

$$m_f(t) := m(\{w \in \Omega : |f(w)| > t\}), \quad t \geq 0.$$

The distribution function m_f is a norm bounded function since the measure m has bounded range. Moreover, for a simple function, say $f = \sum_{k=0}^{n} a_k \chi_{A_k}$, where $0 = a_0 < a_1 < \cdots < a_n$ and $\{A_1, \ldots, A_n\}$ are measurable pairwise disjoint sets, a direct computation shows that its distributions function is the vector valued simple function $m_f = \sum_{k=1}^{n} m(C_k)\, \chi_{[a_{k-1}, a_k)}$, where $C_k := \cup_{j=k}^{n} A_j$, for all $k = 1, 2, \ldots, n$.

In what follows, for a measurable function $f : \Omega \longrightarrow \mathbb{R}$ and a scalar t, we will use the following notation $[f > t] := \{w \in \Omega : f(w) > t\}$.

Lemma 1.1. *If $(f_n)_n$ and f are non–negative measurable functions such that f_n increases to f pointwise m–a.e., then $m_{f_n}(t)$ converges in norm to $m_f(t)$ for all $t \geq 0$. In particular, if f is a measurable function, then its distribution function m_f is strongly measurable.*

Proof. With the above notation, for every $t \geq 0$, observe that $([f_n > t])_n$ is a sequence of measurable sets increasing to $[f > t]$. Consequently (see [6, Chapter IV, §10] for example)

$$\|m_f(t) - m_{f_n}(t)\| = \|m([f > t]) - m([f_n > t])\| \leq \|m\|([f > t] \setminus [f_n > t]) \to 0,$$

when $n \to \infty$. If f is a measurable function, then there exists a sequence $(\varphi_n)_n$ of non–negative simple function increasing to $|f|$ pointwise m–a.e. Thus $m_{\varphi_n}(t)$ converges in norm to $m_f(t)$ for all $t \geq 0$, and the distribution function m_f is strongly measurable. $\square$

In what follows we will consider the σ–algebra $\mathcal{M}$ of measurable Lebesgue subsets of the half line $[0, \infty)$, and λ will denote the Lebesgue measure.

2. Dunford and Pettis integrability of the distribution function.

In this section we will prove that Dunford or Pettis integrability of the distribution function m_f of a scalar function f is strongly related with Bartle–Dunford–Schwartz integrability of the function f with respect to the vector measure m. In order to do that we need the following lemmas that are probably known, but we include their proofs for the sake of completeness.

Lemma 2.1. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function and let $B \in \mathcal{M}$. Then the function $f^B : w \in \Omega \longrightarrow f^B(w) := \lambda(B \cap [0, |f(w)|]) \in \mathbb{R}$ is measurable. Moreover, if $f \in L^1_w(m)$, then $f^B \in L^1_w(m)$, and similarly, if $f \in L^1(m)$, then $f^B \in L^1(m)$.*

Proof. For a simple function, say $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$, where the A_k's are measurable pairwise disjoint subsets of Ω and the a_k's are scalars, it is not difficult to see that φ^B is the simple function

$$\varphi^B = \sum_{k=1}^{n} \lambda(B \cap [0, |a_k|])\, \chi_{A_k}.$$

Now, if f is a measurable function then there exists a sequence $(\varphi_n)_n$ of simple functions increasing to $|f|$ pointwise m–a.e. Then

$$f^B(w) = \lambda\left(B \cap [0, |f(w)|]\right) = \lim_n \lambda\left(B \cap [0, |\varphi_n(w)|]\right) = \lim_n \varphi_n^B(w),$$

for all $w \in \Omega \setminus N$, where $\|m\|(N) = 0$, that is, the sequence $(\varphi_n^B)_n$ increases to f^B pointwise m–a.e., and so f^B is measurable. For the last part of the assertion, take into account that $0 \leq f^B \leq |f|$ and both $L^1_w(m)$ and $L^1(m)$ are ideals of measurable functions. $\square$

Remark 2.2. Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function and take B to be the interval $[0, n]$. Then $f^{[0,n]}(w) = \lambda\left([0, n] \cap [0, |f(w)|]\right) = \inf\{|f(w)|, n\}$. In what follows we denote this function by f_n, that is, $f_n := \inf\{|f|, n\}$, for all $n = 1, 2, \ldots$ Note that $0 \leq f_n \uparrow |f|$ pointwise m–a.e. Moreover it is not difficult to check that $m_{f_n \cdot \chi_A} = m_{f\cdot\chi_A} \cdot \chi_{[0,n]}$, for all $A \in \Sigma$, and $n = 1, 2, \ldots$

Lemma 2.3. *Let h be and $(h_n)_n$ in $L^1(\lambda)$ such that h_n converge to h pointwise λ–a.e., and suppose that there exists $\lim_n \int_B h_n d\lambda$, for each $B \in \mathcal{M}$. Then, $\lim_n \int_B h_n d\lambda = \int_B h\, d\lambda$ for all $B \in \mathcal{M}$.*

Proof. For each $B \in \mathcal{M}$ and $k = 1, 2, \ldots$ let $B_k := B \cap [k-1, k)$, and put $\alpha_{kn} := \int_{B_k} h_n d\lambda$, for all $k, n = 1, 2, \ldots$ Then

$$\sum_{k=1}^{\infty} |\alpha_{kn}| = \sum_{k=1}^{\infty} \left| \int_{B_k} h_n d\lambda \right| \leq \sum_{k=1}^{\infty} \int_{B_k} |h_n|\, d\lambda = \int_B |h_n|\, d\lambda < \infty.$$

This means that the sequence $\alpha^n := (\alpha_{kn})_k$ is in ℓ_1, for all $n = 1, 2, \ldots$ Moreover, for each subset N of natural numbers we have that $\sum_{k\in N} \alpha_{kn} = \int_{\bigcup_{k\in N} B_k} h_n d\lambda$, and, by the hypothesis, there exists $\lim_n \sum_{k\in N} \alpha_{kn}$. As the subset N of natural numbers is arbitrary, this means that the sequence $(\alpha^n)_n$ is weakly convergent in ℓ_1, say to a sequence $\alpha := (\alpha_k)_k \in \ell_1$. Taking into account that the Lebesgue measure of B_k is finite for each $k = 1, 2, \ldots$, an application of the Vitali–Hahn–Saks theorem (see [6, Chapter III, §7] for example) gives us

$$\alpha_k = \lim_n \alpha_{kn} = \lim_n \int_{B_k} h_n d\lambda = \int_{B_k} h d\lambda, \quad k = 1, 2, \ldots$$

Thus, by the Schur theorem, the sequence $(\alpha^n)_n$ converge to α in the norm of ℓ_1. Then $\lim_n \sum_{k=1}^{\infty} \left| \int_{B_k} h_n d\lambda - \int_{B_k} h d\lambda \right| = 0$, and finally

$$\begin{aligned}\lim_n \left| \int_B h_n d\lambda - \int_B h d\lambda \right| &= \lim_n \left| \sum_{k=1}^{\infty} \int_{B_k} h_n d\lambda - \sum_{k=1}^{\infty} \int_{B_k} h d\lambda \right| \\ &\leq \lim_n \sum_{k=1}^{\infty} \left| \int_{B_k} h_n d\lambda - \int_{B_k} h d\lambda \right| = 0,\end{aligned}$$

as we want to see. $\square$

Theorem 2.4. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function. Then $f \in L^1_w(m)$ if and only if $m_g \in \mathcal{D}(\lambda, X)$ for all measurable function g such that $|g| \leq |f|$. In this case we have*

$$\int_B \langle m_f, x' \rangle \, d\lambda = \int_\Omega \lambda\left(B \cap [0, |f|]\right) d \langle m, x' \rangle \,, \quad B \in \mathcal{M}, \; x' \in X'. \tag{2}$$

In particular $\int_0^\infty \langle m_{f\chi_A}, x' \rangle \, d\lambda = \int_A |f| \, d \langle m, x' \rangle$, for all $A \in \Sigma$ and $x' \in X'$.

Proof. Let $f \in L^1_w(m)$. Then there exists a sequence $(\varphi_n)_n$ of simple functions such that $0 \leq \varphi_n \uparrow |f|$ pointwise m–a.e. We know that $m_{\varphi_n} \to m_f$ pointwise in the norm of X by the Lemma 1.1. Then, $|\langle m_{\varphi_n}, x' \rangle| \to |\langle m_f, x' \rangle|$ pointwise for all $x' \in X'$. Moreover

$$\begin{aligned} 0 \leq \int_0^\infty |\langle m_{\varphi_n}, x' \rangle| \, d\lambda &\leq \int_0^\infty |\langle m, x' \rangle|_{\varphi_n} \, d\lambda \\ &= \int_\Omega \varphi_n d \, |\langle m, x' \rangle| \to \int_\Omega |f| \, d \, |\langle m, x' \rangle| \,, \end{aligned} \tag{3}$$

when $n \to \infty$. In the above formula $|\langle m, x' \rangle|_{\varphi_n}$ denotes the distribution function of φ_n with respect to the positive measure $|\langle m, x' \rangle|$. Note that we have used equality (1) to obtain the above expression (3). By applying Fatou's lemma

$$\int_0^\infty \liminf_n |\langle m_{\varphi_n}, x' \rangle| \, d\lambda \leq \liminf_n \int_0^\infty |\langle m_{\varphi_n}, x' \rangle| \, d\lambda < \infty.$$

So, $\liminf_n |\langle m_{\varphi_n}, x' \rangle| = \lim_n |\langle m_{\varphi_n}, x' \rangle| = |\langle m_f, x' \rangle|$ is integrable. Since x' is arbitrary this tells us that $m_f \in \mathcal{D}(\lambda, X)$. Now, for a measurable function g such that $|g| \leq |f|$, we know that $g \in L^1_w(m)$, so by applying what we have just proved for f, we get $m_g \in \mathcal{D}(\lambda, X)$ too. Note that taking limit, when $n \to \infty$, in (3) we obtain

$$\int_0^\infty |\langle m_f, x' \rangle| \, d\lambda \leq \int_\Omega |f| \, d \, |\langle m, x' \rangle| \,. \tag{4}$$

According to Lemma 2.1 the function $w \mapsto \lambda\left(B \cap [0, |f(w)|]\right)$ is in $L^1_w(m)$. Moreover, a direct computation shows that formula (2) is true for simple functions, that is, for every $B \in \mathcal{M}$ we have

$$\int_B \langle m_{\varphi_n}, x' \rangle \, d\lambda = \int_\Omega \lambda\left(B \cap [0, |\varphi_n|]\right) d \langle m, x' \rangle \,, \quad n \geq 1. \tag{5}$$

Also note that

$$\lim_n \int_\Omega \lambda\left(B \cap [0, |\varphi_n|]\right) d \langle m, x' \rangle = \int_\Omega \lambda\left(B \cap [0, |f|]\right) d \langle m, x' \rangle \,.$$

Then formula (2) follows from Lemma 2.3 applied to the functions $h := \langle m_f, x' \rangle$ and $h_n := \langle m_{\varphi_n}, x' \rangle$, by taking limit when $n \to \infty$ in the above equality (5).

Finally, taking $B = [0, \infty)$ and changing f by $f\chi_A$, with $A \in \Sigma$, in (2) we arrive to $\int_0^\infty \langle m_{f\chi_A}, x' \rangle \, d\lambda = \int_A |f| \, d \langle m, x' \rangle$.

Reciprocally, suppose $f : \Omega \longrightarrow \mathbb{R}$ is a measurable function such that $m_g \in \mathcal{D}(\lambda, X)$ for all measurable function g such that $|g| \leq |f|$. Take the sequence $(f_n)_n$ from Remark 2.2 associated to f, and let $x' \in X'$ and $A \in \Sigma$ be arbitrary. By the hypothesis $m_{f \cdot \chi_A}$ and $m_{f_n \cdot \chi_A}$ belong to $\mathcal{D}(\lambda, X)$, for every $n = 1, 2, \ldots$, and consequently the functions $\langle m_{f \cdot \chi_A}, x' \rangle$ and $\langle m_{f_n \cdot \chi_A}, x' \rangle$ are integrable with respect to the Lebesgue measure λ, for all $n = 1, 2, \ldots$ Thus, taking into account Remark 2.2 and also that $f_n \in L^1_w(m)$, we obtain

$$\begin{aligned} \int_A f_n d\langle m, x' \rangle &= \int_0^\infty \langle m_{f_n \cdot \chi_A}, x' \rangle d\lambda = \int_0^\infty \langle m_{f \cdot \chi_A} \cdot \chi_{[0,n]}, x' \rangle d\lambda \\ &= \int_0^n \langle m_{f \cdot \chi_A}, x' \rangle d\lambda \to \int_0^\infty \langle m_{f \cdot \chi_A}, x' \rangle d\lambda, \end{aligned}$$

when $n \to \infty$. The convergence of the integrals in the above expression follows from the integrability of $\langle m_{f \cdot \chi_A}, x' \rangle$ with respect to the Lebesgue measure λ. By applying [10, Theorem 3.5] to the measure $\langle m, x' \rangle$ we conclude that f belongs to $L^1(\langle m, x' \rangle)$, since the sequence $\left(\int_A f_n d\langle m, x' \rangle \right)_n$ of integrals is convergent for all $A \in \Sigma$. Accordingly $f \in L^1_w(m)$ and the proof is complete. □

Remark 2.5. Note that inequality (4) tell us that $\|m_f\|_{\mathcal{P}} \leq \|f\|_{L^1_w(m)}$, for all $f \in L^1_w(m)$. If the space X is a Banach lattice and the measure m is positive, that is, $m(A)$ belongs to the positive cone of X, it suffices that $m_f \in \mathcal{D}(\lambda, X)$ for f belongs to $L^1_w(m)$. Taking into account that $|\langle m, x' \rangle|(A) \leq \langle m, |x'| \rangle(A)$ for every $x' \in X'$ (here $|x'|$ denotes the absolute value of the functional x') and each $A \in \Sigma$, we conclude that

$$\int_\Omega |f| \, d \, |\langle m, x' \rangle| \leq \int_\Omega |f| \, d\langle m, |x'| \rangle = \int_0^\infty \langle m_f, |x'| \rangle d\lambda.$$

and taking supremum in x', with $\|x'\| \leq 1$, we obtain $\|f\|_{L^1_w(m)} \leq \|m_f\|_{\mathcal{P}}$. In summary, for a positive vector measure m, we have that a function $f \in L^1_w(m)$ if and only if $m_f \in \mathcal{D}(\lambda, X)$, and $\|m_f\|_{\mathcal{P}} = \|f\|_{L^1_w(m)}$. For a general measure m the last equality is far from to be true as the next simple example shows.

Example 2.6. Consider the Lebesgue measure λ on the interval $[-1, 1]$ and the measure

$$m : A \in \mathcal{M} \to m(A) := \lambda(A \cap [-1, 0]) - \lambda(A \cap [0, 1]) \in \mathbb{R}.$$

Then $L^1_w(m) = \mathcal{D}(\lambda, \mathbb{R}) = L^1(\lambda)$, the classical Lebesgue space. Take a non–negative measurable even function f such that $f \notin L^1(\lambda)$. Thus m_f is the zero function which obviously is in $\mathcal{D}(\lambda, \mathbb{R})$, but $f \notin L^1_w(m)$.

Now we consider Pettis integrability of the distribution function. The following result is the analogous one of Theorem 2.4.

Theorem 2.7. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function. Then $f \in L^1(m)$ if and only if $m_g \in \mathcal{P}(\lambda, X)$ for all measurable function g such that $|g| \leq |f|$. In this case we have*

$$\int_B m_f d\lambda = \int_\Omega \lambda\left(B \cap [0, |f|]\right) dm, \quad B \in \mathcal{M}. \tag{6}$$

In particular, $\int_0^\infty m_{f\chi_A} d\lambda = \int_A |f|\, dm$, for all $A \in \Sigma$.

Proof. Let $f \in L^1(m)$. Then Theorem 2.4 assures that $m_f \in \mathcal{D}(\lambda, X)$. Now, for every $B \in \mathcal{M}$, the function $w \mapsto \lambda\left(B \cap [0, |f(w)|]\right)$ belongs to $L^1(m)$ and its integral $\int_\Omega \lambda\left(B \cap [0, |f|]\right) dm$ is in X. From (2) we obtain, for all $x' \in X'$,

$$\int_B \langle m_f, x'\rangle\, d\lambda = \int_\Omega \lambda\left(B \cap [0, |f|]\right) d\langle m, x'\rangle = \left\langle \int_\Omega \lambda(B \cap [0\,|f|])dm, x' \right\rangle.$$

This means that $m_f \in \mathcal{P}(\lambda, X)$, and moreover

$$\int_B m_f d\lambda = \int_\Omega \lambda\left(B \cap [0, |f|]\right) dm. \tag{7}$$

Now, if g is a measurable function such that $|g| \leq |f|$, then $g \in L^1(m)$, and so $m_g \in \mathcal{P}(\lambda, X)$. In particular, if $A \in \Sigma$ then $m_{f\chi_A} \in \mathcal{P}(\lambda, X)$, and replacing f by $f\chi_A$ in (7), with $B = [0, \infty)$, we obtain $\int_0^\infty m_{f\cdot\chi_A} d\lambda = \int_A |f|\, dm$.

Reciprocally, suppose $f : \Omega \longrightarrow \mathbb{R}$ is a measurable function such that $m_g \in \mathcal{P}(\lambda, X)$ for all measurable function g such that $|g| \leq |f|$. Take the sequence $(f_n)_n$ of Remark 2.2 and let $A \in \Sigma$. Then $0 \leq f_n \cdot \chi_A \uparrow |f| \cdot \chi_A$ pointwise m–a.e. We know that

$$\begin{aligned}\int_A f_n dm = \int_0^\infty m_{f_n\cdot\chi_A} d\lambda &= \int_0^\infty m_{f\cdot\chi_A} \cdot \chi_{[0,n]} d\lambda \\ &= \int_0^n m_{f\cdot\chi_A} d\lambda \to \int_0^\infty m_{f\cdot\chi_A} d\lambda,\end{aligned}$$

when $n \to \infty$. The norm convergence of the integrals in the above expression follows from the Pettis integrability of $m_{f\cdot\chi_A}$ with respect to the Lebesgue measure λ. By applying [10, Theorem 3.5] to the measure m we conclude that $f \in L^1(m)$, since the sequence $\left(\int_A f_n dm\right)_n$ of integrals is convergent for all $A \in \Sigma$. The proof is finished. □

Remark 2.8. As in the Remark 2.5, if the space X is a Banach lattice and the measure m is positive in Theorem 2.7, it suffices that $m_f \in \mathcal{P}(\lambda, X)$ for f belongs to $L^1(m)$. For a general measure m this is not true as the Example 2.6 also shows.

3. Bochner integrability of the distribution function.

A natural question appears at this point. For which functions f can we assure that its distribution function m_f is Bochner integrable? Given a vector measure $m : \Sigma \to X$, the aim of this section is to describe the set of measurable functions

$f : \Omega \longrightarrow \mathbb{R}$ such that it distribution function m_f belongs to $\mathcal{B}(\lambda, X)$. With the idea to answer this question we introduce the space $L^1(\|m\|)$ of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the real valued function

$$\|m\|_f : t \in [0, \infty) \longrightarrow \|m\|_f(t) := \|m\|([|f| > t]) \in \mathbb{R}$$

is in $L^1(\lambda)$. Note that $\|m\|_f$ is the *distribution function* of f with respect to the semivariation $\|m\|$, which, in general, is not a scalar measure, it is only a *submeasure*. Moreover $\|m(A)\| \leq \|m\|(A)$, for all $A \in \Sigma$, but when X is a Banach lattice and m is a positive vector measure $\|m\|(A) = \|m(A)\|$, for all $A \in \Sigma$. In this way $L^1(\|m\|)$ is the space of all scalar measurable functions f such that the Choquet integral of $|f|$, with respect to $\|m\|$, is finite. See [2] for details. As usual we identify functions which are equal m–a.e. In that follows we will consider the space $L^\infty(m)$ of all real valued measurable functions defined on Ω that are *essentially bounded* with respect to m. As usual, the norm $\|f\|_{L^\infty(m)}$ of a function $f \in L^\infty(m)$ is the essential supremum of the function f.

Proposition 3.1. *The space $L^1(\|m\|)$ is a quasi–Banach lattice with the Fatou property. Moreover, the simple functions are dense in $L^1(\|m\|)$.*

Proof. For $f \in L^1(\|m\|)$ denote $\|f\|_{L^1(\|m\|)} := \int_0^\infty \|m\|_f \, d\lambda$. Given two measurable functions f and g note that

$$[|f + g| > t] \subseteq \left[|f| > \frac{t}{2}\right] \cup \left[|g| > \frac{t}{2}\right], \quad t \geq 0,$$

so $\|m\|_{f+g}(t) \leq \|m\|_f\left(\frac{t}{2}\right) + \|m\|_g\left(\frac{t}{2}\right)$, for all $t \geq 0$, and consequently

$$\|f + g\|_{L^1(\|m\|)} \leq 2\left(\|f\|_{L^1(\|m\|)} + \|g\|_{L^1(\|m\|)}\right), \quad f, g \in L^1(\|m\|).$$

It easy to check that $\|\cdot\|_{L^1(\|m\|)}$ is homogenous and that $\|f\|_{L^1(\|m\|)} = 0$ implies that $f = 0$, m–a.e. This means that $\|\cdot\|_{L^1(\|m\|)}$ is a quasi–norm. Moreover if $f \in L^1(\|m\|)$ and g is a measurable function such that $|g| \leq |f|$, then $g \in L^1(\|m\|)$ and $\|g\|_{L^1(\|m\|)} \leq \|f\|_{L^1(\|m\|)}$. Thus the space $L^1(\|m\|)$ becomes a quasi–normed lattice. Moreover if $(f_n)_n$ is an increasing sequence of measurable function such that $f_n \uparrow f$ pointwise m–a.e., then $0 \leq \|m\|_{f_n}(t) \uparrow \|m\|_f(t)$, for all $t \geq 0$, and so, the space $L^1(\|m\|)$ has the Fatou property. It is standard to check (see [1, Theorem I.1.1.6] for details) that the Fatou property implies the completeness of $L^1(\|m\|)$. Finally, we are going to prove that simple functions are dense. Take $0 \leq f \in L^1(\|m\|)$ and let the sequence $(f_n)_n$ from Remark 2.2. Then,

$$\begin{aligned}
\|f - f_n\|_{L^1(\|m\|)} &= \int_0^\infty \|m\|([f - f_n > t]) \, d\lambda(t) \\
&= \int_0^\infty \|m\|([f > n + t]) \, d\lambda(t) \\
&= \int_n^\infty \|m\|([f > s]) \, d\lambda(s) \to 0,
\end{aligned}$$

when $n \to \infty$. Thus, given $\varepsilon > 0$ there exists n_0 such that

$$\|f - f_{n_0}\|_{L^1(\|m\|)} < \frac{\varepsilon}{4}.$$

Since f_{n_0} is a bounded function, there exists a simple function φ, with $0 \le \varphi \le f_{n_0}$, such that $\|f_{n_0} - \varphi\|_{L^\infty(m)} < \frac{\varepsilon}{4\|m\|(\Omega)}$. Consequently,

$$\begin{aligned}\|f_{n_0} - \varphi\|_{L^1(\|m\|)} &= \int_0^\infty \|m\|\left([f_{n_0} - \varphi > t]\right) d\lambda(t)\\ &= \int_0^{\frac{\varepsilon}{4\,\|m\|(\Omega)}} \|m\|\left([f_{n_0} - \varphi > t]\right) d\lambda(t) \le \frac{\varepsilon}{4},\end{aligned}$$

and $\|f - \varphi\|_{L^1(\|m\|)} \le 2\,\|f - f_{n_0}\|_{L^1(\|m\|)} + 2\,\|f_{n_0} - \varphi\|_{L^1(\|m\|)} < \varepsilon$. For an arbitrary function $f \in L^1(\|m\|)$ it is enough to consider now its positive and negative parts. □

Now, taking into account the previous comments to Proposition 3.1, together with Proposition 3.1 and [7, Theorem 3.7.4] we obtain the following result.

Theorem 3.2. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function.*

1) *If $f \in L^1(\|m\|)$, then $m_g \in \mathcal{B}(\lambda, X)$, for all measurable function g such that $|g| \le |f|$.*
2) *If X is a Banach lattice and m is a positive vector measure, then f belongs to $L^1(\|m\|)$ if and only if $m_f \in \mathcal{B}(\lambda, X)$.*

Remark 3.3. Recall that a vector measure m with values in a Banach lattice X is said to have a *Hahn decomposition* if there exist two disjoint measurable sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, with $m(A \cap \Omega_1) \ge 0$ and $m(A \cap \Omega_2) \le 0$, for every $A \in \Sigma$. For this kind of vector measures we have that

$$\|m\|(A) \le \|m\,(A \cap \Omega_1)\,\| + \|m\,(A \cap \Omega_2)\,\|, \quad A \in \Sigma.$$

Then, for a vector measure with a Hahn decomposition, a function $f \in L^1(\|m\|)$ if and only if $m_g \in \mathcal{B}(\lambda, X)$ for every measurable function g such that $|g| \le |f|$.

We finish this section with some properties of the space $L^1(\|m\|)$ and its connection with the previously considered spaces. Recall that $|m|$ denotes the variation measure of the vector measure m. It is well–known that $\|m\|(A) \le |m|(A)$, for all $A \in \Sigma$.

Proposition 3.4. *The inclusions $L^\infty(m) + L^1(|m|) \subseteq L^1(\|m\|) \subseteq L^1(m)$ hold. Moreover, these inclusions are continuous.*

Proof. To prove the first inclusion take $f \in L^\infty(m)$ and $g \in L^1(|m|)$. Note that

$\|m\|_f(t) = 0$, for all $t \geq \|f\|_{L^\infty(m)}$. Then

$$\begin{aligned}\|f+g\|_{L^1(\|m\|)} &\leq 2\|f\|_{L^1(\|m\|)} + 2\|g\|_{L^1(\|m\|)} \\ &\leq 2\|f\|_{L^\infty(m)}\|m\|(\Omega) + 2\int_0^\infty |m|([|g|>t])\,d\lambda(t) \\ &= 2\|f\|_{L^\infty(m)}\|m\|(\Omega) + 2\|g\|_{L^1(|m|)}. \qquad (8)\end{aligned}$$

From (8) it follows that $\|h\|_{L^1(\|m\|)} \leq 2\max\{\|m\|(\Omega), 1\}\|h\|_{L^\infty(m)+L^1(|m|)}$, for all $h \in L^\infty(m) + L^1(|m|)$, and the inclusion is continuous.

To prove the second inclusion take a function $f \in L^1(\|m\|)$. Then, Theorem 3.2 assures that $m_g \in \mathcal{B}(\lambda, X) \subseteq \mathcal{P}(\lambda, X)$, for all measurable function g such that $|g| \leq |f|$. From Theorem 2.7 we obtain that $f \in L^1(m)$. Moreover, for every $x' \in X'$, with $\|x'\| \leq 1$, we have that

$$\int_\Omega |f|\,d|\langle m, x'\rangle| = \int_0^\infty |\langle m, x'\rangle|_f\,d\lambda \leq \int_0^\infty \|m\|_f\,d\lambda.$$

Taking supremum in x' in the above inequality, we get $\|f\|_{L^1(m)} \leq \|f\|_{L^1(\|m\|)}$, for all $f \in L^1(\|m\|)$. □

The inclusions in the above proposition can be strict as the next example shows.

Example 3.5. Consider the following vector measure m, defined on all subsets of natural numbers, with values in the space c_0 of all null real sequences

$$m : A \in \mathcal{P}(\mathbb{N}) \longrightarrow m(A) := \sum_{n\in A} \frac{e_n}{n^2} \in c_0,$$

where the e_n's are the canonical vectors in c_0. In this case the integrable functions with respect to m are sequences of real numbers, and we can compute easily the associated spaces of integrable functions. Namely,

$$\begin{aligned} L^1_w(m) &= \left\{(f_n)_n \in \mathbb{R}^\mathbb{N} : \left(|f_n|\,n^{-2}\right)_n \in \ell_\infty\right\}, \\ L^1(m) &= \left\{(f_n)_n \in \mathbb{R}^\mathbb{N} : \left(|f_n|\,n^{-2}\right)_n \in c_0\right\}, \\ L^1(|m|) &= \left\{(f_n)_n \in \mathbb{R}^\mathbb{N} : \left(|f_n|\,n^{-2}\right)_n \in \ell_1\right\}. \end{aligned}$$

Here ℓ_∞ is the space of all bounded real sequences and ℓ_1 is the space of all absolutely summable real sequences. Moreover, for a function $f := (f_n)_n$ and a scalar $t \geq 0$, it can be checked that $\|m\|([|f|>t]) = \frac{1}{(\min\{n\in\mathbb{N}:|f_n|>t\})^2}$. The variation measure of m is given by

$$|m|(A) = \sum_{n\in A} \frac{1}{n^2}, \quad A \in \mathcal{P}(\mathbb{N}).$$

Note that the function $f = (f_n)_n$, with $f_n := n$, is in $L^1(\|m\|)$ because

$$\int_0^\infty \|m\|([|f| > t])\, d\lambda(t) = \sum_{n=0}^\infty \int_n^{n+1} \|m\|([|f| > t])\, d\lambda(t) = \sum_{n=0}^\infty \frac{1}{(n+1)^2},$$

but clearly f is not in $L^\infty(m) + L^1(|m|)$. Now take the function $f = (f_n)_n$, given by $f_n := \frac{(n+1)^2}{\log(n+1)}$, which belongs to $L^1(m)$. However it is not in $L^1(\|m\|)$. In fact, if we put $a_k = \frac{(k+1)^2}{\log(k+1)}$, then

$$\int_{a_1}^\infty \|m\|([|f| > t])\, d\lambda(t) = \sum_{k=1}^\infty \frac{a_{k+1} - a_k}{(k+1)^2} = \log 2 + \sum_{k=1}^\infty \frac{2k+3}{(k+1)^2 \log(k+2)},$$

which is a divergent series.

Now we present a sufficient condition for the equality between the spaces $L^1(m)$ and $L^1(\|m\|)$. For a vector measure $m : \Sigma \to X$ and an operator $T : X \to Y$ with values in a second Banach space Y, consider the vector measure

$$m_T : A \in \Sigma \to m_T(A) := T(m(A)) \in Y.$$

It is not difficult to see that $\|m_T\|(A) \le \|T\|\,\|m\|(A)$, for all $A \in \Sigma$. In particular, every m–null set is an m_T–null set, and $L^1(m) \subseteq L^1(m_T)$ as is seen in [6, IV.10.8(f)]. Recall that the operator T is said to be *absolutely summing* if there exists a constant $C > 0$ such that

$$\sum_{k=1}^n \|Tx_k\| \le C \sup\left\{ \sum_{k=1}^n |\langle x_k, x'\rangle| : \|x'\| \le 1 \right\}, \; x_1, \ldots, x_n \in X, \; n \ge 1.$$

For details about absolutely summing operator see the monograph [5]. The following proposition is the counterpart (in the setting of the Bartle–Dunford–Schwartz integral) of the main result of [4] about Pettis and Bochner integrals.

Proposition 3.6. *Let $m : \Sigma \to X$ be a vector measure and $T : X \to Y$ an absolutely summing operator. Then $L^1_w(m) \subseteq L^1(\|m_T\|)$, and this inclusion is continuous.*

Proof. Take a function $f \in L^1_w(m)$ and consider, for all $k \ge 1$, the level sets $[|f| > k]$. For every $x' \in X'$ we have $\int_\Omega |f|\, d\,|\langle m, x'\rangle| = \int_0^\infty |\langle m, x'\rangle|_f \, d\lambda$, and consequently the integral $\int_0^\infty |\langle m, x'\rangle|_f \, d\lambda$ is convergent. Since the integrand $|\langle m, x'\rangle|_f$ is not increasing, this means that the series $\sum_{k=1}^\infty |\langle m, x'\rangle|([|f| > k])$ is convergent. Thus, for every sequence $(A_k)_k$ of measurable sets such that $A_k \subseteq [|f| > k]$, for all $k \ge 1$, we obtain that $\sum_{k=1}^\infty |\langle m, x'\rangle|(A_k)$ is also convergent, and in particular, $\sum_{k=1}^\infty |\langle m(A_k), x'\rangle|$ is convergent. Now apply that T is absolutely summing to obtain that $\sum_{k=1}^\infty \|m_T(A_k)\|$ is convergent. Since the sequence $(A_k)_k$ of subsets of $([|f| > k])_k$ is arbitrary, we conclude that the series $\sum_{k=1}^\infty \|m_T\|([|f| > k])$ is convergent, and consequently also the integral $\int_0^\infty \|m_T\|_f \, d\lambda$ is convergent. This means that $f \in L^1(\|m_T\|)$. The inclusion is continuous because it is a positive operator and $L^1_w(m)$ is a Banach lattice, and the proof is concluded. $\square$

Corollary 3.7. *Let $m : \Sigma \to X$ be a vector measure. If $\dim(X) < \infty$, then $L^1(\|m\|) = L^1(m) = L^1_w(m)$.*

Proof. The first equality follows from Proposition 3.6 and the second one is a very well–known fact (see [8, Theorem II.5.1] or [10, p. 138]). $\square$

Remark 3.8. When equality $L^1(\|m\|) = L^1(m)$ holds, taking into account that $L^1(\|m\|)$ has the Fatou property, it follows the other equality (see [3] for the details), that is, the three spaces coincide. Nevertheless, the situation $L^1(\|m\|) \subsetneq L^1(m) = L^1_w(m)$ is possible as the next example shows.

Example 3.9. Consider the Lebesgue measure λ on the σ–algebra $\mathcal{M}$ of Lebesgue measurable subsets of the interval $(0,1)$, and also the (positive) vector measure $m : A \in \mathcal{M} \to m(A) := \chi_A \in L^2(0,1)$. Since $L^2(0,1)$ is a weakly sequentially complete Banach space we have that $L^1(m) = L^1_w(m) = L^2(0,1)$ isometrically. Moreover $\|m\|(A) = \|m(A)\|_{L^2(0,1)} = \sqrt{\lambda(A)}$, for all $A \in \mathcal{M}$. Let us check that $L^1(\|m\|)$ is a proper subspace of $L^1(m)$. We have to find a function $f \in L^2(0,1)$ such that $f \notin L^1(\|m\|)$. In order to do that let us consider the real function $g(t) := \frac{2(\log 2)^2}{(t+2)(\log(t+2))^2}$. Note that g is a decreasing and continuous function from $(0,\infty)$ onto $(0,1)$. Then, its inverse g^{-1} is a well–defined function from $(0,1)$ onto $(0,\infty)$. Moreover, it is easy to see that $g \in L^1(0,\infty)$. Now consider the function

$$f : w \in (0,1) \longrightarrow f(\omega) := \sqrt{g^{-1}(\omega)} \in \mathbb{R}.$$

We can compute the distribution function of f^2 with respect to the Lebesgue measure. In fact, for all $t > 0$, we have

$$\begin{aligned} \lambda\left(\left\{\omega \in (0,1) : f(\omega)^2 > t\right\}\right) &= \lambda\left(\left\{\omega \in \ (0,1) : g\left(f(\omega)^2\right) < g(t)\right\}\right) \\ &= \lambda\left(\{\omega \in \ (0,1) : \omega < g(t)\}\right) = g(t). \end{aligned}$$

Then, from (1), we obtain $f \in L^2(0,1)$, because $g \in L^1(0,\infty)$. However $f \notin L^1(\|m\|)$ because the distribution function of f with respect to the semivariation

$$\begin{aligned} \|m\|_f(t) &= \|m\|\left(\{\omega \in (0,1) : f(\omega) > t\}\right) = \sqrt{\lambda\left(\{\omega \in (0,1) : f(\omega)^2 > t^2\}\right)} \\ &= \sqrt{g(t^2)} = \frac{\sqrt{2}\log 2}{\sqrt{t^2+2}\,\log(t^2+2)} \end{aligned}$$

is not an integrable function on the interval $(0,\infty)$.

From Proposition 3.1 we know that $\|\cdot\|_{L^1(\|m\|)}$ is a quasi–norm. Under certain conditions on the vector measure m we can assure that it is, in fact, a norm. This is what we prove in the next proposition.

Proposition 3.10. *The following conditions are equivalent:*

1) $\|\cdot\|_{L^1(\|m\|)}$ *is a norm in* $L^1(\|m\|)$.

2) $\|m\|(A \cup B) + \|m\|(A \cap B) \leq \|m\|(A) + \|m\|(B),\ A, B \in \Sigma.$

Proof. *1)* $\Rightarrow$ *2)*. Suppose that $\|\cdot\|_{L^1(\|m\|)}$ is a norm in $L^1(\|m\|)$. Then

$$\|\chi_A + \chi_B\|_{L^1(\|m\|)} \leq \|\chi_A\|_{L^1(\|m\|)} + \|\chi_B\|_{L^1(\|m\|)}, \quad A, B \in \Sigma.$$

But, this is just the inequality of *2)*, because it is not difficult to see that $\|\chi_A + \chi_B\|_{L^1(\|m\|)} = \|m\|(A \cup B) + \|m\|(A \cap B)$.

2) $\Rightarrow$ *1)*. First we are going to prove that for all $f \in L^1(\|m\|)$, $A \in \Sigma$ and $a \in \mathbb{R}$, we have

$$\|f + a\chi_A\|_{L^1(\|m\|)} \leq \|f\|_{L^1(\|m\|)} + \|a\chi_A\|_{L^1(\|m\|)}. \tag{9}$$

Obviously we can assume $f \geq 0$ and $a > 0$. Then

$$\begin{aligned}
&\|f + a\chi_A\|_{L^1(\|m\|)} \\
&= \int_0^a \|m\|(A \cup [f > t])\, d\lambda(t) + \int_a^\infty \|m\|([f \cdot \chi_A > t - a] \cup [f > t])\, d\lambda(t) \\
&\leq \int_0^a (\|m\|(A) + \|m\|([f > t]))\, d\lambda(t) \\
&\qquad - \int_0^a \|m\|([f \cdot \chi_A > t])\, d\lambda(t) + \int_a^\infty \|m\|([f \cdot \chi_A > t - a])\, d\lambda(t) \\
&\qquad + \int_a^\infty \|m\|([f > t])\, d\lambda(t) - \int_a^\infty \|m\|([f \cdot \chi_A > t])\, d\lambda(t) \\
&= \int_0^a \|m\|(A) d\lambda(t) + \int_0^\infty \|m\|([f > t])\, d\lambda(t) = \|f\|_{L^1(\|m\|)} + \|a\chi_A\|_{L^1(\|m\|)}.
\end{aligned}$$

Note that we have used in the above inequality the following equality $\int_a^\infty \|m\|([f \cdot \chi_A > t - a])\, d\lambda(t) = \int_0^\infty \|m\|([f \cdot \chi_A > u])\, d\lambda(u)$.

Now we are going to prove that for all $f \in L^1(\|m\|)$ and all simple function φ, we have

$$\|f + \varphi\|_{L^1(\|m\|)} \leq \|f\|_{L^1(\|m\|)} + \|\varphi\|_{L^1(\|m\|)}. \tag{10}$$

Once again we can assume that $f \geq 0$ and $\varphi \geq 0$. Then, write $\varphi = \sum_{k=1}^n a_k \chi_{A_k}$, where $a_1, \ldots, a_n > 0$, and $A_1 \supseteq \cdots \supseteq A_n$ are measurable sets. It is easy to obtain that $\|\varphi\|_{L^1(\|m\|)} = \sum_{k=1}^n a_k \|m\|(A_k)$. Having in mind (9) we have

$$\|f + \varphi\|_{L^1(\|m\|)} \leq \|f\|_{L^1(\|m\|)} + \sum_{k=1}^n \|a_k \chi_{A_k}\|_{L^1(\|m\|)} = \|f\|_{L^1(\|m\|)} + \|\varphi\|_{L^1(\|m\|)}.$$

Finally, as the simple functions are dense in the space $L^1(\|m\|)$, we have from (10) that $\|f + g\|_{L^1(\|m\|)} \leq \|f\|_{L^1(\|m\|)} + \|g\|_{L^1(\|m\|)}$, for all $f,\ g \in L^1(\|m\|)$, and $\|\cdot\|_{L^1(\|m\|)}$ is a norm. $\square$

Remark 3.11. The condition *2*) in the above proposition is usually called the *inequality of strong subadditivity* [2, pp. 131–132]. Under this condition, the quasi–Banach space $L^1(\|m\|)$ is, in fact, a Banach space. In particular, the measure considered in Example 3.9 satisfies the strong subadditivity inequality, and so the corresponding space $L^1(\|m\|)$ is a Banach space. For a general vector measure, it is possible to see that the quasi–Banach space $L^1(\|m\|)$ has a separating dual, but we don't know in general if it is *normable*.

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