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An Upper Bound for the Convergence of Integral Functionals

Emmanuel Giner
Laboratoire MIP, Institut de Mathématiques, Université Paul Sabatier, UFR MIG, 118 route de Narbonne 31062 Toulouse Cedex 04, France
giner@math.univ-toulouse.fr

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Given a σ-finite measure space $(\Omega, \mathcal{T}, \mu)$ endowed with a μ-complete tribe, a separable Banach space E, we consider a topological vector space (X, T) X being a decomposable subspace of measurable E-valued functions defined on Ω. Under a reasonable assumption on the vector topology T, we show that if $(f_n)_n$ is a sequence of extended real-valued measurable integrands defined on the product $\Omega \times E$, with upper epi-limit (or upper Γ-limit) $f = \mathrm{ls}_e\, f_n$, then I_f is in many cases an upper bound for the T-upper epi-limit of the sequence $(I_{f_n})_n$, where I_f, I_{f_n} are the integral functionals defined on X associated to the integrands f, f_n. The cases of Lebesgue spaces endowed with its strong, weak, or Mackey topologies are reached. We discuss also the necessity of the given conditions.

Keywords: Integral functional, upper epi-limits, Γ-convergence, finally equi-integrable sets

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1. Introduction

Given a topological space (X, T) and a sequence of subsets we can define the classical Painlevé-Kuratowski lower limit, upper limit and limit (when the two preceding coincide) [1], [9]. Considering the Painlevé-Kuratowski lower limit, upper limit, or limit of the epigraphs, it is possible to define the T upper epi-limit $T - \mathrm{ls}_e\, f_n$ and the T lower epi-limit or the T epi-limit (denoted also Γ-limit) of a sequence $(f_n)_n$ of extended real valued functions defined on X, [1], [9]. These convergence notions are very useful in the study of minimization problems and are used by many authors since, under equi-coercivity and uniqueness assumptions, we have the convergence of the infimums and the convergence of the minimizers to the minimizer of the limit problem [1], [9] Theorem 7.4 and Corollary 7.24. They can be used also in non smooth analysis [2], [25], [30]. If we have two topologies on X, T_1 not weaker than T_2, the coincidence of the limits for the two topologies permits to define the (T_1, T_2)-Mosco convergence of a sequence of sets therefore the convergence of a sequence of functions. The case originally considered by U. Mosco was a separable reflexive space endowed with its strong and weak topologies [22], [23]. Another important type of convergence if the slice convergence put in light

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by G. Beer [3]. When the space X is reflexive, this last convergence coincide with Mosco convergence. In convex case, another important property of these convergences is the continuity of the Young-Fenchel-Moreau transform [36], [23], [19], [1], [3], and related results: [26], [27]. The purpose of this article is to obtain a "good" upper bound of the upper epi-limit of a sequence of particular functionals: integral functionals defined on a decomposable (in sense of F. Hiai and H. Umegaki [18]) space. For this kind of functionals its appears that it is interesting to not study directly the convergence, but to study separately the cases of upper epi-limit and lower epi-limit for various topologies and for non necessarily convex or proper functionals, finding an appropriate upper bound (for the first) and lower bound (for the second). In fact the study of Mosco convergence for a sequence of convex integral functionals was first treated by J. L. Joly and F. De Thélin in [20] for Lebesgue spaces of functions valued in a finite dimensional space, then by A. Salvadori [31], [32] for Lebesgue spaces of functions taking values in a separable reflexive space. In [33], [34] A. Truffert made a study in non convex case of strong epi-limits in Orlicz spaces of functions with values in a separable space. The slice convergence for a sequence of convex integral functionals on Lebesgue spaces of functions with values in a Banach with separable dual has been studied more recently by J. Couvreux [8]. This last author in [8] Theorem 4.1, when $1 \leq p < \infty$, gives an upper bound for the strong upper epi-limit of a non necessarily convex sequence of integral functionals defined on L_p. The existence of such criterion for the upper bound is not new: in [20], in convex case, it is already isolated for the norm topology in the second step of J. L. Joly and F. De Thélin's proof of Theorem 1, and also in Theorem 2 for the Mackey topology of L_∞. In [13] Chapter VI, with geometrical proofs, a such criterion is already given for a class of topological vector subspaces of measurable functions with values in a separable Banach space. We will consider a σ-finite measure space $(\Omega, \mathcal{T}, \mu)$ endowed with a μ-complete tribe $\mathcal{T}$, and a separable Banach space E. For a sequence of integral functionals $(I_{f_n})_n$ associated to a sequence $(f_n)_n$ of measurable extended real valued integrands defined on $\Omega \times E$, the integral functionals will be defined on a decomposable topological vector space (X, T) of measurable E-valued functions.

The second section is devoted to some preliminaries. In section three we give a proof of the measurability of the pointwise norm upper epi-limit of the sequence $(f_n)_n$. The section four is a list of practical consequences of Theorem 6.1 in the case of Lebesgue spaces. A key tool in the proof of the main result, is the notion of finally equi-integrable sets respect to a topology T (Definition 5.1), which is an extension of the notion of equi-integrable sets, Definition 2.5 (see [21]). In section five we study this notion when X is a Lebesgue space endowed with the strong, weak or Mackey topologies. For example, the measure being supposed atomless, when $1 \leq p < +\infty$, the strong finally equi-integrable sequences are the p-equi-integrable sequences of L_p (see Proposition 5.4); in L_∞ the norm finally equi-integrable sequences are the sequences converging to the origin (see Proposition 5.5). In the product space $L_\infty \times L_1$ endowed with the product norm, the norm finally equi-integrable sequences are neither equi-integrable sets nor sequences converging to the origin, this proves that this notion takes its full sense on Orlicz or Köthe spaces. When the Lebesgue space L_p, $1 < p \leq \infty$, is endowed with the topology $T = \sigma(L_p, L_q)$, $p^{-1} + q^{-1} = 1$, the T finally equi-integrable sequences are exactly the bounded sequences, Proposition 5.8. The case $p = 1$ and the topology $\sigma(L_1, L_\infty)$ is treated in Proposition 5.6. The case $p = \infty$ when T is the Mackey topology $\tau(L_\infty, L_1)$ is studied in Proposition 5.9. The isolated case where T is the topology of convergence in local measure is also considered. Section six is dedicated to the proof of the main result of this article (Theorem 6.1) and its consequences. The proof techniques are new and they do not use convolution approximations. Theorem 6.1 improves and extends all the previously cited works, and it is an exact extension of the upper Fatou's inequality [15] Theorem 1.12 (a). In the last section, the Theorem 7.1 proves briefly that we have obtained in many cases a necessary an sufficient condition for the interchange between the upper epi-limit and the symbol of integration. A first consequence of this article in optimal Control theory and other applications in non smooth analysis are given in [16].

2. Preliminaries

$\mathbb{R}$ is the set of real numbers, for $r \in \mathbb{R}, |r| = \max(r, -r)$, and we adopt the following notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let (X, T) be a topological vector space, the set of all open neighbourhoods of x in X will be denoted by $\mathcal{N}(x)$. For a function f with extended real values from X to $\overline{\mathbb{R}}$ we consider its domain, $\operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$ and its epigraph, $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. We say that the function f is proper if its domain is nonempty and if the function f does not take the value $-\infty$.

Definition 2.1. Given a sequence $(M_n)_n$ of subsets of X, we define its T lower limit and upper limit in sense of Kuratowski [1], [30], [9] Definition 4.10, as follows:

$$\liminf_n M_n = \{x \in X : \forall U \in \mathcal{N}(x), \exists m \in \mathbb{N} : m \leq n \Rightarrow U \cap M_n \neq \emptyset\},$$

$$\limsup_n M_n = \{x \in X : \forall U \in \mathcal{N}(x), \forall m \in \mathbb{N}, \exists n \geq m : U \cap M_n \neq \emptyset\}.$$

Remark 2.2. Clearly if X is a metric space with a distance d, for a subset M of X setting $d(x, M) = \inf\{d(x, m), m \in M\}$ if $M \neq \emptyset$, $+\infty$ if $M = \emptyset$, then:

$$\liminf_n M_n = \{x \in X : \lim_n d(x, M_n) = 0\},$$
$$\limsup_n M_n = \{x \in X : \liminf_n d(x, M_n) = 0\}.$$

Definition 2.3. Given a sequence $(f_n)_n$ of extended real-valued functions, we define (see [1]) the upper epi-limit (resp. lower epi-limit) (also called Γ-limits see [9]) as the function $T - \operatorname{ls}_e f_n$ (resp. $T - \operatorname{li}_e f_n$) whose epigraph is the lower limit (resp. upper limit) of the sequence of epigraphs of the f_n's. When these two epi-limits coincide the sequence is said epi converging to the common epi-limit.

If $(X, \|.\|)$ is a normed space and T is the topology associated to the norm we will write simply $T - \operatorname{ls}_e f_n$ by $\|.\| - \operatorname{ls}_e f_n$.
Recall that the topological space (X, T) satisfies the first axiom of countability when every point has a countable basis of neighbourhoods.

Remark 2.4. The following formula gives an analytic mean for this limit (see [9] Definition 4.1):

$$T-\mathrm{ls}_e\, f_n(x)=\sup_{V\in\mathcal{N}(x)}\ \limsup_n\ \inf_{x'\in V} f_n(x').$$

We can consider also the sequential upper epi-limit:

$$\mathrm{seq}\, T-\mathrm{ls}_e\, f_n(x)=\inf_{x_n\to x}\ \limsup_n f_n(x_n);$$

when the topology T satisfies the first axiom of countability, by [9] Proposition 8.1, then:

$$T-\mathrm{ls}_e\, f_n(x)=\mathrm{seq}\, T-\mathrm{ls}_e\, f_n(x).$$

The analog properties for the lower epi-limit are true [9] Definition 4.1.

In the sequel $(\Omega,\mathcal{T},\mu)$ is a measure space with a σ-finite positive measure μ and with a μ-complete tribe $\mathcal{T}$. For a measurable subset $A\in\mathcal{T}$, we set $A^c=\{\omega\in\Omega,\ \omega\notin A\}$, and 1_A stands for the characteristic function of A, $1_A(w)=1$ if $w\in A$, 0 if $\omega\notin A$. Let E be a separable Banach space with Borel tribe $\mathcal{B}(E)$.
We consider the space $L_0(\Omega,E)$ of classes of measurable functions (for μ-almost everywhere equality) defined on Ω and with values in E. If X is a subset of $L_0(\Omega,E)$, and α is a measurable non negative real valued function, we denote $X\alpha=\alpha X=\{\alpha x : x\in X\}$. Recall that a subset X of $L_0(\Omega,E)$ is said decomposable (in the sense of [18]), when for every measurable set A, $X1_A+X1_{A^c}\subseteq X$. Let $L_p(\Omega,E,\mu)$, $1\le p\le\infty$, be the Lebesgue space of classes of p-μ-integrable functions (μ essentially bounded functions if $p=\infty$) defined on Ω with values in E. These spaces are also denoted by $L_p(\Omega,E)$ when there is no ambiguity on the measure μ used, and $\|x\|_p$ is the usual norm of an element $x\in L_p(\Omega,E)$. Let us recall the following definition.

Definition 2.5 (see [21]). Let $\mathcal{X}$ be a subset of $L_p(\Omega,E)$ with $1\le p<\infty$.
$\mathcal{X}$ is said to be p-equi-integrable (equi-integrable if $p=1$), if for every positive number ϵ, there exist a positive constant δ and a measurable set K of finite measure such that:

(1) $\sup_{x\in\mathcal{X}}\|x1_A\|_p<\epsilon$ for every measurable set A satisfying $\mu(A)\le\delta$.
(2) $\sup_{x\in\mathcal{X}}\|x1_{K^c}\|_p<\epsilon$.

Definition 2.6 (see [24] II-5). Assume that the measure μ is finite. A subset $\mathcal{X}$ of $L_1(\Omega,E)$ is said uniformly integrable if it satisfies one of the following conditions:

(a) $\mathcal{X}$ is bounded and equi-integrable in $L_1(\Omega,E)$.
(b) $\lim_n\sup_{x\in\mathcal{X}}\int_{\{\|x\|\ge n\}}\|x\|d\mu=0$.

Given and extended real valued function v defined on Ω, the upper integral of v I_v or $\int_\Omega^* v d\mu$ is defined by:

$$I_v=\int_\Omega^* v d\mu=\inf\left\{\int_\Omega u d\mu,\ u\in L_1(\Omega,I\!R),\ v\le u,\ \mu-a.e.\right\}$$

We set $v^+ = \sup(0, v)$, $v^- = (-v)^+$ and $\{v \geq 0\} = \{\omega \in \Omega : v(\omega) \geq 0\}$.

Definition 2.7 (see [15]). Let Σ be the collection of all countable decreasing sequences of measurable sets $\sigma = (S_k)_k$ with a negligible intersection. Given a sequence $(u_n)_n$ of $\overline{I\!R}$-valued measurable functions, let us define the following modulus of "final equi-integrability"

$$\delta^+((u_n)_n) = \sup_{\sigma\in\Sigma,\, \sigma=(S_k)_k} \limsup_k \sup_{n\geq k} \int_{S_k}^* u_n d\mu.$$

Notice (see [15] Proposition 1.7) that a sequence $(u_n)_n$ of integrable non negative functions is equi-integrable if and only if $\delta^+((u_n)_n) = 0$.

Given a measurable integrand $f : \Omega \times E \to \overline{I\!R}$, for $(\omega, e) \in \Omega \times E$ we put $f_\omega(e) = f(\omega, e)$, if x is a measurable E-valued function then $f(x)$ is the map $\omega \mapsto f_\omega(x(\omega))$ and the integral functional I_f associated to f is the functional defined at every point x of $L_0(\Omega, E)$ by:

$$I_f(x) = I_{f(x)} = \int_\Omega^* f(x) d\mu.$$

3. Measurability of the upper epi-limit of a sequence of measurable integrands.

Recall the following notions,

Definition 3.1 (see [28]). Let F be a complete separable metric space. A closed and F-valued multifunction M defined on Ω is said measurable if its graph is $\mathcal{T} \otimes \mathcal{B}(F)$ measurable.

Proposition 3.2 (see [28] Proposition 1, or [16]). *Let (E, d) be a complete separable metric space and $f : \Omega \times E \to \overline{I\!R}$ an integrand. The integrand f is said normal if it satisfies one of the following equivalent assertions:*

(a) *f is $\mathcal{T} \otimes \mathcal{B}(E)$ measurable and for every $\omega \in \Omega$ the function f_ω is lower semicontinuous,*

(b) *the multifunction* epi f *is closed-valued and measurable.*

Notice that in the above definition (a), *the integrand f is not supposed to be proper* as in [28].

Proposition 3.3. *Let $(E, \|.\|)$ be a separable Banach space and $(f_n)_n$ be a sequence of measurable integrands defined on $\Omega \times E$, then the pointwise upper epi-limit integrand $f = \|.\| - \mathrm{ls}_e f_n$ defined for $(\omega, e) \in \Omega \times E$ by $f(\omega, e) = \|.\| - \mathrm{ls}_e f_n(\omega, .)(e)$ is a normal integrand.*

Proof of Proposition 3.3.

Lemma 3.4. *Let (F, d) be a complete separable metric space and $(M_n)_n$ be a sequence of multifunctions from Ω into F with measurable graphs. Then the closed valued multifunction* $\liminf_n M_n$ *is measurable.*

Proof of Lemma 3.4. For a subset C of F and $a \in F$, recall that

$$d(a, C) = \inf_{c \in C} d(a, c), \quad \text{if } C \neq \emptyset, \quad \text{and} \quad +\infty \ \text{ if } C = \emptyset.$$

Given an F-valued multifunction M with measurable graph, from [6] Theorem III 23, its domain D is measurable, and for $e \in F$:

$$d(e, M(\omega)) = d(e, M(\omega)), \ \text{ if } \omega \in D, \qquad d(e, M(\omega)) = +\infty, \ \text{ if } \omega \in D^c.$$

From [6] Theorem III 22 and Lemma III 14, the integrand $(\omega, e) \mapsto d(e, M(\omega))$ is a normal (Caratheodory) integrand on $D \times F$. Since for $\omega \in D^c$, $d(., M(\omega)) = +\infty$, then the integrand $(\omega, e) \mapsto d(e, M(\omega))$ is a measurable integrand. Let $(M_n)_n$ be a sequence of F-valued multifunctions defined on Ω with measurable graph. Due to Remark 2.2, we have:

$$\operatorname{graph}(\liminf_n M_n) = \{(\omega, e) \in \Omega \times F : \limsup_n d(e, M_n(\omega)) = 0\}.$$

The integrand $g_n(\omega, e) = d(e, M_n(\omega))$ being $\mathcal{T} \otimes \mathcal{B}(F)$-measurable so is $g = \limsup_n g_n$. But $\{g = 0\} = \operatorname{graph}(\liminf_n M_n))$ is $\mathcal{T} \otimes \mathcal{B}(E)$-measurable, therefore the multifunction $\liminf_n M_n$ is measurable. The proof of Lemma 3.4 is complete.

End of the proof of Proposition 3.3. Since each f_n is measurable, the graph of the multifunction $\operatorname{epi} f_n$ is measurable. Lemma 3.4 then proves that the multifunction $\operatorname{epi} f = \liminf_n \operatorname{epi} f_n$ is closed-valued and measurable, hence the result follows from Proposition 3.2.

4. Consequences of the main result for Lebesgue spaces

Hereafter $E^{'}$ is the dual of E and $p^{-1} + q^{-1} = 1$. We consider also a sequence $(f_n)_n$ of extended real valued measurable integrands defined on $\Omega \times E$ (respectively $\Omega \times E^{'}$ in Theorem 4.7 and Corollary 4.8), with pointwise upper epi-limit $f = \mathrm{ls}_e\, f_n$.

We will prove the following results in Section 6.

Theorem 4.1. *Let $1 \leq p < \infty$. If for every $\epsilon > 0$ there exists a p-equi-integrable sequence $(y_n)_n$ of elements of $L_p(\Omega, E)$ with $\delta^+((f_n(y_n))_n) \leq \epsilon$, then*

$$\|.\|_p - \mathrm{ls}_e\, I_{f_n} \leq I_f.$$

Since for an equi-integrable sequence $(u_n)_n$ we have $\delta^+((u_n)_n) = 0$, we obtain immediately

Corollary 4.2. *Let $1 \leq p < \infty$. If there exists a p-equi-integrable sequence $(y_n)_n$ of elements of $L_p(\Omega, E)$ such that the sequence $(f_n^+(y_n))_n$ is equi-integrable in $L_1(\Omega, \mathbb{R})$, one has*

$$\|.\|_p - \mathrm{ls}_e\, I_{f_n} \leq I_f.$$

For statements similar to the above corollary see [33] Theorem 2.5, [34] Theorem 3.5, [13] Chapter VI Corollary 2.7 and [8] Theorem 4.1 also in convex case [20] IV Theorem 1 and [32] Theorem 3.1.

Theorem 4.3. *Let $1 < p < \infty$, and E with a separable dual. Let σ be the weak topology $\sigma(L_p(\Omega, E), L_q(\Omega, E^{'}))$. If for every $\epsilon > 0$ there exists a bounded sequence $(y_n)_n$ of elements of $L_p(\Omega, E)$ with $\delta^+((f_n(y_n))_n) \leq \epsilon$ then:*

$$\sigma - \mathrm{ls}_e \, I_{f_n} \leq I_f .$$

If in addition $L_q(\Omega, E^{'})$ is separable then:

$$\mathrm{seq}\, \sigma - \mathrm{ls}_e \, I_{f_n} \leq I_f .$$

Corollary 4.4. *Assume that $1 < p < \infty$, E has a separable dual and $L_q(\Omega, E^{'})$ is separable. If there exists a bounded sequence $(y_n)_n$ of elements of $L_p(\Omega, E)$ with the sequence $(f_n^+(y_n))_n$ equi-integrable in $L_1(\Omega, \mathbb{R})$, then one has:*

$$\mathrm{seq}\, \sigma - \mathrm{ls}_e \, I_{f_n} \leq I_f .$$

Theorem 4.5. *Let $p = \infty$. If for every $\epsilon > 0$ there exists a sequence $(y_n)_n$ of elements of $L_\infty(\Omega, E)$ norm converging to $x \in L_\infty(\Omega, E)$ with $\delta^+((f_n(y_n))_n) \leq \epsilon$ then:*

$$\|.\|_\infty - \mathrm{ls}_e \, I_{f_n}(x) \leq I_f(x).$$

Corollary 4.6. *Let $p = \infty$. If there exists a sequence $(y_n)_n$ of elements of $L_\infty(\Omega, E)$ norm converging to $x \in L_\infty(\Omega, E)$ with the sequence $(f_n^+(y_n))_n$ equi-integrable in $L_1(\Omega, \mathbb{R})$, one has:*

$$\|.\|_\infty - \mathrm{ls}_e \, I_{f_n}(x) \leq I_f(x).$$

For statements similar to the above result see [33] Theorem 3.3 and [34] Theorem 3.5.

Theorem 4.7. *Let $p = \infty$, and E with a separable dual $E^{'}$. Let τ be the Mackey topology $\tau(L_\infty(\Omega, E^{'}), L_1(\Omega, E))$. If for every $\epsilon > 0$ there exists a bounded sequence $(y_n)_n$ of elements of $L_\infty(\Omega, E^{'})$ such $\delta^+((f_n(y_n))_n) \leq \epsilon$ then:*

$$\mathrm{seq}\, \tau - \mathrm{ls}_e \, I_{f_n} \leq I_f .$$

Corollary 4.8. *Suppose $p = \infty$, and E with a separable dual $E^{'}$. If there exists a bounded sequence $(y_n)_n$ of elements of $L_\infty(\Omega, E^{'})$ with the sequence $(f_n^+(y_n))_n$ equi-integrable in $L_1(\Omega, \mathbb{R})$, one has:*

$$\mathrm{seq}\, \tau - \mathrm{ls}_e \, I_{f_n} \leq I_f .$$

For results near the above corollary, but non explicit, see the proofs of [20] IV Theorem 2, [32] Theorem 3.1, and [8] Theorem 4.3.

Let us give a last statement with a non necessarily locally convex topology. Recall that convergence in local measure is convergence in measure on each subset of finite measure [21].

Theorem 4.9. *Let X be a decomposable vector space endowed with the topology $T_{\mu\,\mathrm{loc}}$ associated to the convergence in local measure. Suppose in addition that there exists a positive measurable function β such $\beta L_\infty(\Omega, E) \subseteq X$. Then for every sequence $(y_n)_n$ of elements of X:*

$$T_{\mu\,\mathrm{loc}} - \mathrm{ls}_e \, I_{f_n} \leq I_f + \delta^+((f_n(y_n))_n).$$

5. Equi-integrable and finally equi-integrable sets

The next definition is a useful unifying notion for the sequel.

Definition 5.1. Let a decomposable topological vector subspace (X, T) of $L_0(\Omega, E)$. A subset $Y \subset X$ is said to be T-equi-integrable if for every neighbourhood V of the origin, there exist a positive constant δ and a measurable set K of finite measure such that:

(a) For every measurable set A, $\mu(A) \leq \delta \Rightarrow Y 1_A \subset V$.
(b) For every measurable set $A \subset K^c$, $Y 1_A \subset V$.

The subset Y is said finally equi-integrable if for every neighbourhood V of the origin there exists a finite subset $Z_V \subset Y$ such the two above conditions hold substituting Y by $Y \cap Z_V^c$.
A sequence $(x_n)_n$ will be said T equi-integrable (respectively finally T equi-integrable) if the set $\{x_n : n \in I\!N\}$ is T equi-integrable (respectively finally T equi-integrable).

The following result is used in the next section.

Proposition 5.2. *Let (X, T) be a decomposable topological vector subspace of $L_0(\Omega, E)$, and $(x_n)_n$ be a finally T equi-integrable sequence of elements of X. For every decreasing sequence $(S_n)_n$ of measurable subsets with a negligible intersection, we have* $\lim_n x_n 1_{S_n} = 0$.

Proof of Proposition 5.2. First let us show the following lemma,

Lemma 5.3. *Let (X, T) be a decomposable topological vector subspace of $L_0(\Omega, E)$, and Y be a finally T equi-integrable subset of X. For every decreasing sequence $(S_k)_k$ of measurable subsets with a negligible intersection, for every neighbourhood V of the origin, there exist a finite subset $Z_V \subset Y$ and an integer k_V such:*

$$k \geq k_V, A \subset S_k \Rightarrow (Y \cap Z_V^c) 1_A \subseteq V.$$

Proof of Lemma 5.3. Let a neighbourhood W of the origin such $W + W \subset V$. From the above definition there exist a positive number δ, a set of finite measure K, a finite subset $Z_W \subset Y$ such:

(a) For every measurable set A, $\mu(A) \leq \delta \Rightarrow (Y \cap Z_W^c) 1_A \subset W$.
(b) For every measurable set $A \subset K^c$, $(Y \cap Z_W^c) 1_A \subset W$.

Since the intersection of the decreasing sequence $(S_k)_k$ is negligible, we have

$$\lim_k \mu(S_k \cap K) = 0.$$

Hence there exists an integer k_V such for every $k \geq k_V$, $\mu(S_k \cap K) \leq \delta$. Remarking that when $A \subset S_k$, we have

$$(Y \cap Z_V^c) 1_A \subset (Y \cap Z_V^c) 1_{A \cap K} + (Y \cap Z_V^c) 1_{A \cap K^c},$$

using this last inclusion and the properties (a) and (b) we obtain:

$$k \geq k_V, A \subset S_k \Rightarrow (Y \cap Z_V^c)1_A \subset W + W \subset V.$$

This achieves the proof of Lemma 5.3.

End of the proof of Proposition 5.2. Due to Lemma 5.3, for every neighbourhood V of the origin there exists an integer $k_V^{'}$, such for $k \geq k_V^{'}$,

$$\{x_n, n \geq k_V^{'}\}1_{S_k} \subseteq V,$$

and therefore:

$$n \geq k_V^{'} \Rightarrow x_n 1_{S_n} \in V,$$

this proves that the sequence $(x_n 1_{S_n})_n$ converges to the origin. The proof of Proposition 5.2 is complete.

As useful examples, in the sequel we will characterize the sequences (finally) equi-integrable for the norm, the weak and the Mackey topologies in the usual Lebesgue spaces.

Proposition 5.4. *Let $1 \leq p < \infty$. A sequence $(x_n)_n$ of elements of $L_p(\Omega, E)$ is finally norm equi-integrable if and only if the set $\{x_n,\ n \in \mathbb{N}\}$ is p-equi-integrable.*

Proof of Proposition 5.4. In such spaces every finite set is norm equi-integrable.

Proposition 5.5. *Let $p = \infty$. In $L_\infty(\Omega, E)$ a norm-converging sequence $(x_n)_n$ to the origin is finally norm equi-integrable. The converse is true when the measure is non atomic.*

Proof of Proposition 5.5. The first part is obvious. Conversely, let $(x_n)_n$ be a sequence of $L_\infty(\Omega, E)$ which is $\|.\|_\infty$-finally equi-integrable. For every $\epsilon > 0$ there exist an integer n_ϵ, a positive constant δ and a measurable set K of finite measure such that the two properties of Definition 5.1 hold with the sets $V = \{x \in L_\infty(\Omega, E) : \|x\|_\infty \leq \epsilon\}$ and $Y \cap Z_V^c = \{x_n, n \geq n_\epsilon\}$. Since the measure is non atomic, there exists a finite covering $(K_i)_{1 \leq i \leq l}$ of K by disjoint sets of measure less than δ. Thus for every $n \geq n_\epsilon$,

$$\|x_n\|_\infty \leq \sup(\|x_n 1_K\|_\infty, \|x_n 1_{K^c}\|_\infty) \leq \sup(\sup_{1 \leq i \leq l} \|x_n 1_{K_i}\|_\infty, \|x_n 1_{K^c}\|_\infty) \leq \epsilon.$$

Therefore the sequence $(x_n)_n$ converges to the origin.

Proposition 5.6. *Let $p = 1$. Any norm-equi-integrable sequence $(x_n)_n$ of elements of $L_1(\Omega, E)$ is $\sigma(L_1(\Omega, E), L_\infty(\Omega, E^{'}))$-equi-integrable. Conversely, suppose E is reflexive separable and that the measure μ is atomless; a finally $\sigma(L_1(\Omega, E), L_\infty(\Omega, E^{'}))$-equi-integrable sequence of integrable functions is norm-equi-integrable.*

Proof of Proposition 5.6. Since the norm topology is finer that the weak topology, a norm-equi-integrable sequence $(x_n)_n$ of elements of $L_1(\Omega, E)$ is $\sigma(L_1(\Omega, E), L_\infty(\Omega, E'))$-equi-integrable.
Conversely suppose that the measure μ is atomless. Let us first show that a such sequence $(x_n)_n$ is norm bounded. The sequence $(x_n)_n$ being finally equi-integrable, considering $y \in L_\infty(\Omega, E')$ and $V = \{x \in L_1(\Omega, E) : x(y) = \int \langle x, y\rangle d\mu \leq 1\}$, then there exist a positive constant δ and a set K of finite measure, an integer n_V such that:

(a) For every measurable set A, $\mu(A) \leq \delta \Rightarrow \{x_n 1_A,\ n \geq n_V\} \subset V$.
(b) For every measurable set $A \subset K^c$, $\{x_n 1_A,\ n \geq n_V\} \subset V$.

Since K is of finite atomless measure there exists a finite partition of K, $(K_i)_{i\leq l}$ by sets of measure less than δ. Since for $n \geq n_V$

$$x_n(y) = \int \langle x_n, y\rangle d\mu = \sum_{i\leq l} \int_{K_i} \langle x_n, y\rangle d\mu + \int_{K^c} \langle x_n, y\rangle d\mu \leq l + 1,$$

we deduce that the sequence $(x_n(y))_n$ is bounded above. This last property being valid for arbitrary $y \in L_\infty(\Omega, E')$, we obtain from Banach-Steinhaus's theorem that the sequence $(x_n)_n$ is norm bounded in $L_\infty(\Omega, E')'$ hence in $L_1(\Omega, E)$.
Let us now prove that the sequence $(x_n)_n$ is weakly relatively compact. Let $L_\infty(\Omega, E')'$ be the topological dual of $L_\infty(\Omega, E')$. Clearly the sequence $(x_n)_n$ being bounded in $L_1(\Omega, E)$, it is $\sigma(L_\infty(\Omega, E')', L_\infty(\Omega, E'))$ relatively compact. Let x' be an arbitrary $\sigma(L_\infty(\Omega, E')', L_\infty(\Omega, E'))$ cluster point of the set $M = \{x_n,\ n \in I\!N\}$. Then since E is reflexive, $x' = x^* + x_s^*$, where x^* is in $L_1(\Omega, E)$, and x_s^* is a singular linear form on $L_\infty(\Omega, E')$. It suffices to prove that $x_s^* = 0$. From [35] §1 Proposition 9, x_s^* is supported by a decreasing sequence $(A_n)_n$ of measurable subsets with a negligible intersection. Let us show that for every $y \in L_\infty(\Omega, E')$, $x_s^*(y) = 0$. Let $V = \{x \in L_1(\Omega, E) :\ x(y) \leq 1\}$ as above. Then V is a $\sigma(L_1(\Omega, E), L_\infty(\Omega, E'))$ neighbourhood of the origin. Since the sequence $(x_n)_n$ is finally equi-integrable, the above properties (a) and (b) are valid. We can always suppose that there exists a directed family $(x_i)_i$ of elements of the subset $\{x_n,\ n \geq n_V\}$ of M such $x' = \sigma(L_\infty(\Omega, E')', L_\infty(\Omega, E')) - \lim_i x_i$ (if not, then $x_s^* = 0$), hence

$$x_s^* = \sigma(L_\infty(\Omega, E')', L_\infty(\Omega, E')) - \lim_i x_i - x^*.$$

For every integer n we have for every element y of $L_\infty(\Omega, E')$,

$$x_s^*(y) = x_s^*(y1_{A_n}) = x_s^*(y1_{A_n \cap K}) + x_s^*(y1_{A_n \cap K^c}).$$

Therefore for every integer n:

$$x_s^*(y) = x_s^*(y1_{A_n}) = \lim_i x_i(y1_{A_n \cap K}) + \lim_i x_i(y1_{A_n \cap K^c}) - x^*(y1_{A_n}),$$

or

$$x_s^*(y) = x_s^*(y1_{A_n}) = \lim_i x_i 1_{A_n \cap K}(y) + \lim_i x_i 1_{A_n \cap K^c}(y) - x^* 1_{A_n}(y). \tag{1}$$

Since K is of finite measure, $\lim_n \mu(A_n \cap K) = 0$. Since $|x^* 1_{A_n}(y)| \leq \|y\|_\infty \|x^* 1_{A_n}\|_1$, we have $\lim_n |x^* 1_{A_n}(y)| = 0$. Choose m such $\mu(A_m \cap K) \leq \delta$, and $|x^* 1_{A_m}(y)| \leq 1$. From (1) and properties (a) and (b) used precedently we obtain:

$$x_s^*(y) = x_s^*(y 1_{A_m}) \leq 3.$$

This last upper bound being valid for every $y \in L_\infty(\Omega, E')$, then $x_s^* = 0$. Moreover we obtain the weak convergence of the family $(x_i)_i$ to x^*. Therefore every $\sigma(L_\infty(\Omega, E')', L_\infty(\Omega, E'))$ cluster point of M is in $L_1(\Omega, E)$. This proves that the set M is $\sigma(L_1(\Omega, E), L_\infty(\Omega, E'))$ relatively compact, hence, with the following lemma, norm equi-integrable.

Lemma 5.7. *Let a σ-finite measure space $(\Omega, \mathcal{T}, \mu)$, and E be a separable Banach space. A weakly compact subset of $L^1(\Omega, E)$ is bounded and equi-integrable.*

Proof of Lemma 5.7. Let M a such subset. Indeed for a finite measure μ, the result is proved with various proofs by N. Dunford (see [11]), J. Diestel and J. J. Uhl [10] IV Theorem 2.4. For the extension to the σ-finite case, keeping an integrable positive valued function α, then $\alpha^{-1}M$ is weakly compact in $L_1(\Omega, E, \alpha\mu)$ (this due to the form of the dual space $L_1(\Omega, E)'$, [12] Theorem 2.112). Hence $\alpha^{-1}M$ is uniformly equi-integrable in $L_1(\Omega, E, \alpha\mu)$, then the use of Bourass's Lemma [4] IV § 1, (see also [15] Section 2) allows us to obtain the equi-integrability of M.

The proof of Proposition 5.6 is complete.

Proposition 5.8. *Let $1 < p \leq \infty$ and suppose the dual of the Banach space E is separable. Any bounded sequence $(x_n)_n$ of elements of $L_p(\Omega, E)$ is $\sigma(L_p(\Omega, E), L_q(\Omega, E'))$-equi-integrable. Conversely suppose the measure μ atomless, then a finally $\sigma(L_p(\Omega, E), L_q(\Omega, E'))$-equi-integrable sequence is bounded.*

Proof of Proposition 5.8. Since E is with separable dual, notice first that $L_p(\Omega, E)' = L_q(\Omega, E')$ (see [10]). Let a bounded sequence $(x_n)_n$ of elements of $L_p(\Omega, E)$ and $m = \sup_n \|x_n\|_p$. A $\sigma(L_p(\Omega, E), L_q(\Omega, E'))$- neighbourhood V of the origin is given by

$$V = \bigcap_{i \leq l} \left\{ x \in L_p(\Omega, E) : x_i^*(x) = \int \langle x, x_i^* \rangle d\mu \leq 1 \right\},$$

where $x_i^* \in L_q(\Omega, E')$. Remark that by the Hölder inequality, for every measurable set A:

$$\int \langle x 1_A, x_i^* \rangle d\mu \leq \|x\|_p \max_{i \leq l} \|x_i^* 1_A\|_q.$$

then for every integer n,

$$x_i^*(x_n 1_A) = \int \langle x_n 1_A, x_i^* \rangle d\mu \leq m \max_{i \leq l} \|x_i^* 1_A\|_q,$$

since every finite family of elements of $L_q(\Omega, E')$ is q-equi-integrable, we deduce that every bounded sequence in $L_p(\Omega, E)$ is $\sigma(L_p(\Omega, E), L_q(\Omega, E'))$-equi-integrable.

Conversely, suppose that the sequence $(x_n)_n$ is finally $\sigma(L_p(\Omega, E), L_q(\Omega, E^{'}))$ equi-integrable, given $x^* \in L_q(\Omega, E^{'})$, set $V = \{x \in L_p : x^*(x) = \int \langle x, x^* \rangle d\mu \leq 1\}$. Then there exist a positive constant δ and a set K of finite measure, an integer n_V such:

(a) For every measurable set A, $\mu(A) \leq \delta \Rightarrow \{x_n 1_A,\ n \geq n_V\} \subset V$.
(b) For every measurable set $A \subset K^c$, $\{x_n 1_A,\ n \geq n_V\} \subset V$.

Since K is of finite atomless measure there exists a finite partition of K, $(K_i)_{i \leq l}$ by sets of measure less than δ. Since for $n \geq n_V$

$$x^*(x_n) = \int \langle x_n, x^* \rangle d\mu = \sum_{i \leq l} \int_{K_i} \langle x_n, x^* \rangle d\mu + \int_{K^c} \langle x_n, x^* \rangle d\mu \leq l + 1,$$

we deduce that the sequence $(x^*(x_n))_n$ is bounded above. This last property being valid for arbitrary $x^* \in L_q(\Omega, E^{'})$, due to the Banach-Steinhaus's theorem we deduce that the sequence $(x_n)_n$ is norm bounded in $L_p(\Omega, E)$, for $1 < p \leq \infty$.

Proposition 5.9. *Suppose that the dual of the Banach space E is separable. Any bounded sequence $(x^*_n)_n$ of elements of $L_\infty(\Omega, E^{'})$ is equi-integrable for the Mackey topology τ respect to $L_\infty(\Omega, E^{'})$ and $L_1(\Omega, E)$. Conversely if the measure μ is atomless, any sequence $(x^*_n)_n$ of elements of $L_\infty(\Omega, E^{'})$ which is τ finally equi-integrable is bounded.*

Proof of Proposition 5.9. In this case $L_\infty(\Omega, E^{'})$ is the topological dual of $L_1(\Omega, E)$, [10]. The Mackey topology $\tau(L_\infty(\Omega, E^{'}), L_1(\Omega, E))$ of $L_\infty(\Omega, E^{'})$ is the topology of uniform convergence on weakly convex symetric compact sets of $L_1(\Omega, E)$. Let M be such a subset of $L_1(\Omega, E)$, then by Lemma 5.7, M is equi-integrable.
Considering $V(M, t) = \{x^* \in L_\infty(\Omega, E^{'}) : \sup_{x \in M} \int \langle x, x^* \rangle d\mu \leq t\}$, we know that the sets $\{V(M, t) : M$ convex symetric, weakly compact, $t > 0\}$, is a basis of the Mackey topology τ. Let $(x^*_n)_n$ be a bounded sequence of elements of $L_\infty(\Omega, E^{'})$ and $m = \sup_n \|x^*_n\|_\infty$. Then for every integer n:

$$\sup_{x \in M} \int \langle x, x^*_n 1_A \rangle d\mu \leq m \sup_{x \in M} \|x 1_A\|_1.$$

Since M is norm equi-integrable, for every $t > 0$, there exist $\delta > 0$ and a measurable set K of finite measure such:

(a) For every measurable set A satisfying $\mu(A) \leq \delta$, $\{x^*_n, n \in I\!N\} 1_A \in V(M, t)$.
(b) For every measurable set $A \subset K^c$, $\{x^*_n, n \in I\!N\} 1_A \in V(M, t)$.

We have proved that $(x^*_n)_n$ is a τ equi-integrable sequence.
Conversely, let us suppose that $(x^*_n)_n$ is τ finally equi-integrable, then it is $\sigma(L_\infty(\Omega, E^{'}), L_1(\Omega, E))$ finally equi-integrable thus by Proposition 5.8 it is bounded. The proof of Proposition 5.9 is complete.

Let us conclude this section with a singular example:

Proposition 5.10. *Let X be a decomposable vector space endowed with the topology $T_{\mu\,\mathrm{loc}}$ associated to the convergence in local measure. Every subset of X is $T_{\mu\,\mathrm{loc}}$ equi-integrable.*

Proof of Proposition 5.10. It is known that for any integrable positive valued function α, the topology $T_{\mu\,\mathrm{loc}}$ associated to the convergence in local measure is the vector topology associated to the distance

$$d_{\mu\,\mathrm{loc}}(x, y) = \int_\Omega \frac{\|x - y\|}{1 + \|x - y\|} \alpha d\mu.$$

If for $r > 0$ $B_r = \{x \in X : d_{\mu\,\mathrm{loc}}(0, x) \leq r\}$, clearly a fundamental system of neighbourhoods of the origin is the family $\{B_r,\ r > 0\}$. Remark that for any subset $Y \subset X$ and any measurable set A we have:

$$\sup_{x \in Y} d_{\mu\,\mathrm{loc}}(0, x1_A) \leq \int_A \alpha d\mu,$$

and since the singleton $\{\alpha\}$ is equi-integrable, we obtain the desired result.

6. Main result and proofs

Remark first that if α is a positive real valued measurable function, the vector space $\alpha L_\infty(\Omega, E)$ endowed with $\|x\|_{\alpha,\,\infty} = \inf\{t > 0 : \|x\| \leq t\alpha \ a.e.\}$ becomes a normed vector space with unit ball $B_{\alpha,\,\infty} = \{x \in L_0(\Omega, E) : \ \|x\| \leq \alpha \ a.e.\}$.
As in the preceding section we consider a decomposable topological vector subspace (X, T) of $L_0(\Omega, E)$ satisfying also the following property:

($\mathcal{H}$) *There exists a positive real valued measurable function α such $\alpha L_\infty(\Omega, E) \subseteq X$ with a continuous embedding.*

Clearly, every Lebesgue space endowed with its strong, therefore with its weak or Mackey topology, satisfies the property ($\mathcal{H}$).

In the sequel, given $x \in X$ and a sequence $(y_n)_n$ of elements of X, we will use also this last assumption for a given topology T:

($\mathcal{B}[x, (y_n)_n]$) *There exist a vector topology T' satisfying ($\mathcal{H}$), a positive real valued measurable function β such that the restriction of the topology T' to the subspace Z defined by*

$$Z_{\beta,(y_n)_n} = \bigcup_{A \in \mathcal{T}} \{(x + B_{\beta,\,\infty})1_A + (\{x\} \cup \{y_n,\ n \in I\!N\})1_{A^c}\},$$

is not weaker than T and satisfies the first axiom of countability.

In the statement of the following main result $(f_n)_n$ is a sequence of extended real valued measurable integrands defined on $\Omega \times E$ with upper epi-limit $f = \mathrm{ls}_e\, f_n$.

Theorem 6.1. *Suppose that the assumption $(\mathcal{H})$ is fulfilled and let $x \in X$. Then for every sequence $(y_n)_n$ of elements of X for which the sequence $(y_n - x)_n$ is finally T-equi-integrable we have the inequality:*

$$T - \operatorname{ls}_e I_{f_n}(x) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

If in addition the property $(\mathcal{B}[x,(y_n)_n])$ is true, then:

$$\operatorname{seq} T - \operatorname{ls}_e I_{f_n}(x) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

Proof of Theorem 6.1. Let us prove first the following result:

Lemma 6.2. *Suppose that (X,T) satisfies the assumption $(\mathcal{H})$ with the positive real valued measurable function α. Let β be any positive real valued measurable function, set $\eta = \min(\alpha, \beta)$. For every $\epsilon > 0$, for every neighbourhood V of $x \in X$, for every sequence $(y_n)_n$ of elements of X such the sequence $(y_n - x)_n$ is finally T-equi-integrable, there exist an integer $k_{V,\,\epsilon}$, a sequence $(z_n)_n$ of elements of $Z_{\eta,(y_n)_n}$ such for all integers $n \geq k_{V,\,\epsilon}$, $z_n \in V$ and verifies*

$$I_{f_n}(z_n) \leq \max(-\epsilon^{-1}, I_f(x)) + \delta^+((f_n(y_n))_n) + 4\epsilon.$$

Proof of Lemma 6.2. We can always suppose that $I_f(x) < +\infty$ and $\delta^+((f_n(y_n))_n) < +\infty$. Let us consider $\epsilon > 0$ and a neighbourhood V of $x \in X$. Due to the definition of the upper integral let us consider an integrable function u satisfying for almost every $\omega \in \Omega$, $f(x)(\omega) < u(\omega)$ with

$$\int_\Omega u d\mu \leq \max(-\epsilon^{-1}, I_f(x)) + \epsilon.$$

Denote $w = (x, u)$. Let the product norm on $E \times I\!R$ and d the associated distance. If $s_n(\omega) = d(w(\omega), \operatorname{epi} f_n(\omega,.))$, then since for almost every $\omega \in \Omega$, $w(\omega) \in \operatorname{epi} f(\omega,.) = \liminf_n \operatorname{epi} f_n(\omega,.)$, from Remark 2.2 we have:

$$\lim_n s_n(\omega) = 0 \tag{2}$$

Let W be a neighbourhood of the origin in X such that $W + W \subset V - x$. Since the assumption $(\mathcal{H})$ is true with the function α, it is true with the measurable positive valued function $\eta = \min(\alpha, \beta)$, thus there exists a real number t such that $0 < t < 1$ and

$$tB_{\eta,\,\infty} \subseteq W. \tag{3}$$

Given an integrable function θ with positive values such that $\int_\Omega \theta d\mu = 1$, define the measurable positive valued function $\gamma = \min(\epsilon\theta, t\eta)$. The integrand f_n being measurable, the multifunction $\omega \rightrightarrows \operatorname{epi} f_n(\omega,.)$ is with measurable graph. From [6] Theorem III 22 and Lemma III 14, the function s_n is measurable. Let the measurable sets

$$\Omega_n = \bigcap_{k \geq n} \{\omega \in \Omega : s_k(\omega) < \gamma(\omega)\}.$$

Then due to (2) $(\Omega_n)_n$ is up to a negligible set an increasing covering of Ω.
Let B be the closed unit ball of $E \times \mathbb{R}$. The multifunction $C(\omega) = w(\omega) + \gamma(\omega)B$ is measurable, thus for every integer n, the multifunction $M_n(\omega) = C(\omega) \cap \operatorname{epi} f_n(\omega, .)$ is with measurable graph and with non empty values on Ω_n. In view of [6] Theorem III 22, there exists a measurable selection (x_n, u_n) defined on Ω_n. This selection (x_n, u_n) satisfies for every $\omega \in \Omega_n$:

$$(x_n(\omega), u_n(\omega)) \in \operatorname{epi} f_n(\omega, .) \quad \text{and} \quad \|x(\omega) - x_n(\omega)\| + |u(\omega) - u_n(\omega)| \leq \gamma(\omega).$$

Therefore for $\omega \in \Omega_n$:

$$\|x(\omega) - x_n(\omega)\| \leq t\eta(\omega), \quad \text{and} \tag{4}$$
$$u_n(\omega) \leq u(\omega) + \gamma(\omega) \leq u(\omega) + \epsilon\theta(\omega) \tag{5}$$

Denote by $\overline{x}_n$ (respectively $\overline{u}_n$) the null extension to Ω of x_n (respectively u_n). For every integers n, set $z_n = \overline{x}_n 1_{\Omega_n} + y_n 1_{\Omega_n^c}$ and $v_n = \overline{u}_n 1_{\Omega_n} + f_n(y_n) 1_{\Omega_n^c}$, then $f_n(z_n) \leq v_n$. Since the sequence $(y_n - x)_n$ is finally T equi-integrable, from Proposition 5.2 there exists an integer k_W such that $(y_n - x)1_{\Omega_n^c} \in W$ for $n \geq k_W$. Moreover $z_n = \overline{x}_n 1_{\Omega_n} + y_n 1_{\Omega_n^c}$, from (4) we have $z_n \in (x + B_{\eta,\infty})1_{\Omega_n} + \{y_n,\ n \in \mathbb{N}\}1_{\Omega_n^c} \subset Z_{\eta,(y_n)_n}$. Using (3) and (4) we obtain for $n \geq k_W$,

$$\begin{aligned} z_n - x &= (z_n - x)1_{\Omega_n} + (z_n - x)1_{\Omega_n^c} \\ &= (\overline{x}_n - x)1_{\Omega_n} + (y_n - x)1_{\Omega_n^c} \in W + W \subset V - x. \end{aligned}$$

Thus for $n \geq k_W$, $z_n \in V$ and

$$I_{f_n}(z_n) \leq \int_\Omega v_n d\mu = \int_{\Omega_n} u_n d\mu + \int^*_{\Omega_n^c} f_n(y_n) d\mu,$$

therefore using (5), for $n \geq k_W$ $z_n \in V$ and

$$I_{f_n}(z_n) \leq \int_{\Omega_n} (u + \epsilon\theta) d\mu + \sup_{k \geq n} \int^*_{\Omega_n^c} f_k(y_k) d\mu. \tag{6}$$

Since $u + \epsilon\theta$ is integrable, and due to Definition 2.7, there exists an integer $k_V \geq k_W$ such that for $n \geq k_V$:

$$\int_{\Omega_n} (u + \epsilon\theta) d\mu \leq \int_\Omega (u + \epsilon\theta) d\mu + \epsilon = \int_\Omega u d\mu + 2\epsilon \quad \text{and}$$
$$\sup_{k \geq n} \int^*_{\Omega_n^c} f_k(y_k) d\mu \leq \delta^+((f_n(y_n))_n) + \epsilon,$$

With (6) and the above inequalities then for $n \geq k_V$, $w_n \in V$ and

$$I_{f_n}(z_n) \leq \int_\Omega u \, d\mu + \delta^+((f_n(y_n))_n) + 3\epsilon.$$

Due to the choice of u we get for $n \geq k_V$, $w_n \in V$ and

$$I_{f_n}(z_n) \leq \max(-\epsilon^{-1}, I_f(x)) + \delta^+((f_n(y_n))_n) + 4\epsilon. \tag{7}$$

the inequality (7) achieves the proof of Lemma 6.2.

End of the proof of Theorem 4.9. We begin by the proof of the first statement. Let $(y_n)_n$ be a sequence of elements of X such that the sequence $(y_n - x)_n$ is finally T-equi-integrable, $\epsilon > 0$ and V be a neighbourhood of $x \in X$. Then by Lemma 6.2, there exist an integer $k_{V,\,\epsilon}$, a sequence $(z_n)_n$ of elements of X such that for $n \geq k_{V,\,\epsilon}$, $z_n \in V$, and

$$\inf_{x^{'} \in V} I_{f_n}(x^{'}) \leq I_{f_n}(z_n) \leq \max(-\epsilon^{-1}, I_f(x)) + \delta^+((f_n(y_n))_n) + 4\epsilon.$$

Therefore

$$\limsup_n \inf_{x^{'} \in V} I_{f_n}(x^{'}) \leq \max(-\epsilon^{-1}, I_f(x)) + \delta^+((f_n(y_n))_n) + 4\epsilon.$$

The above inequality being valid for arbitrary $\epsilon > 0$ we obtain:

$$\limsup_n \inf_{x^{'} \in V} I_{f_n}(x^{'}) \leq I_f(x) + \delta^+((f_n(y_n))_n),$$

this last inequality being valid for every neighbourhood V of x, we deduce with Remark 2.4:

$$T - \mathrm{ls}_e\, I_{f_n}(x) = \sup_{V \in \mathcal{N}(x)} \limsup_n \inf_{x^{'} \in V} I_{f_n}(x^{'}) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

This completes the proof of the first statement of Theorem 6.1. Remark that since z_n is in $Z_{\eta,(y_n)_n}$, we have proved that the above inequaliy is true on the topological subspace $Z_{\eta,(y_n)_n}$.
For the proof of the second statement, suppose that $T^{'}$ is not weaker than T, that $T^{'}$ verifies $\mathcal{H}$ with the measurable positive valued function $\alpha^{'}$, and satisfies $(\mathcal{B}[x,(y_n)_n])$ with the measurable positive valued function $\beta^{'}$. Then the assumption $(\mathcal{B}[x,(y_n)_n])$ is fulfilled for $T^{'}$ with $\eta^{'} = \min(\alpha^{'}, \beta^{'})$. For every $\epsilon > 0$, for every neighbourhood V of x for $T^{'}$, the sequences $(z_n)_n$ of Lemma 6.2, with $\alpha := \alpha^{'}$, $\beta := \beta^{'}$, used in the above proof are elements of the subspace of X

$$Z_{\eta^{'},(y_n)_n} = \bigcup_{A \in \mathcal{T}} \{(x + B_{\eta^{'},\,\infty}))1_A + (\{x\} \cup \{y_n,\ n \in I\!N\})1_{A^c}\}.$$

Let $T^{'}_{Z_{\eta^{'},(y_n)_n}}$ be the restriction of $T^{'}$ to the space $Z_{\eta^{'},(y_n)_n}$ (which contains x). Due to the remark at the end of the proof of the first statement of Theorem 4.9 we can show that:

$$T^{'}_{Z_{\eta^{'},(y_n)_n}} - \mathrm{ls}_e\, I_{f_n}(x) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

Since the assumption $(\mathcal{B}[x,(y_n)_n])$ is true for $T^{'}$ with $\eta^{'}$, the restriction of $T^{'}$ to the space $Z_{\eta^{'},(y_n)_n}$ verifies the first axiom of countability and by [9] Proposition 8.1,

$$\mathrm{seq}\, T^{'}_{Z^{'}_{\eta^{'},(y_n)_n}} - \mathrm{ls}_e\, I_{f_n}(x) = T^{'}_{Z^{'}_{\eta^{'},(y_n)_n}} - \mathrm{ls}_e\, I_{f_n}(x).$$

Therefore $T^{'}$ being not weaker than T:

$$\mathrm{seq}\, T - \mathrm{ls}_e\, I_{f_n}(x) \leq \mathrm{seq}\, T^{'}_{Z_{\eta^{'},(y_n)_n}} - \mathrm{ls}_e\, I_{f_n}(x) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

The proof of Theorem 6.1 is achieved.

Now let us prove the results of Section 5.

From Propositions 5.4, 5.5, due to the first part of Theorem 6.1, we obtain as direct consequences Theorems 4.1 and 4.5.

The Banach space E being with separable dual, let us consider the space $L_p(\Omega, E)$ endowed with the weak topology $\sigma = \sigma(L_p(\Omega, E), L_q(\Omega, E^{'}))$. The measure being σ-finite, there exists an element α_0 in $L_p(\Omega, \mathbb{R})$ which is positive valued and $\alpha_0 L_\infty(\Omega, E) \subseteq L_p(\Omega, E)$ with a continuous embedding, therefore $(\mathcal{H})$ is true. From Proposition 5.8, every bounded sequence $(y_n)_n$ in $L_p(\Omega, E)$ is σ equi-integrable, this with the first part of Theorem 6.1 give the first part of Theorem 4.3. If $L_q(\Omega, E^{'})$ is separable then σ is metrisable on bounded subsets. Since $(y_n)_n$ is bounded, for every x the set

$$Z = \bigcup_{A \in \mathcal{T}} \{(x + B_{\alpha_0,\, \infty})1_A + \{y_n,\ n \in \mathbb{N}\}1_{A^c}\}$$

is bounded therefore the restriction of σ to Z is metrisable, this proves the property $(\mathcal{B}[x, (y_n)_n])$. With the second part of Theorem 6.1 we deduce that for every bounded sequence $(y_n)_n$ we have:

$$\operatorname{seq} \sigma - \operatorname{ls}_e I_{f_n}(x) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

and this proves entirely Theorem 4.3.

Let us prove now Theorem 4.7.
The Banach space E being with separable dual, let us consider the dual space $L_\infty(\Omega, E^{'})$ of $L_1(\Omega, E)$ endowed with the Mackey topology $\tau = \tau(L_\infty(\Omega, E^{'}), L_1(\Omega, E))$. We will use the following lemma:

Lemma 6.3. *On bounded sets of $L_\infty(\Omega, E^{'})$ the convergence in local measure is not weaker than τ convergence.*

The links between convergence in measure and Mackey convergence have been originally discovered by A. Grothendieck in [17] (Prop. 1, p. 298 and exercise, p. 300), and extended by C. Castaing [5]. For the convenience of the reader we give the proof of the above result which treats the case of convergence in local measure.

Proof of Lemma 6.3. Indeed let a sequence $(x_n^*)_n$ of elements of $L_\infty(\Omega, E^{'})$ norm bounded by m, converging in local measure to the origin and $\mathcal{K}$ be a weakly compact set in $L_1(\Omega, E)$. For any measurable set A:

$$\sup_{x \in \mathcal{K},\, n \in \mathbb{N}} \int_A |\langle x_n^*, x\rangle| d\mu \leq m \sup_{x \in \mathcal{K}} \int_A \|x\| d\mu. \tag{8}$$

From Lemma 5.7 the set $\mathcal{K}$ is equi-integrable. For every $\epsilon > 0$ there exist $\delta > 0$ and a measurable set K of finite measure such that:

$$\mu(A) \leq \delta \Rightarrow \sup_{x \in \mathcal{K}} \int_A \|x\| d\mu \leq \epsilon, \quad \text{and} \quad \sup_{x \in \mathcal{K}} \int_{K^c} \|x\| d\mu \leq \epsilon.$$

$\mathcal{K}$ being weakly compact in $L_1(\Omega, E)$, it is norm bounded, let $M = \sup_{x\in\mathcal{K}} \|x\|_1$. Since the sequence $(x_n^*)_n$ converges in measure on K, there exists an integer n_ϵ such that for $n \geq n_\epsilon$, $\mu(A_{n,\,\epsilon}) \leq \delta$ with $A_{n,\,\epsilon} = K \cap \{\omega \in \Omega : \|x_n^*\|(\omega) > \epsilon\}$. Using (8) and the following inequality

$$\begin{aligned}&\sup_{x\in\mathcal{K}} \int_\Omega |\langle x_n^*, x\rangle| d\mu \\ \leq\ &\sup_{x\in\mathcal{K}} \int_{A_{n,\,\epsilon}} |\langle x_n^*, x\rangle| d\mu + \sup_{x\in\mathcal{K}} \int_{K\cap A^c_{n,\,\epsilon}} |\langle x_n^*, x\rangle| d\mu + \sup_{x\in\mathcal{K}} \int_{K^c} |\langle x_n^*, x\rangle| d\mu,\end{aligned}$$

we obtain for $n \geq n_\epsilon$,

$$\sup_{x\in\mathcal{K}} \int_\Omega |\langle x_n^*, x\rangle| d\mu \leq (2m + M)\epsilon.$$

Therefore for every $\epsilon > 0$,

$$\limsup_n \sup_{x\in\mathcal{K}} \int_\Omega |\langle x_n^*, x\rangle| d\mu \leq (2m + M)\epsilon,$$

This proves that $\lim_n \sup_{x\in\mathcal{K}} \int_\Omega |\langle x_n^*, x\rangle| d\mu = 0$, this equality being valid for arbitrary weakly compact set $\mathcal{K}$ in $L_1(\Omega, E)$, the sequence $(x_n^*)_n$ τ-converges to the origin. The proof of Lemma 6.3 is achieved.

End of the proof of Theorem 4.7. Let $(y_n)_n$ be a bounded sequence in $L_\infty(\Omega, E')$. Clearly if $T' = T_{\mu\,\mathrm{loc}}$ is the topology of convergence in local measure, the topological vector space $(L_\infty(\Omega, E'), T')$ verifies $(\mathcal{H})$. Since the sequence $(y_n)_n$ is bounded, so is the space

$$Z = \bigcup_{A\in\mathcal{T}} \{(x + B_\infty)1_A + (\{x\} \cup \{y_n,\ n \in I\!N\})1_{A^c}\}.$$

The topology of convergence in local measure being metrisable, Lemma 6.3 proves that the property $(\mathcal{B}[x, (y_n)_n])$ is true for the Mackey topology, therefore the second part of Theorem 6.1 shows that for every bounded sequence $(y_n)_n$,

$$\operatorname{seq}\tau - \mathrm{ls}_e\, I_{f_n}(x) \leq I_f(x) + \delta^+((f_n(y_n))_n).$$

This last property proves entirely Theorem 4.7.

We end this section by proving Theorem 4.9. Indeed by assumptions there exists a measurable positive valued function β such $\beta L_\infty(\Omega, E) \subseteq X$, let any integrable positive valued function α, and set $\gamma = \inf(1, \beta)$. Then with the notations of the proof of Proposition 5.10,

$$\|x\|_{\gamma,\infty} \leq t \Leftrightarrow \|x\| \leq t\gamma \leq t \ a.e. \ \Rightarrow d_{\mu,\mathrm{loc}}(x,\ 0) = \int_\Omega \frac{\|x\|}{1 + \|x\|} \alpha d\mu \leq t \int_\Omega \alpha d\mu.$$

This proves that $\gamma L_\infty(\Omega, E)$ is continuously embedded in $(X, T_{\mu\,\mathrm{loc}})$. This shows that $(X, T_{\mu\,\mathrm{loc}})$ verifies $(\mathcal{H})$ moreover the first part of Theorem 6.1 and Proposition 5.10 give Theorem 4.9.

7. A characteristic condition for the obtaining of the inequality $\mathrm{ls}_e\, I_{f_n}(x) \leq I_f(x)$, with $f = \mathrm{ls}_e\, f_n$.

In this section (X, T) is a decomposable topological space of measurable E valued functions with the following property:

$(\mathcal{M})$ *Sequential T-convergence is not weaker than convergence in local measure.*

Given a sequence $(f_n)_n$ of measurable integrands let us recall the following criterion (see [15], [16]):

(Ioffe's criterion) *For every subsequence $(f_{n_p})_p$ of $(f_n)_n$, for every T-converging sequence $(x_p)_p$ to x such the sequence $(I_{f_{n_p}}(x_p))_p$ is bounded above, the sequence $(f^-_{n_p}(x_p))_p$ is eventually weakly precompat in $L_1(\Omega, I\!R)$.*

In many cases ([15] Lemma 4.1) the Ioffe' criterion is equivalent to: *for every T-converging sequence $(x_n)_n$ to x, the sequence $(f^-_n(x_n))_n$ is eventually weakly precompact in $L_1(\Omega, I\!R)$.*

Theorem 7.1. *Let (X, T) be a decomposable topological space verifying the assumptions $(\mathcal{M})$, $(\mathcal{H})$ and the first axiom of countability. Let a sequence $(f_n)_n$ of measurable integrands, $f = \mathrm{ls}_e\, f_n$, $x \in X$ with $f(x) = \mathrm{li}_e\, f_n(x)$ and $I_f(x)$ finite; then when the Ioffe's criterion at x is true, the following conditions are equivalent:*

(a) $\mathrm{ls}_e\, I_{f_n}(x) \leq I_f(x)$.

(b) *For every $\epsilon > 0$ there exists a sequence $(x_n)_n$ such the sequence $(x_n - x)_n$ is finally T-equi-integrable and moreover $\delta^+((f_n(x_n))_n) < \epsilon$.*

(c) *For every $\epsilon > 0$ there exists a sequence $(x_n)_n$ T-converging to x and verifying moreover $\delta^+((f_n(x_n))_n) < \epsilon$.*

Proof of Theorem 7.1. For the proof of $(a) \Rightarrow (c)$ we suppose that (c) is not true and we will prove that in every cases the inequality (a) is false.
If $\mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x) = \infty > I_f(x)$ then the inequality is not true.
From the first axiom of countability, we have $\mathrm{ls}_e\, I_{f_n}(x) = \mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x)$. I claim that $\mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x) \neq -\infty$. If not, because $I_f(x)$ is finite, there exists a sequence $(x_n)_n$ T-converging to x such that:

$$\limsup_n I_{f_n}(x_n) < I_f(x).$$

From [15] Theorem 4.5 and since $f(x) = \mathrm{li}_e\, f_n(x)$, we get the contradiction:

$$-\infty < I_f(x) \leq \liminf_n I_{f_n}(x_n) < I_f(x).$$

Let us consider now the case where $\mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x)$ is finite. Since (c) is not true there exists $\epsilon_0 > 0$ with $\delta^+((f_n(x_n))_n) > \epsilon_0$ for every sequence $(x_n)_n$ T-converging to x. Due to the definition of $\mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x)$, there exists a sequence $(x_n)_n$ T-converging to x such:

$$\limsup_n I_{f_n}(x_n) < \mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x) + \frac{1}{3}\epsilon_0.$$

As a consequence of Ioffe's criterion we deduce that the sequence $(f_n^-(x_n))_n$ is eventually bounded an equi-integrable in $L_1(\Omega, I\!R)$. Due to [15] Theorem 3.6 (a) we obtain:

$$\delta^+((f_n(x_n))_n) + I_{\liminf_n f_n(x_n)} \le \limsup_n I_{f_n}(x_n) \le \mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x) + \frac{1}{3}\epsilon_0 < +\infty,$$

this proves that $\delta^+((f_n(x_n))_n)$ is finite.
From [15] Lemma 3.10, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with the following property: for each subsequence $(x_{n_{k_l}})_l$ of $(x_{n_k})_k$,

$$\frac{2}{3}\delta^+((f_n(x_n))_n) \le \delta^+((f_{n_{k_l}}(x_{n_{k_l}}))_l)$$

Extracting from $(x_{n_k})_k$ a subsequence $(x_{n_{k_l}})_l$ almost everywhere converging to x, and using [15] Theorem 3.6 (a) we obtain:

$$\delta^+((f_{n_{k_l}}(x_{n_{k_l}}))_l) + I_{\liminf_l f_{n_{k_l}}(x_{n_{k_l}})} \le \limsup_l I_{f_{n_{k_l}}}(x_{n_{k_l}}) \le \mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x) + \frac{1}{3}\epsilon_0,$$

but $f(x) = \mathrm{li}_e\, f_n(x)$, hence

$$\frac{2}{3}\delta^+((f_n(x_n))_n) + I_f(x) \le \mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x) + \frac{1}{3}\epsilon_0,$$

an therefore:

$$\frac{1}{3}\epsilon_0 + I_f(x) \le \mathrm{seq}-\mathrm{ls}_e\, I_{f_n}(x)$$

This proves that the inequality (a) of Theorem 7.1 is false.
Clearly $(c) \Rightarrow (b)$. The assertion $(b) \Rightarrow (a)$ is an immediate consequence of Theorem 6.1. The proof of Theorem 7.1 is complete.

Notice that with the assumptions of Theorem 7.1, due to [15] Theorem 4.5 we have more precisely under the assertions (b) or (c):

$$\lim_e I_{f_n}(x) = I_f(x).$$

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