

Remarks on Diameter 2 Properties

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If X is an infinite-dimensional uniform algebra, if X has the Daugavet property or if X is a proper M -embedded space, every relatively weakly open subset of the unit ball of the Banach space X is known to have diameter 2, i.e., X has the diameter 2 property. We prove that in these three cases even every finite convex combination of relatively weakly open subsets of the unit ball have diameter 2. Further, we identify new examples of spaces with the diameter 2 property outside the formerly known cases; in particular we observe that forming ℓ_p -sums of diameter 2 spaces does not ruin diameter 2 structure.

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1. Introduction

Let X be a (real) Banach space and B_X its unit ball. By a slice of B_X we mean a set of the type $S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}$ where x^* is in the unit sphere S_{X^*} of X^* and $\varepsilon > 0$. A finite convex combination of slices of B_X is then a set of the form

$$S = \sum_{i=1}^n \lambda_i S(x_i^*, \varepsilon_i), \quad \lambda_i \geq 0, \quad \sum_{i=1}^n \lambda_i = 1,$$

where $x_i^* \in S_{X^*}$ and $\varepsilon_i > 0$ for $i = 1, 2, \dots, n$.

Let us consider the following three properties:

Definition 1.1. A Banach space X has the

- 1) *local diameter 2 property* if every slice of B_X has diameter 2.

- 2) *diameter 2 property* if every non-empty relatively weakly open subset of B_X has diameter 2.
- 3) *strong diameter 2 property* if every finite convex combination of slices of B_X has diameter 2.

Before going further, let us just remark that the following implications hold true for the properties in Definition 1.1:

$$\text{strong diameter 2} \Rightarrow \text{diameter 2} \Rightarrow \text{local diameter 2}.$$

The second implication is clear since every slice of B_X is a non-empty relatively weakly open subset of B_X . The first implication is a consequence of a lemma by Bourgain saying that every non-empty relatively weakly open subset of B_X contains a finite convex combination of slices (see [9, Lemma II.1 p. 26]). Note that a finite convex combination of slices need not be relatively weakly open (it may be contained in δB_X , where $\delta < 1$, see [9, Remark IV.5 p. 48], and recall that any non-empty relatively weakly open subset of B_X must intersect S_X when X is infinite-dimensional.) Note also that the strong diameter 2 property implies that every non-empty finite convex combination of relatively weakly open sets in B_X has diameter 2.

To the best of our knowledge it is not known whether any of the reverse implications hold. Nor do we know of any general characterizations of the diameter 2 properties. The diameter 2 property as a term has been in use for some years now. The idea of studying its local and strong versions appears to be new.

It is an interesting exercise to show that the classical spaces c_0 , $C[0, 1]$ and $L_1[0, 1]$ have the diameter 2 properties. Using dentability it is clear that spaces with the Radon-Nikodým property cannot have the local diameter 2 property. It is also clear that spaces with the point of continuity property cannot have the diameter 2 property. Thus diameter 2 properties are at the opposite side of the spectrum from the Radon-Nikodým and point of continuity properties. Note that the predual B of the James tree space has the point of continuity property but lacks the Radon-Nikodým property (see [8, Example 6.(1)]).

Since the point of continuity property is preserved by renormings, B can not be renormed to have the diameter 2 property. It is an open question (see e.g. [5, p. 553]) whether every Banach space failing the Radon-Nikodým property can be renormed to have the local diameter 2 property.

Section 2 is a survey of examples and results on diameter 2 properties. Particular emphasis has been put on checking which of the diameter 2 properties that in fact are known to hold in each case. Through this survey we will motivate the research we have done, and the reader can easily see our main theorems in the light of known results.

Section 3 contains our perhaps most surprising result. While in Section 2 spaces with L - and M -structure dominate the landscape of diameter 2 spaces, we show that the diameter 2 property is actually preserved when taking any ℓ_p -sum, $1 \leq$

$p \leq \infty$, of diameter 2 spaces. In particular, $c_0 \oplus_2 c_0$ has the diameter 2 property, and we show that this example is not covered by the results in Section 2.

In Section 4 we show that spaces with the Daugavet property, infinite-dimensional uniform algebras, proper M -embedded spaces, and biduals of proper M -embedded spaces all have the strong diameter 2 property.

We end the paper with some concluding remarks and questions.

We use standard Banach space terminology and notation.

2. A survey of examples and results on diameter 2 properties

We will now try to give an overview of *general* results on diameter 2 properties for Banach spaces.

Recall that a Banach space X has the Daugavet property if for every rank one operator $T : X \rightarrow X$ the equation $\|I + T\| = 1 + \|T\|$ holds where I is the identity operator on X . One can show that the Daugavet property is equivalent to each of the following two statements (see [19] or [17]):

1. For every slice $S = S(x_0^*, \varepsilon_0)$ of B_X , every $x_0 \in S_X$, and every $\varepsilon > 0$ there exists a point $x \in S$ such that $\|x + x_0\| \geq 2 - \varepsilon$.
2. For every $\varepsilon > 0$, every $x \in S_X$, $B_X = \overline{\text{conv}}\Delta_\varepsilon(x)$ where $\Delta_\varepsilon(x) = \{y \in B_X : \|y - x\| \geq 2 - \varepsilon\}$.

The first general results date some ten years back. From [17] and [16] we have:

Theorem 2.1. *Let X be a space with the Daugavet property or an infinite-dimensional uniform algebra. Then X has the diameter 2 property.*

In Section 4 we will prove that spaces with the Daugavet property and infinite-dimensional uniform algebras in fact have the strong diameter 2 property.

Note that the Daugavet property can be weakened as in the proposition below (see also Problem (7) in [19]) and still imply the local diameter 2 property.

Proposition 2.2. *Let X be a Banach space such that $x \in \overline{\text{conv}}\Delta_\varepsilon(x)$ for every $x \in S_X$ and $\varepsilon > 0$. Then X has the local diameter 2 property.*

Proof. Let $x^* \in S_{X^*}$ and $\varepsilon > 0$. Find $x \in B_X$ such that $x^*(x) \geq 1 - \varepsilon/2$ and a convex combination $\sum_{i=1}^n \lambda_i y_i$ where $y_i \in \Delta_\varepsilon(x)$ such that $\|\sum_{i=1}^n \lambda_i y_i - x\| \leq \varepsilon/2$. In particular $x^*(\sum_{i=1}^n \lambda_i y_i) \geq 1 - \varepsilon$. If $x^*(y_i) < 1 - \varepsilon$ for every $i = 1, 2, \dots, n$, then

$$x^* \left(\sum_{i=1}^n \lambda_i y_i \right) < 1 - \varepsilon.$$

This is a contradiction, so there is at least one i such that $\|x - y_i\| \geq 2 - \varepsilon$ with x and y_i in the slice $S(x^*, \varepsilon) = \{z \in B_X : x^*(z) \geq 1 - \varepsilon\}$. $\square$

Recall that a closed subspace X of a Banach space Y is an L -summand (u-summand) in Y if there is a subspace Z , the L -complement (u-complement) of X ,

so that $X \oplus Z = Y$ and if $x \in X, z \in Z$ then $\|x+z\| = \|x\| + \|z\|$ ($\|x+z\| = \|x-z\|$). If the annihilator $X^\perp$ of X is an L -summand (u-summand) in Y^* , then X is said to be an M -ideal (u-ideal) in Y . Banach spaces which are M -ideals in their biduals are called M -embedded. Strict u-ideals are u-ideals in their biduals where the u-complement of $X^\perp$ is X^* .

In [6] a very important observation is the following:

Theorem 2.3. *If X^* is a proper L -summand in X^{***} , then X has the diameter 2 property. In particular, proper M -embedded spaces have the diameter 2 property.*

The authors of [6] prove that real JB^* -triples over non-reflexive Banach spaces get the diameter 2 property from Theorem 2.3. The role of M -structure is taken further in [15]:

Theorem 2.4. *Assume Y is a proper M -ideal in X and put $X^* = Z \oplus_1 Y^\perp$. If Z is 1-norming for X , then both X and Y have the diameter 2 property. In particular, if Y is a proper M -embedded space, then both Y and Y^{**} have the diameter 2 property. In fact, in the latter case, every subspace Z of Y^{**} which contains Y has the diameter 2 property.*

In [7] the centralizer is introduced to the study of the diameter 2 property. The centralizer of X (we write $Z(X)$), is the set of those multipliers T on X such that there exists a multiplier S on X satisfying $a_S(p) = \overline{a_T(p)}$ for every extreme point p of B_{X^*} . Recall that a multiplier on X is a bounded linear operator T on X such that every extreme point of B_{X^*} becomes an eigenvector for the adjoint T^* of T . Thus given a multiplier T on X and an extreme point p of B_{X^*} , there exists a unique number $a_T(p)$ satisfying $p \circ T = a_T(p)p$.

Let X be a Banach space. Using the notation of [2] consider the increasing sequence of even duals

$$X \subseteq X^{**} \subseteq X^{(4)} \dots \subseteq X^{(2n)} \subseteq \dots,$$

and define $X^{(\infty)}$ as the completion of the normed space $\cup_{n=0}^{\infty} X^{(2n)}$.

In [7] the following result is proved:

Theorem 2.5. *If $Z(X^{(\infty)})$ of a Banach space X is infinite-dimensional, then X has the diameter 2 property. In fact, if X fails the diameter 2 property, then $\sup_n \dim Z(X^{(2n)}) < \infty$.*

Theorem 2.5 includes Theorem 2.3 and a lot of other cases (see [2, Proposition 3.3]). As Theorem 2.5 indicates, $Z(X^{(\infty)})$ being infinite-dimensional more than suffices to assure that X has the diameter 2 property. In [2] this becomes very clear; we state the main result:

Theorem 2.6. *If $\dim Z(X^{(\infty)}) = \infty$, then the completed n -fold symmetric projective tensor product of X has the diameter 2 property.*

The results involving the centralizer are strong and powerful, but they do not

contain the L_1 -cases. We have now stated the most fundamental theorems of sufficiency for diameter 2 properties. Let us see how diameter 2 properties are transferred to some basic structures:

Theorem 2.7. *Let X and Y be Banach spaces. Then the following hold.*

- (i) *If X or Y has the local diameter 2 property, then the projective tensor product $X \widehat{\otimes}_\pi Y$ has the local diameter 2 property.*
- (ii) *If X or Y has the diameter 2 property, then $X \oplus_\infty Y$ has the diameter 2 property.*
- (iii) *If X and Y both have the strong diameter 2 property, then $X \oplus_1 Y$ has the strong diameter 2 property.*

Here (i) follows by using that $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{L}(Y, X^*)$. (ii) is Lemma 2.1 of [15]. The proof of (iii) is the proof of Lemma 2.1 (ii) in [5].

Remark 2.8. Note that the statement and proof of Lemma 2.1 (ii) in [5] do not match. What they actually prove is statement (iii) in Theorem 2.7 above. We will show in Theorem 3.2 that their statement is also true.

We will end this section by mentioning that the interpolation spaces $L_1(\mathbb{R}^+) + L_\infty(\mathbb{R}^+)$ (endowed with their two natural norms) and $L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$ (endowed with the maximum norm) all have the diameter 2 property, but they do not have the Daugavet property [4].

3. Some examples not covered by the known classes of diameter 2 spaces

In [15] it is proved that if a space has the diameter 2 property then also the ℓ_∞ -sum with any other space has this property. Also, as pointed out in Theorem 2.7, the authors of [5] proved that the ℓ_1 -sum of two spaces with the strong diameter 2 property inherits this property. We will now prove that the (local) diameter 2 property is in fact stable by taking ℓ_p -sums for every $1 \leq p \leq \infty$. First we need a lemma.

Lemma 3.1. *Suppose every non-empty relatively weakly open subset (resp. every slice, every finite convex combination of slices) of the unit ball B_X of a Banach space X has diameter 2. Then every non-empty relatively weakly open subset (resp. every slice, every finite convex combination of slices) of δB_X has diameter 2δ , where $1 \geq \delta > 0$.*

Proof. We first prove the result for non-empty relatively weakly open subsets. To this end, let $\varepsilon > 0$, $W \subset X$ be weakly open, and suppose $W \cap \delta B_X \neq \emptyset$. Pick some $x_0 \in B_X$ such that $\delta x_0 \in W \cap \delta B_X$ and some weak neighborhood $W_0 = \{x \in X : |x_i^*(x - \delta x_0)| < \varepsilon, 1 \leq i \leq n\}$ of δx_0 with $W_0 \subset W$. Using the assumption there are $x_1, x_2 \in W'_0 = \{x \in X : |x_i^*(x - x_0)| < \delta^{-1}\varepsilon, 1 \leq i \leq n\} \cap B_X$ such that $\|x_1 - x_2\| > 2 - \varepsilon$. Now $\delta x_1, \delta x_2$ are in $W_0 \cap \delta B_X$ and $\|\delta x_1 - \delta x_2\| > (2 - \varepsilon)\delta$. Since ε was arbitrary we are done.

The same argument works also for slices.

We now prove the result for finite convex combinations of slices. So let $\varepsilon > 0$, $S = \sum_{i=1}^n \lambda_i S_i$ a convex combination of slices of δB_X , and write $S_i = \{x \in \delta B_X : x_i^*(x) > \delta - \varepsilon_i\}$ with $x_i^* \in S_{X^*}$. Let $S'_i = \{x \in B_X : x_i^*(x) > 1 - \delta^{-1}\varepsilon_i\}$. By assumption there are $x_1, x_2 \in \sum_{i=1}^n \lambda_i S'_i$ with $\|x_1 - x_2\| > 2 - \varepsilon$. It is easy to see that δx_1 and δx_2 are in S and $\|\delta x_1 - \delta x_2\| > (2 - \varepsilon)\delta$. Since ε was arbitrary we are done. $\square$

Theorem 3.2. *The (local) diameter 2 property is stable by taking ℓ_p -sums for all $1 \leq p \leq \infty$.*

Proof. We will only prove it for the diameter 2 property and for the case when we sum two spaces. The general case is similar. Also the proof for the local diameter 2 property is similar.

To begin with note that the case $p = \infty$ is Theorem 2.7 (ii). So let $1 \leq p < \infty$, $\varepsilon > 0$, X_1 and X_2 Banach spaces with the diameter 2 property, and put $Z = X_1 \oplus_p X_2$. Let W be a non-empty relatively weakly open subset of B_Z . Since Z is infinite-dimensional every weakly open set is unbounded in some direction. Thus there is some $z_0 = (x_1, x_2) \in W \cap S_Z$. Now find some relatively weakly open neighborhood $W_0 = \{z \in B_Z : |z_i^*(z - z_0)| < \varepsilon, i = 1, 2, \dots, n\}$ of z_0 where $z_i^* = (x_{1,i}^*, x_{2,i}^*) \in S_{Z^*}$ such that $W_0 \subset W$. Put

$$\begin{aligned} W_0^1 &= \{x \in \|x_1\|B_{X_1} : |x_{1,i}^*(x - x_1)| < \varepsilon/2, i = 1, 2, \dots, n\}, \\ W_0^2 &= \{x \in \|x_2\|B_{X_2} : |x_{2,i}^*(x - x_2)| < \varepsilon/2, i = 1, 2, \dots, n\}. \end{aligned}$$

Since, for $k = 1, 2$, W_0^k is non-empty relatively weakly open in $\|x_k\|B_{X_k}$, we can by Lemma 3.1 find points x_k^1 and x_k^2 in W_0^k with $\|x_k^1 - x_k^2\| > (2 - \varepsilon)\|x_k\|$. Moreover, $W_0^\times = W_0^1 \times W_0^2 \subset W_0$. Indeed, suppose $y = (y_1, y_2) \in W_0^\times$. Then

$$\|y\|^p = \|y_1\|^p + \|y_2\|^p \leq \|x_1\|^p + \|x_2\|^p = 1,$$

so $y \in B_Z$. Also for every $i = 1, 2, \dots, n$ we have

$$|z_i^*(y - z_0)| = |x_{1,i}^*(y_1 - x_1) + x_{2,i}^*(y_2 - x_2)| < \varepsilon,$$

so $y \in W_0^\times$. Now put $x^1 = (x_1^1, x_2^1)$ and $x^2 = (x_1^2, x_2^2)$. Then we get

$$\begin{aligned} \|x^1 - x^2\|^p &= \|x_1^1 - x_1^2\|^p + \|x_2^1 - x_2^2\|^p \\ &\geq (2 - \varepsilon)^p (\|x_1\|^p + \|x_2\|^p) \\ &= (2 - \varepsilon)^p, \end{aligned}$$

and since ε is arbitrary, we are done. $\square$

Using Theorem 3.2 it is easy to produce an example which falls outside the theorems in Section 2.

Example 3.3. The space $X = c_0 \oplus_2 c_0$ is a strict u-ideal in its bidual and has the diameter 2 property. However, X^* is not an L -summand in its bidual, $\sup_n \dim Z(X^{(2n)}) = 1$, X is not a uniform algebra, and X lacks the Daugavet property.

Proof. X is a strict u-ideal in its bidual. This can be shown as in [10, Example (5)] or it can be shown directly using [10, Lemma 2.2].

Since every dual of X contains an ℓ_2 -sum, X^* is in particular not an L -summand in its bidual and none of the duals of X can contain any non-trivial M -ideal ([11, p. 45]). Thus $Z(X^{(2n)}) = 1$ for all n ([11, p. 39]), hence $\sup_n \dim Z(X^{(2n)}) = 1$. X does not have the Daugavet property since no separable space with the Daugavet property can have an unconditional basis ([12, Corollary 2.7]). Further, X is definitely not a uniform algebra. Thus X is not contained in any of the main cases covered in Section 2. $\square$

Theorem 2.3 tells us that proper M -embedded spaces have the diameter 2 property. Strict u-ideals in their biduals share many of the properties of M -embedded spaces (see [10], [13], [1], and [11]). However the next example shows that spaces which are strict u-ideals in their biduals need not even have the local diameter 2 property.

Example 3.4. $X = \mathbb{R} \oplus_1 c_0$ is a strict u-ideal in X^{**} which does not have the local diameter 2 property.

Proof. First, X is a strict u-ideal by [10, Example (6)]. To finish, let $\phi = (1, (0)) \in X^*$ and $S = \{x \in B_X : \phi(x) > 1 - \varepsilon\}$. Then S has diameter less than 2ε . $\square$

4. Spaces with the strong diameter 2 property

The object of this section is to prove that infinite-dimensional uniform algebras, spaces with the Daugavet property, proper M -embedded spaces, and biduals of proper M -embedded spaces have the strong diameter 2 property. We start by proving this for uniform algebras. Let us first note the following simple, sufficient condition for a space to have the strong diameter 2 property. It will also be used in the proof of Theorem 4.10.

Lemma 4.1. *Suppose X has the local diameter 2 property in the following sense: Whenever $\{S_j\}_{j=1}^n$ is a finite family of slices of B_X and $\varepsilon > 0$, then there exist points $h_j \in S_j$ and $\varphi \in B_X$, independent of j , such that $h_j \pm \varphi \in S_j$ and $\|\varphi\| > 1 - \varepsilon$. Then X has the strong diameter 2 property.*

Proof. Let S be a finite convex combination of slices, i.e., $S = \sum_{j=1}^n \lambda_j S_j$ where $0 < \lambda_j < 1$ and $\sum_{j=1}^n \lambda_j = 1$. For every $1 \leq j \leq n$, take $h_j \in S_j$ and $\varphi \in B_X$ such that $h_j \pm \varphi \in S_j$ and $\|\varphi\| > 1 - \varepsilon$. Define

$$\psi_+ = \sum_{j=1}^n \lambda_j h_j + \varphi \quad \text{and} \quad \psi_- = \sum_{j=1}^n \lambda_j h_j - \varphi.$$

Then $\psi_+, \psi_- \in S$ and $\|\psi_+ - \psi_-\| = \|2\varphi\| > 2 - 2\varepsilon$. $\square$

Theorem 4.2. *Infinite-dimensional uniform algebras have the strong diameter 2 property.*

Proof. An inspection of the proof of [16, Theorem 2] shows that a uniform algebra fulfills the conditions in Lemma 4.1. Using their notation, let $\varepsilon = \min_j \varepsilon_j$ and choose $0 < \delta \leq \varepsilon/12$ instead (to simplify for point (ii) below) and write

$$\frac{h_j}{1+4\delta} \pm \frac{\varphi}{1+4\delta} = h'_j \pm \varphi'.$$

Then $h'_j \pm \varphi' \in S_j(\ell_j, \varepsilon_j)$, since by using facts from the proof of [16, Theorem 1], we get

- (i) $\|h'_j\| \leq (1+3\delta)/(1+4\delta) < 1$,
- (ii) $1 > \|\varphi'\| = \frac{1}{1+4\delta} > \frac{1}{1+\frac{\varepsilon}{3}} > 1-\varepsilon$,
- (iii) $\|h'_j \pm \varphi'\| \leq 1$,
- (iv) $\ell_j(h_j) \geq 1-\delta-4\delta = 1-5\delta$, and
- (v) $|\ell_j(\varphi)| < 2\delta$.

Finally (iv) and (v) imply that

$$\ell_j(h'_j \pm \varphi') = \frac{\ell_j(h_j) - |\ell_j(\varphi)|}{1+4\delta} > \frac{1-5\delta-2\delta}{1+4\delta} \geq \frac{1-\frac{7\varepsilon}{12}}{1+\frac{\varepsilon}{3}} \geq 1-\varepsilon_j. \quad \square$$

For the proof that spaces with the Daugavet property have the strong diameter 2 property we will need a version of [12, Lemma 2.1]. The lemma is used in the proof of [17, Lemma 3], but is not stated explicitly. We include a proof for easy reference.

Lemma 4.3. *If X has the Daugavet property, then for every $\varepsilon > 0$, every $y_0 \in X$ and every slice $S(x_0^*, \alpha_0)$ of B_X there is another slice $S(x_1^*, \alpha_1) \subset S(x_0^*, \alpha_0)$ such that for every $x \in S(x_1^*, \alpha_1)$ the inequality $\|\lambda x + y_0\| \geq \lambda + \|y_0\| - \varepsilon$ holds for every $0 \leq \lambda \leq 1$.*

Proof. Only the case $y_0 \neq 0$ needs proof. By making the slice $S(x_0^*, \alpha_0)$ smaller we may assume without loss of generality that $2\alpha_0\|y_0\| < \varepsilon$.

Define $T = x_0^* \otimes y_0$, so that $\|T\| = \|y_0\|$. By the Daugavet equation $\|I^* + T^*\| = \|I + T\| = 1 + \|y_0\|$. In order to guarantee that $0 < \alpha_1 < 1$, one can choose $y^* \in S_{X^*}$ such that $\|(I+T)^*y^*\| \geq 1 + \|y_0\|(1-\alpha_0)$ and $y^*(y_0) \geq 0$. Define

$$x_1^* = \frac{(I+T)^*y^*}{\|(I+T)^*y^*\|} \quad \text{and} \quad \alpha_1 = 1 - \frac{1 + \|y_0\|(1-\alpha_0)}{\|(I+T)^*y^*\|}.$$

Then, for all $x \in S(x_1^*, \alpha_1)$,

$$\begin{aligned} y^*(x) + y^*(y_0)x_0^*(x) &= \langle (I+T)^*y^*, x \rangle \geq (1-\alpha_1)\|(I+T)^*y^*\| \\ &= 1 + \|y_0\|(1-\alpha_0). \end{aligned}$$

We get

$$\|y_0\|x_0^*(x) \geq y^*(y_0)x_0^*(x) \geq 1 + \|y_0\|(1 - \alpha_0) - y^*(x) \geq \|y_0\|(1 - \alpha_0), \quad (1)$$

hence $x_0^*(x) \geq 1 - \alpha_0$ which shows that $S(x_1^*, \alpha_1) \subseteq S(x_0^*, \alpha_0)$. Since $x_0^*(x) \leq 1$ we get

$$y^*(x) + y^*(y_0) \geq y^*(x) + y^*(y_0)x_0^*(x) \geq 1 + \|y_0\|(1 - \alpha_0), \quad (2)$$

and thus $\|x + y_0\| \geq 1 + \|y_0\|(1 - \alpha_0)$.

Finally, by (1) and (2),

$$\begin{aligned} y^*(\lambda x) + y^*(y_0) &= \lambda y^*(x) + y^*(y_0) \\ &\geq \lambda(1 + \|y_0\|(1 - \alpha_0) - y^*(y_0)x_0^*(x)) + \|y_0\|(1 - \alpha_0) \\ &\geq \lambda(1 + \|y_0\|(1 - \alpha_0) - \|y_0\|) + \|y_0\|(1 - \alpha_0) \\ &= \lambda + \|y_0\| - \alpha_0\|y_0\|(1 + \lambda). \end{aligned}$$

Hence $\|\lambda x + y_0\| \geq \lambda + \|y_0\| - \alpha_0\|y_0\|(1 + \lambda) > \lambda + \|y_0\| - \varepsilon$. $\square$

Now we are ready to show that spaces with the Daugavet property have the strong diameter 2 property. The idea of the proof is due to Shvydkoy (see [17, Lemma 3]), but we have to apply this idea twice.

Theorem 4.4. *If X has the Daugavet property, then X has the strong diameter 2 property.*

Proof. Let $S_j = \{x \in B_X : x_j^*(x) > 1 - \varepsilon_j\}$ be slices of B_X and let $0 < \lambda_j < 1$ such that $\sum_{j=1}^n \lambda_j = 1$.

Let $0 < \varepsilon < 1$ and $y \in S_X$. Using Lemma 4.3 we can find $x_1 \in S_1$ such that $\|\lambda_1 x_1 + y\| > \lambda_1 + 1 + \varepsilon/n$. Using Lemma 4.3 repeatedly we find $x_j \in S_j$ such that

$$\left\| \sum_{j=1}^n \lambda_j x_j + y \right\| > \sum_{j=1}^n \lambda_j + 1 - \varepsilon = 2 - \varepsilon.$$

In particular,

$$1 \geq \left\| \sum_{j=1}^n \lambda_j x_j \right\| \geq \left\| \sum_{j=1}^n \lambda_j x_j + y \right\| - \|y\| \geq 2 - \varepsilon - 1 = 1 - \varepsilon.$$

Define $y_0 = \frac{\sum_{j=1}^n \lambda_j x_j}{\|\sum_{j=1}^n \lambda_j x_j\|}$. Then $\|y_0\| = 1$ and $\|y_0 - \sum_{j=1}^n \lambda_j x_j\| \leq \varepsilon$.

Repeat the procedure above using $-y_0$ instead of y and find $z_j \in S_j$ such that

$$\left\| \sum_{j=1}^n \lambda_j z_j - y_0 \right\| \geq 2 - \varepsilon.$$

We get

$$\left\| \sum_{j=1}^n \lambda_j z_j - \sum_{j=1}^n \lambda_j x_j \right\| \geq \left\| \sum_{j=1}^n \lambda_j z_j - y_0 \right\| - \left\| y_0 - \sum_{j=1}^n \lambda_j x_j \right\| \geq 2 - 2\varepsilon.$$

This shows the existence of points in the convex combination of the slices with distance arbitrarily close to 2. $\square$

We will end this section by showing that the bidual of proper M -embedded spaces have the strong diameter 2 property. To do this we will need some results inspired by [15]. The first lemma is contained in [15, Lemma 2.1], but we provide a short elementary proof.

Lemma 4.5. *Let X and Y be Banach spaces, W a weakly open subset in $Z = X \oplus_\infty Y$, and $(x_0, y_0) \in W$. Then there exist weakly open subsets U of X and V of Y such that $(x_0, y_0) \in U \times V \subset W$. Moreover, if W is a relatively weakly open subset of B_Z , then U and V can be chosen to be relatively weakly open subsets of B_X and B_Y respectively.*

Proof. There exists $f_i = (x_i^*, y_i^*) \in X^* \times Y^*$ where $i = 1, 2, \dots, n$ such that

$$W_0 = \{(x, y) \in Z : |f_i(x, y) - f_i(x_0, y_0)| < 1, i = 1, 2, \dots, n\} \subset W.$$

Let $U_0 = \{x \in X : |x_i^*(x) - x_i^*(x_0)| < \frac{1}{2}, i = 1, 2, \dots, n\}$ and $V_0 = \{y \in Y : |y_i^*(y) - y_i^*(y_0)| < \frac{1}{2}, i = 1, 2, \dots, n\}$. Then U_0 and V_0 are weakly open in X and Y respectively and $U_0 \times V_0 \subset W_0 \subset W$.

For the last part, just write $U = U_0 \cap B_X$ and $V = V_0 \cap B_Y$. $\square$

Lemma 2.1 in [15] shows that an ℓ_∞ -sum of two Banach spaces has the diameter 2 property if one of the components has. Next we show that a similar result holds for the strong diameter 2 property.

Proposition 4.6. *Let X and Y be Banach spaces and suppose X has the strong diameter 2 property. Then $X \oplus_\infty Y$ has the strong diameter 2 property.*

Proof. Let $Z = X \oplus_\infty Y$ and let $P : Z \rightarrow X$ be the natural projection onto X . Let $W = \sum_{i=1}^n \lambda_i S_i$ be a convex combination of slices $S_i = \{(x, y) \in B_Z : g_i(x, y) > 1 - \varepsilon_i\}$ for some $g_i \in S_{Z^*}$ and $\varepsilon_i > 0$. Obviously $W \subset B_Z$. It is enough to show that $P(W)$ is non-empty and contains a non-empty finite convex combination of relatively weakly open subsets of B_X . Indeed, the conclusion then follows from Bourgain's result (see [9, Lemma II.1 p. 26]) that every relatively weakly open subset of the unit ball contains a convex combination of slices and from the assumption, since $\|P\| = 1$.

Since W is non-empty, there is some $(x_0, y_0) \in W$. Thus $x_0 = P(x_0, y_0) \in P(W)$, so $P(W)$ is non-empty. Finally, write $(x_0, y_0) = \sum_{i=1}^n \lambda_i (x_0^i, y_0^i)$ where $(x_0^i, y_0^i) \in S_i$. Since each slice S_i is relatively weakly open, it is possible by Lemma 4.5 to find

relatively weakly open sets $U_i \subset B_X$ and $V_i \subset B_Y$ such that $(x_0^i, y_0^i) \in U_i \times V_i \subset S_i$. Thus

$$\begin{aligned} P(W) &\supset P\left(\sum_{i=1}^n \lambda_i(U_i \times V_i) \cap B_Z\right) \\ &= P\left(\left(\sum_{i=1}^n \lambda_i U_i\right) \times \left(\sum_{i=1}^n \lambda_i V_i\right) \cap (B_X \times B_Y)\right) = \sum_{i=1}^n \lambda_i U_i \cap B_X. \end{aligned}$$

Since each U_i is weakly open, we are done. $\square$

Using the above proposition we can strengthen [15, Proposition 2.6].

Proposition 4.7. *Let X be a Banach space containing a subspace isomorphic to c_0 . Then there exists an equivalent norm on X such that X has the strong diameter 2 property in the new norm.*

Proof. The proof is similar to the proof of [15, Proposition 2.6]. The only difference is that we use that ℓ_∞ has the strong diameter 2 property and Proposition 4.6. $\square$

Next we have dual versions of Lemma 4.5 and Proposition 4.6.

Lemma 4.8. *Let X be a Banach space and Y a closed subspace of X . Assume that $X^{**} = Y^{\perp\perp} \oplus_\infty Z^\perp$ for some closed subspace Z of X^* . Let W be a weak* open subset in X^{**} and $(x_0^{**}, y_0^{**}) \in W$. Then there exist weak* open subsets U of $Y^{\perp\perp}$ and V of $Z^\perp$ such that $(x_0^{**}, y_0^{**}) \in U \times V \subset W$. Moreover, if W is a relatively weak* open subset of $B_{X^{**}}$, then U and V can be chosen to be relatively weak* open subsets of $B_{Y^{\perp\perp}}$ and $B_{Z^\perp}$ respectively.*

Proof. The proof is similar to the proof of Lemma 4.5. $\square$

Proposition 4.9. *Let Y be a closed subspace of a Banach space X such that $X^{**} = Y^{\perp\perp} \oplus_\infty Z^\perp$ for some closed subspace Z of X^* . If every finite convex combination of weak* slices of $Y^{\perp\perp}$ has diameter 2 then every finite convex combination of weak* slices of X^{**} has diameter 2.*

Proof. The proof is similar to the proof of Proposition 4.6. $\square$

We conclude this section by proving that if Y is a proper M -ideal in X and the range of the L -projection in X^* is 1-norming for X , then both X and Y have the strong diameter 2 property. The proof is inspired by the proof of [15, Proposition 2.3].

Theorem 4.10. *Let Y be a proper M -ideal in X , i.e. $X^* = Z \oplus_1 Y^\perp$ for some subspace Z of X^* . If Z is 1-norming for X , then both X and Y have the strong diameter 2 property.*

*In particular, if X is proper M -embedded, then both X and X^{**} have the strong diameter 2 property.*

Proof. We have $X^{**} = Y^{\perp\perp} \oplus_{\infty} Z^{\perp}$. Let $i = 1, 2, \dots, n$, $\varepsilon_i > 0$, and $z_i \in S_Z$, and consider the weak* slices

$$S_i^* = \{y^{\perp\perp} \in B_{Y^{\perp\perp}} : y^{\perp\perp}(z_i) > 1 - \varepsilon_i\}.$$

Let $\lambda_i > 0$ such that $\sum_{i=1}^n \lambda_i = 1$. First we prove that the convex combination $S^* = \sum_{i=1}^n \lambda_i S_i^*$ has diameter 2. Note that by Goldstine's theorem the slice $S_i^* = S(z_i, \varepsilon_i)$ of B_Y is weak* dense in S_i^* for each i . In particular $S_i^* \cap B_Y \neq \emptyset$ for every $i = 1, 2, \dots, n$. Choose $y_0^i \in S_i^* \cap B_Y$.

Given $\delta > 0$ it is possible to find $x \in S_X$ such that $\|x + Y\| > 1 - \delta$ since Y is a proper subspace of X . Using the M -ideal property, by [18, Proposition 2.3] there is a net (y_d) in Y such that $y_d \rightarrow x$ in the $\sigma(X, Z)$ -topology with $\limsup_d \|y_0^i \pm (x - y_d)\| \leq 1$. Thus for any given $0 < r_i < 1$ we can find d_0^i such that

$$r_i \|y_0^i \pm (x - y_d)\| \leq 1$$

whenever $d \geq d_0^i$. Since each S_i is weakly open in Y and the $\sigma(X, Z)$ -topology on Y is just the weak topology we may assume that

$$r_i (y_0^i \pm (x - y_d)) \in S_i \subset S_i^*$$

for $d \geq d_0^i$. Let $r = \max_i r_i$ and $d_0 \geq d_0^i$ for every $i = 1, 2, \dots, n$. Using Lemma 4.1 with $h_i = r y_0^i$ and $\varphi = r(x - y_d)$ we get that S^* has diameter 2 since

$$\|\varphi\| \geq r \|x + Y\| > r(1 - \delta),$$

and r and δ can be chosen arbitrarily close to 1 and 0 respectively.

From Proposition 4.9 and $X^{**} = Y^{\perp\perp} \oplus_{\infty} Z^{\perp}$ we get that every finite convex combination of weak* slices of $B_{X^{**}}$ has diameter 2. By the weak* density of B_X in $B_{X^{**}}$ and the weak* lower semi-continuity of the norm, it follows that X has the strong diameter 2 property.

Since Z is 1-norming the norm on X is $\sigma(X, Z)$ -lower semi-continuous. That Y has the strong diameter 2 property is immediate since a functional $y^* \in S_{Y^*}$ uniquely extends to a functional in S_{X^*} and the slice $S(y^*, \varepsilon)$ of B_Y is $\sigma(X, Z)$ -dense in the slice $S(y^*, \varepsilon)$ of B_X . $\square$

5. Some concluding remarks and questions

As remarked in Section 1 we do not know if the three diameter 2 properties really are different. Having an answer to this question is clearly important for future research on diameter 2 spaces. Our conjecture is that they are not equal.

Meanwhile, and especially if our conjecture is correct, some questions naturally come to mind:

- (a) Can one conclude diameter 2 property or even strong diameter 2 property in Proposition 2.2?
- (b) From Theorems 2.6 and 2.7(i), how are diameter 2 properties in general preserved by tensor products? (An important recent contribution here is [3].)

- (c) Is Theorem 3.2 true for the strong diameter 2 property?
- (d) Does $\dim Z(X^{(\infty)}) = \infty$ imply strong diameter 2 property?
- (e) Note that X inherits all of the three diameter 2 properties from X^{**} . In general, which subspaces of a space with the (local, strong) diameter 2 property inherits this property?

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