

Santaló Region of a Log-Concave Function

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In this paper we define the *Santaló-region* and the *Floating body* of a log-concave function on $\mathbb{R}^n$. We then study their properties. Our main result is that any relation of *Floating body* and *Santaló region* of a convex body is translated to a relation of *Floating body* and *Santaló region* of an even log-concave function.

1. Introduction

In $\mathbb{R}^n$ we fix a scalar product $\langle \cdot, \cdot \rangle$. A convex body in $\mathbb{R}^n$ is a compact convex set which includes 0 in its interior. We denote by B_2^n the *Euclidean* unit ball of radius 1 in $\mathbb{R}^n$, namely $B_2^n = \{x \in \mathbb{R}^n : \sum_i x_i^2 \leq 1\}$. We let the boundary of the ball, namely the unit sphere, be denoted by S^{n-1} . We call a body centrally symmetric if $K = -K$.

Let K be a convex body in $\mathbb{R}^n$. For $x \in \text{int}(K)$, the interior of K , let K^x be the polar body of K with respect to x , i.e.

$$K^x = \{x + y : y \in \mathbb{R}^n, \langle y, z - x \rangle \leq 1 \ \forall z \in K\} = x + (K - x)^\circ$$

where $K^\circ := \{y \in \mathbb{R}^n : \langle y, z \rangle \leq 1 \ \forall z \in K\}$, is the standard polar of a body.

The *Blaschke-Santaló inequality* asserts that the quantity $\inf_x \text{Vol}(K) \text{Vol}(K^x)$ is maximal for balls and ellipsoids. Moreover, it is well known that there exists a unique $x_0 \in \text{int}(K)$ on which this infimum is attained. This unique x_0 is called the *Santaló point* of K .

The above gives rise to a natural geometric construction which was invented by E. Lutwak and consists of all the points which are “witnesses” of the validity of the *Blaschke-Santaló inequality*. This is called the *Santaló-region* of K (with constant 1) and is defined by $S(K, 1) = \{x \in K : \text{Vol}(K) \text{Vol}(K^x) \leq \text{Vol}(B_2^n)^2\}$. More generally, one defines the *Santaló-region* of K with constant t , denoted $S(K, t)$, by

$$S(K, t) = \{x \in K : \text{Vol}(K) \text{Vol}(K^x) \leq t \text{Vol}(B_2^n)^2\}. \quad (1)$$

It follows from the *Blaschke-Santaló inequality* that the *Santaló point* $x_0 \in S(K, 1)$, and that $S(K, 1) = \{x_0\}$ if and only if K is an ellipsoid.

We turn to the definition of the *Floating body*. For $0 < \lambda < \frac{1}{2}$ and $\theta \in S^{n-1}$ we define $H_{\lambda, \theta} := \{x \in \mathbb{R}^n : \langle x, \theta \rangle = a(\theta)\}$, where $a(\theta) \in \mathbb{R}$ is such that $\text{Vol}(H_{\lambda, \theta}^+ \cap$

$K) = \lambda \text{Vol}(K)$, and $H_{\lambda,\theta}^+ = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \geq a(\theta)\}$. Meaning, $H_{\lambda,\theta}$ is a hyperplane in the direction θ that ‘cuts’ $\lambda \text{Vol}(K)$ volume from K . In what follows we will use the notation $H_{\lambda,\theta}^-$, which means $H_{\lambda,\theta}^- = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq a(\theta)\}$.

Let K be a convex body in $\mathbb{R}^n$ and $0 < \lambda < \frac{1}{2}$. We define

$$F(K, \lambda) = \bigcap_{\theta \in S^{n-1}} H_{\lambda,\theta}^-. \quad (2)$$

This body is called the *Floating body* of K , and was first considered by Dupin in 1822. A vast literature exists on the topic of Floating bodies, see e.g. [6], [7], [8]. We list some known results about Floating bodies which exemplify their importance in Section 2. For us here it is important to note that $F(K, \lambda)$ is a closed convex set which is contained in K (possibly empty). In the centrally symmetric case it is known that the family supporting hyperplanes touching $F(K, \lambda)$ at its boundary point are exactly all of $H_{\lambda,\theta}$ above, so that $F(K, \lambda)$ is the envelope of these hyperplanes. We state this result precisely in Section 2 below. In fact, we state there a slightly more general theorem valid for measures, not just bodies.

From convex bodies we turn to log-concave functions. A function $f : \mathbb{R}^n \rightarrow [0, \infty)$ is called log-concave if

$$\lambda \log f(x) + (1 - \lambda) \log f(y) \leq \log f(\lambda x + (1 - \lambda)y)$$

for any $0 < \lambda < 1$ and $x, y \in \text{supp}(f)$, where $\text{supp}(f)$ denotes the support of the function f , $\text{supp}(f) = \{x \in \mathbb{R}^n : f(x) > 0\}$. We say that a function is even if $f(x) = f(-x)$ for all $x \in \mathbb{R}^n$. The idea of transferring results about convex bodies to results about log-concave functions is not new, and its roots can be found in [3]. We will elaborate on the importance of this procedure, and give examples for it, in Section 2 below. Let us just mention that although, so far, it seems that almost every geometric result on convex bodies has a counterpart which is valid for log-concave functions, and although several methods for obtaining such results have been found, there is so far no general procedure for doing this, and this is more of a meta-mathematical fact than an actual mathematical truth.

Motivated by [1], we define the dual of a log-concave function f by

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left[\frac{e^{-\langle x, y \rangle}}{f(y)} \right].$$

We explain the motivation for this definition and equivalent formulations in Section 2. It turns out that a *Santaló type inequality* holds also for functions. Again, one should choose correctly the center of the function. Thus, for a log-concave function f , we let f_a stand for the translation of f by the vector a , namely $f_a(x) = f(x - a)$.

When “volume” is exchanged with “integral” and “an ellipsoid” is exchanged with “a Gaussian function”, we have a *Santaló type inequality* for functions.

Theorem 1.1 ([1]). Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an integrable function such that $0 < \int f < \infty$. Then, for some vector x_0 , one has

$$\int f_{x_0} \int (f_{x_0})^\circ \leq (2\pi)^n.$$

If f is log-concave, one may choose $x_0 = \frac{\int xf(x)}{\int f(x)}$, the center mass of f . The minimum over x_0 of the left side product equals $(2\pi)^n$ if and only if f is proportional to Gaussian, namely $f = ce^{-(Ax, x)}$, where $0 < c$ and A is a positive-definite matrix.

As in the case of bodies, we call the point x_0 which minimizes $a \rightarrow \int (f_a)^\circ dx$ the *Santaló point* of f . The case of even functions f (in which the *Santaló point* is 0) was proven much earlier in [4]. It is natural to define for a log-concave function its *Santaló region* (with parameter t) by

$$S(f, t) = \left\{ a \in \mathbb{R}^n : \int f_a \int (f_a)^\circ \leq (2\pi)^n t \right\}. \quad (3)$$

It follows from Theorem 1.1 that the *Santaló point* $x_0 \in S(f, 1)$, and that $S(f, 1) = \{x_0\}$ if and only if f is proportional to Gaussian.

Similarly to the *Floating body* of a convex body, one may define the *Floating body* of a log concave function f . Let $0 < \lambda < \frac{1}{2}$, and consider $H_{\lambda, \theta} = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = a(\theta)\}$, where $a(\theta)$ satisfies $\int_{H_{\lambda, \theta}^+} f dx = \lambda \int f dx$. We define the *Floating body* of f

$$F(f, \lambda) = \bigcap_{\theta \in S^{n-1}} H_{\lambda, \theta}^-. \quad (4)$$

Notice that if f is an even function, we have that its *Floating body* is a nonempty set for any $0 < \lambda < \frac{1}{2}$, since $0 \in F(f, \lambda)$. Moreover, it was proved by Meyer-Reisner that for every even log-concave function $f : \mathbb{R}^n \rightarrow [0, \infty)$, the family of supporting hyperplanes touching $F(f, \lambda)$ at its boundary point are exactly all of $H_{\lambda, \theta}$ above, so that $F(f, \lambda)$ is the envelope of these hyperplanes (Section 2).

The main result of this paper is that any 'relation' of *Floating body* and *Santaló region* of a convex body is translated to a relation of *Floating body* and *Santaló region* of an even log-concave function, namely

Theorem 1.2. Let $0 < \lambda < \frac{1}{2}$ and $0 < d$ such that $F(K, \lambda) \subseteq S(K, d)$ for every centrally symmetric convex body K . Then

$$F(f, \lambda) \subseteq S(f, d) \text{ for every even log-concave function } f : \mathbb{R}^n \rightarrow [0, \infty).$$

In the case of convex bodies, the following inclusion holds

Theorem 1.3 ([8]). Let K be a convex body in $\mathbb{R}^n$. Then

$$F(K, \lambda) \subseteq S\left(K, \frac{1}{4\lambda(1-\lambda)}\right) \text{ for any } 0 < \lambda < 1/2.$$

From Theorem 1.2 and Theorem 1.3 we get

Corollary 1.4. *For every even log-concave function $f : \mathbb{R}^n \rightarrow [0, \infty)$ and any $0 < \lambda < \frac{1}{2}$ we have that*

$$F(f, \lambda) \subseteq S\left(f, \frac{1}{4\lambda(1-\lambda)}\right).$$

It is not known whether – in some form – the *Floating body* and the *Santaló region* of a convex body are isomorphic. However, we can show that any inclusion in the direction opposite to that of Theorem 1.3 will transfer immediately to the realm of functions. That is, the following “converse” of Theorem 1.2 holds.

Theorem 1.5. *Let $0 < \lambda < \frac{1}{2}$ and $0 < d$ such that $S(K, d) \subseteq F(K, \lambda)$ for every centrally symmetric convex body K . Then*

$$S(f, d) \subseteq F(f, \lambda) \text{ for every even log-concave function } f : \mathbb{R}^n \rightarrow [0, \infty).$$

The rest of the paper is organized as follows. In Section 2 we explore in depth the notions defined above. We quote the relevant known theorems, and discuss several other notions which we will need in the proofs of our main theorems, such as s -concave functions. In Section 3 we explore the properties of the *Santaló region* of a function. In Section 4 we prove Theorem 1.2 and Theorem 1.5.

2. Background

The following theorem, namely Theorem 2.1, is the main ingredient for several important things. First, the *Blaschke-Santaló inequality* is an immediate result of it. Second, it gives us a connection (Theorem 1.3) between the *Floating body* and the *Santaló region* which related to a convex body, which afterwards we translate (Theorem 1.2) to a connection between the *Floating body* and the *Santaló region* related to an even log-concave function.

Theorem 2.1 ([5]). *Let K be a convex body in $\mathbb{R}^n$ and let $H = \{x \in \mathbb{R}^n : \langle x, u \rangle = a\}$ be an affine hyperplane ($u \in \mathbb{R}^n, u \neq 0, a \in \mathbb{R}$), such that $\text{int}(K) \cap H \neq \emptyset$. Then there exists $z \in \text{int}(K) \cap H$ such that*

$$\text{Vol}(K) \text{Vol}(K^z) \leq \text{vol}(B_2^n)^2 / 4\lambda(1-\lambda),$$

where $\lambda \in [0, 1]$ is defined by $\lambda \text{Vol}(K) = \text{Vol}(\{x \in K : \langle x, u \rangle \geq a\})$.

The starting point of the results on the *Santaló-region* of a convex body is, as stated above, the *Blaschke-Santaló inequality*. This inequality states, roughly, that with the right choice of a center, the product of the volumes of a body and its polar is maximized by *Euclidean* balls (and also ellipsoids, since this is an affine invariant). More precisely

Theorem 2.2. *Let K be a convex body in $\mathbb{R}^n$ then*

$$\inf_x \text{Vol}(K) \text{Vol}(K^x) \leq \text{Vol}(B_2^n)^2$$

with equality if and only if K is an ellipsoid.

In the special case where K is centrally symmetric, the infimum is attained at 0, and in 1981 Saint-Raymond [11] gave a simple proof for this case, namely, for centrally symmetric K one has

$$\text{Vol}(K) \text{Vol}(K^\circ) \leq \text{Vol}(B_2^n)^2,$$

and equality is attained if and only if K is an ellipsoid. The case of equality for a general convex body was proven by Petty [9].

Remark 2.3 ([8]). For every K , a convex body in $\mathbb{R}^n$, there is a unique point x_0 for which $\text{Vol}(K) \text{Vol}(K^{x_0})$ is minimal and this unique x_0 is called the *Santaló point* of K .

We turn now to the *Floating body* of convex body. From definition, $F(K, \lambda)$ is the intersection of all half spaces whose hyperplanes cut off from K a set of volume $\lambda \text{Vol}(K)$. Note that when K is a centrally symmetric convex body, we have that its *Floating body* $F(K, \lambda) \neq \emptyset$, for every $0 < \lambda < \frac{1}{2}$, since $0 \in F(K, \lambda)$. Moreover, we have $F(K, \lambda) \rightarrow \{0\}$, in the sense of the *Hausdorff distance*, when $\lambda \rightarrow \frac{1}{2}$ from below. For a general convex body K we have $F(K, \lambda)$ converges to K when $\lambda \rightarrow 0$ from above.

For a centrally symmetric convex body K there is a theorem which was proved by M. Meyer and S. Reisner in [7]. This theorem shows the connection between the supporting hyperplanes of the *Floating body* $F(K, \lambda)$ and the hyperplanes $H_{\lambda, \theta}$, where $\theta \in S^{n-1}$. Before we state this theorem let us first define $H_{\lambda, K}$ to be the family of all hyperplanes $H_{\lambda, \theta}$, where $\theta \in S^{n-1}$.

Theorem 2.4. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$, and let $0 < \lambda < \frac{1}{2}$. Then there exists a centrally symmetric convex body K_λ with the following property: For every supporting hyperplane H of K_λ we have $\text{Vol}(K \cap H^+) = \lambda \text{Vol}(K)$.*

For our discussion, we conclude from Theorem 2.4 that the *Floating body* $F(K, \lambda)$, for $K = -K$, is enveloped by the family $H_{\lambda, K}$. That is, the family of supporting hyperplanes of $F(K, \lambda)$ coincide with the family $H_{\lambda, K}$, and $F(K, \lambda)$ lies on the negative side of every $H \in H_{\lambda, K}$. Theorem 2.4 is a result of a more general theorem related to measures, namely Theorem 2.5.

Before we state Theorem 2.5, let us begin with the following definition related to measures. Let μ be a non-negative *Borel measure* on $\mathbb{R}^n$, and let $0 < \lambda < \frac{1}{2}$. We define $H_{\lambda, \mu}$ to be the family of all hyperplanes H in $\mathbb{R}^n$ such that $\mu(H^+) = \lambda$, where as before H^+ is the half space bounded by H which does not include 0.

Theorem 2.5 ([6]). *Let μ be a finite, even, log-concave and non-degenerate measure on $\mathbb{R}^n$. Then there exists a convex body K_λ whose boundary is the envelope of the family $H_{\lambda, \mu}$. That is, the family of supporting hyperplanes of K_λ coincides with the family $H_{\lambda, \mu}$ and K_λ lies on the negative side of every $H \in H_{\lambda, \mu}$.*

Corollary 2.6. *Theorem 2.4 follows from Theorem 2.5 when we consider measures of the form $\mu_K(L) = \text{Vol}(K \cap L)$, where K is a centrally symmetric convex body in $\mathbb{R}^n$.*

In order to show the connection between convex bodies and log-concave functions, we need to define the right analogue for addition and multiplication by scalar of functions. We consider another known operation, which is related to the *Legendre transform*, namely the *Asplund product* ([10]). Given two functions $f, g : \mathbb{R}^n \rightarrow [0, \infty)$, their *Asplund product* is defined by

$$(f * g)(x) = \sup_{x_1 + x_2 = x} f(x_1)g(x_2).$$

It is easy to check that $1_K * 1_T = 1_{K+T}$ and $(f * g)^\circ = f^\circ g^\circ$, where 1_K is the indicator function. Moreover we argue that the *Asplund product* of log-concave functions is the right analogue for Minkowski addition of convex bodies.

To define the λ -homothet of a function $f(x)$, which we denote $(\lambda \cdot f)(x)$, we use

$$(\lambda \cdot f)(x) = f^\lambda\left(\frac{x}{\lambda}\right).$$

One should note the difference between f° , which represent the dual of f , and f^λ with $\lambda = 0$, which means f to the power of $\lambda = 0$. Note that, for a log-concave function, one has $f * f = 2 \cdot f$ and $(\lambda \cdot f)^\circ = (f^\circ)^\lambda$.

To check whether the definitions of Minkowski addition, homothety and duality for log-concave functions are meaningful and make sense, it is natural to ask whether the basic inequalities for convex bodies such as the *Brunn-Minkowski inequality* and the *Santaló inequality* remain true, where the role of volume will be played of course by the integral. We first discuss the *Brunn-Minkowski inequality*.

In its dimension-free form, the *Brunn-Minkowski inequality* states that

$$\text{Vol}(\lambda A + (1 - \lambda)B) \geq \text{Vol}(A)^\lambda \text{Vol}(B)^{1-\lambda},$$

where $A, B \subseteq \mathbb{R}^n$ measurable sets and $0 < \lambda < 1$.

We have a functional analogue of the *Brunn-Minkowski inequality*, namely the *Prékopa-Leindler inequality* (see, e.g. [1]). In the above notation, it states precisely that

Theorem 2.7 (Prékopa-Leindler). *Given $f, g : \mathbb{R}^n \rightarrow [0, \infty)$ and $0 < \lambda < 1$,*

$$\int (\lambda \cdot f) * ((1 - \lambda) \cdot g) \geq \left(\int f \right)^\lambda \left(\int g \right)^{1-\lambda}.$$

The standard *Brunn-Minkowski inequality* follows directly from *Prékopa-Leindler* by considering indicator functions of sets.

Now, we turn to definition of duality on the class of log-concave functions, its connection with the *Legendre transform*. As before we define the dual of a log-concave function $f : \mathbb{R}^n \rightarrow [0, \infty)$ by $f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left[\frac{e^{-\langle x, y \rangle}}{f(y)} \right]$. The above definition

of polarity of a log-concave function, is connected with the *Legendre transform* in the following way:

$$-\log(f^\circ) = \mathcal{L}(-\log f),$$

where the *Legendre transform* defined by

$$(\mathcal{L}\varphi)(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(y)).$$

So, one may conclude that

$$f^\circ = e^{-\mathcal{L}\varphi},$$

where $f = e^{-\varphi}$ and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function. For more properties and connection between the *Legendre transform* and the dual of a function see [1] and [2].

2.1. Properties of s -concave functions

In what follows, we will use very strongly the notion of an s -concave function, and the fact that such functions can be used to approximate, quite effectively, log-concave functions. This technique was used extensively in [1] to prove the *Blaschke-Santaló inequality* for functions.

We start with the definition of s -concave function. For $s > 0$, a function $g : \mathbb{R}^n \rightarrow [0, \infty)$ is called an s -concave function if $g^{\frac{1}{s}}$ is concave on the support of g . In other words,

$$\lambda g^{\frac{1}{s}}(x) + (1 - \lambda)g^{\frac{1}{s}}(y) \leq g^{\frac{1}{s}}(\lambda x + (1 - \lambda)y)$$

for any $x, y \in \text{supp}(g)$ and any $0 \leq \lambda \leq 1$.

A function on $\mathbb{R}^n$ is s -concave if and only if it is a marginal of a uniform measure on a convex body in $\mathbb{R}^{n+s}$. Indeed, the Brunn concavity principle implies that such a marginal is s -concave, whereas for the other direction, given an s -concave measure one can easily construct a body K in $\mathbb{R}^{n+s}$ which has it as its marginal (there can be many such bodies).

It is easy to see that any s -concave function, for $0 < s$ integer, is also log-concave. The converse is of course not true – a gaussian is log-concave but not s -concave for any $s > 0$. However, the family of all functions which are s -concave for some $s > 0$ is dense in the family of log-concave functions in any reasonable sense.

More precisely, let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. It is possible to create a series, $\{f_s\}_{s=1}^\infty$ of s -concave functions, which converges to f , as s tends to infinity, locally uniformly as follows:

$$f_s(x) := \left(1 + \frac{\log f(x)}{s}\right)_+^s$$

where $z_+ = \max\{z, 0\}$. The log-concavity of f implies the s -concavity of f_s . Note also that $f_s \leq f$ for any $0 < s$, and since a log-concave function is continuous on its support, one has $f_s \rightarrow f$ locally uniformly on $\mathbb{R}^n$ as $s \rightarrow \infty$.

Let $g : \mathbb{R}^n \rightarrow [0, \infty)$ be an s -concave function. Following [1], we define for $s > 0$

$$\mathcal{L}_s g(x) = \inf_{\{y: g(y) > 0\}} \frac{(1 - \frac{\langle x, y \rangle}{s})_+^s}{g(y)}.$$

Moreover we define

$$K_s(g) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s : \sqrt{s}x \in \overline{\text{supp}(g)}, |y| \leq g^{\frac{1}{s}}(\sqrt{s}x)\}.$$

Note that $K_s(g)$ is a centrally symmetric convex body in $\mathbb{R}^{n+s}$ for any even s -concave function $g : \mathbb{R}^n \rightarrow [0, \infty)$ such that $\text{supp}(g) \subseteq \mathbb{R}^n$ is a centrally symmetric convex body.

The following lemma clarifies the relations between our various definitions.

Lemma 2.8 ([1]). *For any $f : \mathbb{R}^n \rightarrow [0, \infty)$,*

$$K_s(f)^\circ = K_s(\mathcal{L}_s(f)).$$

As appears in [1], it is easy to check that

- $\text{Vol}(K_s(g)) = \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int g dx,$
- $K_s(g_a) = K_s(g) + \frac{1}{\sqrt{s}}(a, 0).$

For s -concave function $g : \mathbb{R}^n \rightarrow [0, \infty)$, we call $\mathcal{L}_s g$ the polar of g . From the definition of $\mathcal{L}_s$ it is easy to see that $\mathcal{L}_s g \leq g^\circ$ (see [2]).

As may seem natural by now, we have a *Santaló-type inequality* also for s -concave functions, where we use $\mathcal{L}_s$ for the duality operation. This inequality states that

Theorem 2.9 ([1]). *Let g be an s -concave function on $\mathbb{R}^n$, with $0 < \int g < \infty$, whose center of mass is at the origin (i.e., $\int xg(x) = 0$). Then*

$$\int g \int \mathcal{L}_s(g) \leq \frac{s^n \text{Vol}(B_2^{n+s})^2}{\text{Vol}(B_2^s)^2},$$

with equality if and only if g is a marginal of a uniform distribution of an $(n+s)$ -dimensional ellipsoid. And B_2^{n+s} , B_2^s are the Euclidean balls of radius 1 in $\mathbb{R}^{n+s}$ and $\mathbb{R}^s$ respectively.

3. The Santaló Region of a log-concave function

Let $\varphi : \mathbb{R}^n \rightarrow (-\infty, \infty]$ be a convex function. We discuss some properties of the set

$$S(e^{-\varphi}, t) = \left\{ a \in \mathbb{R}^n : \int e^{-\varphi_a} \int e^{-\mathcal{L}\varphi_a} \leq (2\pi)^n t \right\}.$$

According to [8], we call this set the *Santaló region* of the function $e^{-\varphi}$, with constant t . It follows from the functional version of the *Santaló inequality*, that the set $S(e^{-\varphi}, 1)$ is non-empty for every convex φ , see [1].

Note that the *Legendre transform* of the translate φ_a is simply

$$\begin{aligned} (\mathcal{L}\varphi_a)(x) &= \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - \varphi_a(y)) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(y - a)) \\ &= \sup_{z \in \mathbb{R}^n} (\langle x, z + a \rangle - \varphi(z)) = \langle x, a \rangle + \sup_{z \in \mathbb{R}^n} (\langle x, z \rangle - \varphi(z)) \\ &= \langle x, a \rangle + (\mathcal{L}\varphi)(x). \end{aligned}$$

Therefore, using that $\int e^{-\varphi} = \int e^{-\varphi_a}$ for every a , we may write

$$S(e^{-\varphi}, t) = \left\{ a \in \mathbb{R}^n : \int e^{-\langle a, x \rangle} e^{-(\mathcal{L}\varphi)(x)} dx \leq \frac{(2\pi)^n t}{\int e^{-\varphi} dx} \right\}.$$

For every $0 < \lambda < 1$, $a, b \in \mathbb{R}^n$

$$\begin{aligned} (\mathcal{L}\varphi_{\lambda a + (1-\lambda)b})(x) &= \langle x, \lambda a + (1-\lambda)b \rangle + (\mathcal{L}\varphi)(x) \\ &= \lambda(\langle x, a \rangle + (\mathcal{L}\varphi)(x)) + (1-\lambda)(\langle x, b \rangle + (\mathcal{L}\varphi)(x)) \\ &= \lambda(\mathcal{L}\varphi_a(x)) + (1-\lambda)(\mathcal{L}\varphi_b(x)). \end{aligned}$$

If $a \neq b$ in $\mathbb{R}^n$ and $\varphi \not\equiv 0$ (meaning there exists $x \in \mathbb{R}^n$ such that $\varphi(x) \neq 0$) then there exists $x_0 \in \mathbb{R}^n$ such that $\mathcal{L}\varphi_a(x_0) \neq \mathcal{L}\varphi_b(x_0)$. Indeed, we have

$$\begin{aligned} \mathcal{L}\varphi_a(x_0) &= \langle x_0, a \rangle + (\mathcal{L}\varphi)(x_0), \\ \mathcal{L}\varphi_b(x_0) &= \langle x_0, b \rangle + (\mathcal{L}\varphi)(x_0). \end{aligned}$$

So one may choose $x_0 \in \mathbb{R}^n$ such that $\langle x_0, a \rangle \neq \langle x_0, b \rangle$ and $\mathcal{L}\varphi(x_0) \neq \infty$.

Define $F : \mathbb{R}^n \rightarrow \mathbb{R}^+$ by $F(a) = \int e^{-\mathcal{L}\varphi_a(x)} dx$. Note that F is a strictly convex function. Indeed

$$\begin{aligned} \int e^{-\mathcal{L}\varphi_{\lambda a + (1-\lambda)b}(x)} dx &= \int (e^{-\mathcal{L}\varphi_a(x)})^\lambda (e^{-\mathcal{L}\varphi_b(x)})^{1-\lambda} dx, \\ &< \lambda \int e^{-\mathcal{L}\varphi_a(x)} dx + (1-\lambda) \int e^{-\mathcal{L}\varphi_b(x)} dx, \end{aligned}$$

where the last inequality follows from the *arithmetic-geometric mean inequality* and the existence of $x_0 \in \mathbb{R}^n$ such that $\mathcal{L}\varphi_a(x_0) \neq \mathcal{L}\varphi_b(x_0)$, where $a \neq b$ and $\varphi \not\equiv 0$.

Consider $\varphi \not\equiv 0$. We call a point $a \in \mathbb{R}^n$ a *Santaló point* of $e^{-\varphi}$ if $F(a) \leq F(b)$ for every b . Since F is a strictly convex function, the *Santaló point* is unique. It is easy to see that F is a continuous function on $\{x \in \mathbb{R}^n : F(x) < \infty\}$ (Lemma 3.2 in [1]) and differentiable (since the differentiation under the integral sign is allowed, as the derivative is locally bounded and integrals converge), and that $\nabla F(a) = \int x e^{-\mathcal{L}\varphi_a(x)} dx$, which is equal to 0 if and only if a is the *Santaló point* of $e^{-\varphi}$. Moreover, one may conclude that $\nabla F(b) \neq 0$ for every $b \in \partial(S(e^{-\varphi}, t))$ whenever $\text{int}(S(e^{-\varphi}, t)) \neq \emptyset$. From strict convexity of F we have $0 \in \text{int}(S(f, t))$ whenever $\text{int}(S(f, t)) \neq \emptyset$.

Unless stated otherwise we will always assume that $e^{-\varphi}$ has its *Santaló-point* at the origin and that $\int e^{-\varphi} dx < \infty$.

Lemma 3.1. *For every convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and any $t > 0$, the Santaló region $S(e^{-\varphi}, t)$ is either a strictly convex body, or the empty set.*

Proof. Consider $t > 0$. Let $a \neq b$ and for them to be two points at the boundary of $S(e^{-\varphi}, t)$. We have to show that for any $0 < \lambda < 1$, the point $\lambda a + (1 - \lambda)b$ is in the interior of $S(e^{-\varphi}, t)$. That is, we know that $F(a) = F(b) = \frac{(2\pi)^n t}{\int e^{-\varphi}}$ and need to show that $F(\lambda a + (1 - \lambda)b) < \frac{(2\pi)^n t}{\int e^{-\varphi}}$, which follows from strict convexity of F . $\square$

Fact 3.2. *The mapping $t \rightarrow S(e^{-\varphi}, t)$ is increasing and it is also concave, i.e. for every $0 < \lambda < 1$ we have $\lambda S(e^{-\varphi}, t) + (1 - \lambda)S(e^{-\varphi}, s) \subseteq S(e^{-\varphi}, \lambda t + (1 - \lambda)s)$.*

Proof. The fact that the map is increasing follows immediately from definition. To prove the concavity consider $a \in S(e^{-\varphi}, t)$ and $b \in S(e^{-\varphi}, s)$. We have

$$F(a) \leq \frac{(2\pi)^n t}{\int e^{-\varphi}} dx, \quad F(b) \leq \frac{(2\pi)^n s}{\int e^{-\varphi}} dx.$$

Thus, from strict convexity of F we conclude

$$F(\lambda a + (1 - \lambda)b) < \lambda F(a) + (1 - \lambda)F(b) \leq (\lambda t + (1 - \lambda)s) \frac{(2\pi)^n}{\int e^{-\varphi}}. \quad \square$$

Let us check how linear transformations affect the *Santaló region*. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a convex function and let A be an invertible linear map, $A \in GL_n$.

From definition

$$\begin{aligned} A(S(e^{-\varphi}, \lambda)) &= \left\{ a \in \mathbb{R}^n : \int e^{-\varphi(x)} dx \int e^{-\mathcal{L}\varphi_{A^{-1}a}(x)} dx \leq (2\pi)^n \lambda \right\}, \\ S(e^{-\varphi \circ A^{-1}}, \lambda) &= \left\{ a \in \mathbb{R}^n : \int e^{-\varphi \circ A^{-1}(x)} dx \int e^{-\mathcal{L}((\varphi \circ A^{-1})_a)(x)} dx \leq (2\pi)^n \lambda \right\}. \end{aligned}$$

Note that these two regions coincide.

Fact 3.3. *For every convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the boundary of $S(e^{-\varphi}, t)$, denoted by $\partial(S(e^{-\varphi}, t))$, is infinitely smooth, whenever $\text{int}(S(e^{-\varphi}, t)) \neq \emptyset$.*

Proof. Assume that $\text{int}(S(e^{-\varphi}, t)) \neq \emptyset$, and define as before $F(a) = \int e^{-\mathcal{L}\varphi_a(x)} dx$. The boundary of $S(e^{-\varphi}, t)$ is given by

$$\left\{ a \in \mathbb{R}^n : F(a) = \frac{(2\pi)^n t}{\int e^{-\varphi} dx} \right\} = \partial(S(e^{-\varphi}, t)).$$

Differentiation under the integral sign gives

$$\frac{\partial F(a)}{\partial a_j} = (-1) \int x_j e^{\langle x, a \rangle} e^{-\mathcal{L}\varphi(x)} dx \quad \text{for every } 1 \leq j \leq n.$$

Note that the differentiation under the integral sign is allowed, as the derivative is locally bounded and the integrals converge. Thus as we said before we have $\nabla F(b) \neq 0$ for every $b \in \partial(S(e^{-\varphi}, t))$. Therefore Fact 3.3 follows from the *implicit function theorem*. $\square$

4. The relation between the Santaló region and the Floating body of an even log-concave function

Throughout this section we consider only even log-concave functions. We will compare two different bodies, the *Santaló region* and the *Floating body*, related to a specific even log-concave function. For any positive integer s we consider $\mathbb{R}^n \subset \mathbb{R}^{n+s}$ as the first n coordinates. Moreover, we denote $P_{\mathbb{R}^n}$ as the projection onto $\mathbb{R}^n$.

We start with some basic notations. Let $f = e^{-\varphi}$ be an even log-concave function and let $t > 0$. We define

$$A_t = \{x \in \mathbb{R}^n : \varphi(x) \leq t\}.$$

It is easy to check that A_t is a centrally symmetric body which is convex in $\mathbb{R}^n$, also since we require $\int f < \infty$, we get that A_t is a bounded set. We define

$$\begin{aligned} \varphi_{[t]}(x) &:= \begin{cases} \varphi(x) & \text{if } x \in A_t \\ \infty & \text{if } x \notin A_t, \end{cases} \\ f_{[t]}(x) &:= e^{-\varphi_{[t]}(x)}. \end{aligned}$$

Obviously $f_{[t]}$ is an even log-concave function such that $\text{supp}(f_{[t]}) = A_t$. In addition, we have $f_{[t]}(x) \geq e^{-t} > 0$ for any $x \in \text{supp}(f_{[t]})$.

As before we define $f_{[t],s}(x) = (1 + \frac{\log f_{[t]}(x)}{s})_+^s$. Note that

- $\text{supp}(f_{[t],s}) = \text{supp}(f_{[t]})$,
- $(f_{[t]})_s(x) > \frac{e^{-t}}{2} > 0$ for every $x \in \text{supp}(f_{[t],s})$ for s big enough.

In addition, for $a \in \mathbb{R}^n$ we define

$$f_{[t],s,a}(x) := f_{[t],s}(x - a).$$

It is obvious that for any even log-concave function f , $f_{[t]}$ is an even log-concave function for any positive t , with compact support, which converges locally uniformly to f as $t \rightarrow \infty$. Our goal is to prove Theorem 1.2 and Theorem 1.5 for any function of the form $f_{[t]}$, $t > 0$. Then from continuity (with respect to *Hausdorff distance* of bodies) of the *Floating body* and the *Santaló region* with respect to locally uniform convergence of even log-concave functions (Lemma 4.1 and Lemma 4.2) we will complete the proof of Theorem 1.2 and Theorem 1.5 for any even log-concave function.

The following lemma will be the main ingredient in the proof of Theorem 1.2 and Theorem 1.5. As mentioned above, Lemma 4.1 deals with continuity of the

Floating body with respect to locally uniformly convergence of even log-concave functions.

Lemma 4.1. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function. Then*

$$F(f_{[t]}, \lambda) \xrightarrow{t \rightarrow \infty} F(f, \lambda) \text{ for any } 0 < \lambda < 1/2.$$

Proof. Let $\theta \in S^{n-1}$ and let $H_f, H_{f_{[t]}}$ be two hyperplanes perpendicular to θ and such that $\int_{H_f^+} f dx = \lambda \int f dx$, and $\int_{H_{f_{[t]}}^+} f_{[t]} dx = \lambda \int f_{[t]} dx$.

From definition of $f_{[t]}$, it is easy to see that $\lambda \int f_{[t]} dx \xrightarrow{t \rightarrow \infty} \lambda \int f dx$. Since H_f and $H_{f_{[t]}}$ are parallel, we conclude $H_{f_{[t]}} \xrightarrow{t \rightarrow \infty} H_f$ with respect to the *Hausdorff distance* of bodies, which implies $F(f_{[t]}, \lambda) \xrightarrow{t \rightarrow \infty} F(f, \lambda)$. $\square$

Lemma 4.2 deals with the continuity of the *Santaló region* with respect to locally uniformly convergence of even log-concave functions.

Lemma 4.2. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function. Then*

$$S(f_{[t]}, d) \xrightarrow{t \rightarrow \infty} S(f, d) \text{ for any } d > 0.$$

Proof. Consider, as before, $f = e^{-\varphi}$, where φ is an even convex function, and $f_{[t]} = e^{-\varphi_{[t]}}$. From definition we know

$$S(f_{[t]}, d) = \left\{ a \in \mathbb{R}^n : \int f_{[t]} \int ((f_{[t]})_a)^\circ \leq (2\pi)^n d \right\},$$

$$S(f, d) = \left\{ a \in \mathbb{R}^n : \int f \int (f_a)^\circ \leq (2\pi)^n d \right\}.$$

Moreover, we define

- $F_t(a) := \int ((f_{[t]})_a)^\circ dx = \int e^{-\langle x, a \rangle} e^{-\mathcal{L}(\varphi_{[t]})(x)} dx,$
- $F(a) := \int (f_a)^\circ dx = \int e^{-\langle x, a \rangle} e^{-\mathcal{L}(\varphi)(x)} dx.$

Our general idea is as follows. First we prove $F_t \xrightarrow{t \rightarrow \infty} F$ locally uniformly on $\{x \in \mathbb{R}^n : F(x) < \infty\}$. Then, since $S(f, d) \subseteq \{x \in \mathbb{R}^n : F(x) < \infty\}$ is a closed bounded set, from strict convexity and continuity of F in $S(f, d)$ (Lemma 3.2 [1]) we complete the proof.

In order to show locally uniformly convergence, let us first prove $F_t(a) \xrightarrow{t \rightarrow \infty} F(a)$ for any $a \in \{x \in \mathbb{R}^n : F(x) < \infty\}$. It is easy to check that $\mathcal{L}(\varphi_{[t]})(x) \xrightarrow{t \rightarrow \infty} \mathcal{L}(\varphi)(x)$ for any $x \in \mathbb{R}^n$, hence

$$e^{-\langle x, a \rangle} e^{-\mathcal{L}(\varphi_{[t]})(x)} \xrightarrow{t \rightarrow \infty} e^{-\langle x, a \rangle} e^{-\mathcal{L}(\varphi)(x)} \text{ for every } x \in \mathbb{R}^n.$$

Since $e^{-\langle x, a \rangle} e^{-\mathcal{L}(\varphi_{[t]})(x)}$ and $e^{-\langle x, a \rangle} e^{-\mathcal{L}(\varphi)(x)}$ are both log-concave functions (Lemma 3.2 [1]), we conclude $F_t(a) \xrightarrow{t \rightarrow \infty} F(a)$.

Moreover, for any $a \in \{x \in \mathbb{R}^n : F(x) < \infty\}$ it is easy to see that $F_{t_2}(a) \leq F_{t_1}(a)$ whenever $0 < t_1 < t_2$. Indeed, since $\varphi_{[t_2]}(x) \leq \varphi_{[t_1]}(x)$ for any $x \in \mathbb{R}^n$ one has $\mathcal{L}(\varphi_{[t_1]})(x) \leq \mathcal{L}(\varphi_{[t_2]})(x)$ for any $x \in \mathbb{R}^n$.

Since $F_t \xrightarrow{t \rightarrow \infty} F$ pointwise in $\{x \in \mathbb{R}^n : F(x) < \infty\}$, and $(F_t)_{t>0}$ is a decreasing sequence of functions, we conclude from continuity of F in $\{x \in \mathbb{R}^n : F(x) < \infty\}$ that $F_t \xrightarrow{t \rightarrow \infty} F$ locally uniformly in $\{x \in \mathbb{R}^n : F(x) < \infty\}$. In addition, since F is a strictly convex function in $S(f, d)$, then $S(f, d) \subseteq \{x \in \mathbb{R}^n : F(x) < \infty\}$ is a bounded convex set in $\mathbb{R}^n$. $\square$

Let us prove the following lemma.

Lemma 4.3. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function such that $\text{supp}(f)$ is a convex compact body in $\mathbb{R}^n$, and there exist $0 < t_0$ such that $t_0 < f(x)$ for all $x \in \text{supp}(f)$. Define as before $f_s(x) = (1 + \frac{\log f(x)}{s})_+^s$ and let $0 < d$. Then*

$$\left\{ a \in \mathbb{R}^n : \int f_s \int \mathcal{L}_s((f_s)_a) \leq (2\pi)^n d \right\} \xrightarrow{s \rightarrow \infty} S(f, d),$$

with respect to the Hausdorff distance of convex bodies.

Corollary 4.4. *From Lemma 4.3 we conclude that*

$$\left\{ a \in \mathbb{R}^n : \int f_{[t],s} \int \mathcal{L}_s(f_{[t],s,a}) \leq (2\pi)^n d \right\} \xrightarrow{s \rightarrow \infty} S(f_{[t]}, d)$$

for any even log-concave function $f : \mathbb{R}^n \rightarrow [0, \infty)$ and any $0 < t$.

Proof of Lemma 4.3. For s big enough, $\text{supp}(f_s) = \text{supp}(f)$. We define

- $F(a) := \int (f_a)^\circ(x) dx = \int \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x, y \rangle}}{f(y-a)} dx,$
- $F_s(a) := \int \mathcal{L}_s((f_s)_a) dx = \int \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x, y \rangle}{s})_+^s}{f_s(y-a)} dx.$

First, we prove $F_s \xrightarrow{s \rightarrow \infty} F$ locally uniformly in $\{x \in \mathbb{R}^n : F(x) < \infty\}$. Then, since $S(f, d) \subseteq \{x \in \mathbb{R}^n : F(x) < \infty\}$ is a bounded convex set in $\mathbb{R}^n$, from the fact that F is a continuous and strictly convex function in $S(f, d)$, we complete the proof.

Let $D \subseteq \{x \in \mathbb{R}^n : F(x) < \infty\}$ be a closed, bounded set and let $\epsilon > 0$. We will prove that there exists $s_\epsilon > 0$ such that for every $s \geq s_\epsilon$

$$|F_s(a) - F(a)| < \epsilon \quad \text{for any } a \in D.$$

Since F is a continuous function in $\{x \in \mathbb{R}^n : F(x) < \infty\}$, there exists $r_{\epsilon, D} > 0$ such that

$$\left| \int_{\mathbb{R}^n - r_{\epsilon, D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x, y \rangle}}{f(y-a)} dx \right| < \frac{\epsilon}{8}, \quad \text{for any } a \in D. \quad (5)$$

For this moment let's assume there exists $s_\epsilon > 0$ such that for every $s \geq s_\epsilon$

$$\left| \int_{r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} dx - \int_{r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| < \frac{\epsilon}{4} \quad (6)$$

and

$$\left| \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| < \frac{\epsilon}{4} \quad \text{for any } a \in D. \quad (7)$$

From (5), (6), (7) and the *triangle-inequality* we get

$$\begin{aligned} & \left| \int \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} dx - \int \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| \\ & \leq \left| \int_{r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} dx - \int_{r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| \\ & \quad + \left| \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} dx \right| \\ & \quad + \left| \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| < \epsilon, \end{aligned}$$

which shows $F_s \xrightarrow{s \rightarrow \infty} F$ locally uniformly in $\{x \in \mathbb{R}^n : F(x) < \infty\}$.

Let us show (6). Let $x \in r_{\epsilon,D} B_2^n$, $a \in D$ and $y \in \text{supp}(f) + a \subseteq \text{supp}(f) + D$. Since $t_0 < f$ we get

$$\left| \frac{e^{-\langle x,y \rangle}}{f(y-a)} - \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} \right| \leq \frac{1}{t_0} \left| e^{-\langle x,y \rangle} - \frac{f(y-a)}{f_s(y-a)} \left(1 - \frac{\langle x,y \rangle}{s} \right)_+^s \right|.$$

Now, since $r_{\epsilon,D} B_2^n$ and $\text{supp}(f) + D$ are bounded sets, the value of $\langle x, y \rangle$ is bounded whenever $x \in r_{\epsilon,D} B_2^n$ and $y \in \text{supp}(f) + D$. Since $f_s \xrightarrow{s \rightarrow \infty} f$ locally uniformly in $\text{supp}(f)$, and since $\frac{t_0}{2} < f_s$ for s is big enough, we conclude that for any $\epsilon_0 > 0$ there exists $s_{\epsilon_0} > 0$ such that for any $s \geq s_{\epsilon_0}$

$$\frac{1}{t_0} \left| e^{-\langle x,y \rangle} - \frac{f(y-a)}{f_s(y-a)} \left(1 - \frac{\langle x,y \rangle}{s} \right)_+^s \right| < \epsilon_0$$

for any $x \in r_{\epsilon,D} B_2^n$, $a \in D$ and $y \in \text{supp}(f) + a$.

Thus

$$\left| \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} - \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} \right| \leq \epsilon_0.$$

Choose $0 < \epsilon_0 < \frac{\epsilon}{4r_{\epsilon,D}^n \text{Vol}(B_2^n)}$. Then

$$\begin{aligned} & \left| \int_{r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} dx - \int_{r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| \\ & \leq \int_{r_{\epsilon,D} B_2^n} \left| \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f(y-a)} - \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} \right| dx \\ & \leq r_{\epsilon,D}^n \text{Vol}(B_2^n) \epsilon_0 < \frac{\epsilon}{4}. \end{aligned}$$

We continue with (7). We know $(1 - \frac{\langle x,y \rangle}{s})_+^s \leq e^{-\langle x,y \rangle}$, which implies

$$\begin{aligned} & \left| \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{(1 - \frac{\langle x,y \rangle}{s})_+^s}{f_s(y-a)} dx \right| \\ & \leq \left| \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f_s(y-a)} dx \right|. \end{aligned}$$

Since $f = e^{-\varphi}$ and $f_s = e^{-\varphi_s}$ where $\varphi_s, \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ are convex functions, and since $f_s \leq f$ and $f_s \rightarrow f$ locally uniformly, we conclude that $\varphi \leq \varphi_s$ and that $\varphi_s \rightarrow \varphi$ locally uniformly in $\text{supp}(f)$. Since $\text{supp}(f)$ is a bounded set in $\mathbb{R}^n$, for $\epsilon_1 > 0$ there exists $s_{\epsilon_1} > 0$ such that for any $s \geq s_{\epsilon_1}$

$$\varphi_s(x) \leq \varphi(x) + \epsilon_1 \quad \text{for every } x \in \text{supp}(f).$$

Hence

$$-\mathcal{L}(\varphi_s)(x) \leq -\mathcal{L}(\varphi + \epsilon_1)(x) = \epsilon_1 - \mathcal{L}(\varphi)(x) \quad \text{for every } x \in \text{supp}(f).$$

From (5)

$$\begin{aligned} & \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} \inf_{y \in \text{supp}(f)+a} \frac{e^{-\langle x,y \rangle}}{f_s(y-a)} dx \\ & = \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} e^{-\langle x,a \rangle} e^{-\mathcal{L}(\varphi_s)(x)} dx \\ & \leq e^{\epsilon_1} \int_{\mathbb{R}^n - r_{\epsilon,D} B_2^n} e^{-\langle x,a \rangle} e^{-\mathcal{L}(\varphi)(x)} dx < e^{\epsilon_1} \frac{\epsilon}{8}. \end{aligned}$$

We choose $0 < \epsilon_1 < \ln(2)$ and $s_\epsilon = \max\{s_{\epsilon_0}, s_{\epsilon_1}\}$. □

As mentioned before, any even s -concave function $f_s : \mathbb{R}^n \rightarrow [0, \infty)$ such that $\text{supp}(f_s)$ is convex, bounded and symmetric with respect to the origin, one has $K_s(f_s) \subseteq \mathbb{R}^{n+s}$ is a centrally symmetric convex body. Lemma 4.5 shows the connection between the projection of $F(K_s(f_s), \lambda)$ on $\mathbb{R}^n$ ($\mathbb{R}^n \subseteq \mathbb{R}^{n+s}$ the first n coordinates) and the projection of the intersection of all half spaces which contain $F(K_s(f_s), \lambda)$ and also determined by a hyperplane in $\mathbb{R}^{n+s}$ that is perpendicular to $\mathbb{R}^n$ and supports $F(K_s(f_s), \lambda)$.

Lemma 4.5. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function. Define as before $f_s(x) = (1 + \frac{\log f(x)}{s})_+^s$. Then*

$$P_{\mathbb{R}^n}[F(K_s(f_s), \lambda)] = P_{\mathbb{R}^n} \left[\bigcap_{\theta=(\theta_1, \dots, \theta_n, 0, \dots, 0) \in S^{n+s-1}} H_{\lambda, \theta}^- \right],$$

where $\lambda \text{Vol}(K_s(f_s)) = \text{Vol}(K_s(f_s) \cap H_{\lambda, \theta}^+)$ for any $\theta = (\theta_1, \dots, \theta_n, 0, \dots, 0) \in S^{n+s-1}$.

Proof. Let H be an affine hyperplane in $\mathbb{R}^{n+s}$ ($1 \leq \dim(H) \leq n + s - 1$) and let K be a centrally symmetric convex body in $\mathbb{R}^{n+s}$. We denote by $P_H(K)$ as the projection of K on H . It is easy to check

$$P_H(K) = P_H \left(\bigcap_{\{\theta \in S^{n+s-1} \cap H\}} H_{\theta}^- \right), \quad (8)$$

where H_{θ} is a supporting hyperplane to K and perpendicular to θ such that $K \subseteq H_{\theta}^-$.

From Theorem 2.5, $F(K_s(f_s), \lambda)$ is enveloped by all hyperplanes $H_{\lambda, \theta}$ such that

$$\text{Vol}_{n+s}(K_s(f_s) \cap H_{\lambda, \theta}^+) = \lambda \text{Vol}_{n+s}(K_s(f_s)).$$

Meaning, any hyperplane $\tilde{H}$, such that $F(K_s(f_s), \lambda) \subseteq \tilde{H}^-$, is a supporting hyperplane to $F(K_s(f_s), \lambda)$ if and only if $\tilde{H}$ satisfies $\text{Vol}(K_s(f_s) \cap \tilde{H}^+) = \lambda \text{Vol}(K_s(f_s))$. We take $K = F(K_s(f_s), \lambda)$, $H = \mathbb{R}^n$ ($\mathbb{R}^n \subset \mathbb{R}^{n+s}$ the first n coordinates). $\square$

The next lemma shows the connection between the *Floating body* which is related to a convex body and the *Floating body* which is related to an s -concave function.

Lemma 4.6. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function and let $0 < \lambda < \frac{1}{2}$. Define as before $f_s(x) = (1 + \frac{\log f(x)}{s})_+^s$. Then*

$$F(f_s, \lambda) = \sqrt{s} P_{\mathbb{R}^n}[F(K_s(f_s), \lambda)].$$

Proof. Let $0 < \lambda < \frac{1}{2}$ and $\theta = (\theta_1, \dots, \theta_n, 0, \dots, 0) \in S^{n+s-1}$. We will compute $\text{Vol}(K_s(f_s) \cap H_{\lambda, \theta}^+)$ in two different ways. On the one hand

$$\text{Vol}(K_s(f_s) \cap H_{\lambda, \theta}^+) = \lambda \text{Vol}(K_s(f_s)) = \lambda \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int f_s dx.$$

On the other hand, simple change of variables $x \rightarrow \frac{1}{\sqrt{s}}x$ yields

$$\begin{aligned} \text{Vol}(K_s(f_s) \cap H_{\lambda, \theta}^+) &= \int_{P_{\mathbb{R}^n}(H_{\lambda, \theta})^+} \text{Vol}(f_s^{\frac{1}{s}}(\sqrt{s}x) B_2^s) dx \\ &= \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int_{\sqrt{s} P_{\mathbb{R}^n}(H_{\lambda, \theta})^+} f_s(x) dx. \end{aligned}$$

We conclude

$$\lambda \int f_s dx = \int_{\sqrt{s}P_{\mathbb{R}^n}(H_{\lambda,\theta})^+} f_s(x) dx, \quad \forall \theta = (\theta_1, \dots, \theta_n, 0, \dots, 0) \in S^{n+s-1}.$$

From definition of *Floating body* and Lemma 4.5 we get

$$\begin{aligned} F(f_s, \lambda) &= \bigcap_{\theta=(\theta_1, \dots, \theta_n, 0, \dots, 0)} \sqrt{s}P_{\mathbb{R}^n}(H_{\lambda,\theta})^- \\ &= \sqrt{s} \bigcap_{\theta=(\theta_1, \dots, \theta_n, 0, \dots, 0)} P_{\mathbb{R}^n}(H_{\lambda,\theta}^-) \\ &= \sqrt{s}P_{\mathbb{R}^n} \left[\bigcap_{\theta=(\theta_1, \dots, \theta_n, 0, \dots, 0)} H_{\lambda,\theta}^- \right] \\ &= \sqrt{s}P_{\mathbb{R}^n}[F(K_s(f_s), \lambda)]. \end{aligned} \quad \square$$

Let $K \subseteq \mathbb{R}^{n+s}$ be a centrally symmetric convex body, which is symmetric in respect to $\mathbb{R}^n$ ($\mathbb{R}^n \subseteq \mathbb{R}^{n+s}$ the first n coordinates). For any $a \in \mathbb{R}^n, b \in \mathbb{R}^s$ we define

$$\psi(a, b) := \text{Vol}((K - (a, b))^\circ).$$

The next two lemmas deal with two properties of ψ . The first property is the inequality $\psi(a, 0) \leq \psi(a, b)$, while the other is $\psi(-(a, b)) = \psi(a, b)$, where $a \in \mathbb{R}^n, b \in \mathbb{R}^s$.

Lemma 4.7. *Let $K \subseteq \mathbb{R}^{n+s}$ be a centrally symmetric convex body, such that K is symmetric with respect to $\mathbb{R}^n$. Then, for any $(a, b) \in \mathbb{R}^{n+s}$*

$$\text{Vol}((K - (a, 0))^\circ) \leq \text{Vol}((K - (a, b))^\circ).$$

Proof. Define as before $\psi(z) := \text{Vol}((K - z)^\circ)$, $\psi : \text{int}(K) \rightarrow \mathbb{R}^+$. It is clear that ψ is strictly convex (Remark 2.3), and from the symmetry of the body K with respect to $\mathbb{R}^n$ we have $\text{Vol}((K - (a, b))^\circ) = \text{Vol}((K - (a, -b))^\circ)$, where $a \in \mathbb{R}^n, b \in \mathbb{R}^s$.

Thus, we conclude

$$\begin{aligned} \text{Vol}((K - (a, 0))^\circ) &= \psi(a, 0) \\ &= \psi\left(\frac{1}{2}(a, b) + \frac{1}{2}(a, -b)\right) \\ &\leq \frac{1}{2}\psi((a, b)) + \frac{1}{2}\psi((a, -b)) \\ &= \psi(a, b) = \text{Vol}((K - (a, b))^\circ). \end{aligned} \quad \square$$

Corollary 4.8. *Let $K \subseteq \mathbb{R}^{n+s}$ be a centrally symmetric convex body. If $(x, y) \in S(K, t)$ then $(x, 0) \in S(K, t)$. Indeed, assumption implies*

$$\text{Vol}(K) \text{Vol}((K - (x, y))^\circ) \leq \text{Vol}(B_2^{n+s})^2 t.$$

Hence, from Lemma 4.7

$$\text{Vol}(K) \text{Vol}((K - (x, 0))^\circ) \leq \text{Vol}(B_2^{n+s})^2 t,$$

which leads to $(x, 0) \in S(K, t)$.

Lemma 4.9. *Let $K \subseteq \mathbb{R}^{n+s}$ be a centrally symmetric convex body. Then*

$$\text{Vol}((K - z)^\circ) = \text{Vol}((K + z)^\circ) \text{ for any } z \in \text{int}(K).$$

Proof. Since $K = -K$ we have $K + z = -K + z = -(K - z)$, which implies

$$(K + z)^\circ = (-(K - z))^\circ = -(K - z)^\circ.$$

Hence $\text{Vol}((K + z)^\circ) = \text{Vol}((K - z)^\circ)$. □

The next lemma deals with the continuity of the *Floating body* with respect to locally uniformly convergence of s -concave functions.

Lemma 4.10. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function. Define as before $f_s(x) = (1 + \frac{\log f(x)}{s})_+$. Then*

$$F(f_s, \lambda) \xrightarrow{s \rightarrow \infty} F(f, \lambda) \text{ for any } 0 < \lambda < 1/2.$$

Proof. Let $\theta \in S^{n-1}$, and let $H_{f_s, \theta}$, $H_{f, \theta}$ be the two hyperplanes perpendicular to θ such that

$$\int_{H_{f_s, \theta}^+} f_s(x) dx = \lambda \int f_s(x) dx \quad \text{and} \quad \int_{H_{f, \theta}^+} f(x) dx = \lambda \int f(x) dx.$$

Since $f_s \xrightarrow{s \rightarrow \infty} f$ locally uniformly we conclude

$$\int_{H_{f_s, \theta}^+} f_s dx = \lambda \int f_s dx \xrightarrow{s \rightarrow \infty} \lambda \int f dx = \int_{H_{f, \theta}^+} f dx.$$

Since $H_{f_s, \theta}$ and $H_{f, \theta}$ are parallel, we conclude that $H_{f_s, \theta} \xrightarrow{s \rightarrow \infty} H_{f, \theta}$, with respect to the *Hausdorff distance* of bodies. □

Before we proceed with the proof of Theorem 1.2, let us prove the following technical lemma.

Lemma 4.11. *Let n, s be a positive integers. Then*

$$\left[\frac{\text{Vol}(B_2^s)^2}{\text{Vol}(B_2^{s+n})^2} \left(\frac{2\pi}{s} \right)^n \right] \xrightarrow{s \rightarrow \infty} 1 \text{ from above.}$$

Proof. For $n \in \mathbb{N}$ even, $\text{Vol}(B_2^n) = \frac{(2\pi)^{\frac{n}{2}}}{n!!}$, while for $n \in \mathbb{N}$ odd $\text{Vol}(B_2^n) = \frac{\pi^{\frac{n-1}{2}} 2^{\frac{n+1}{2}}}{n!!}$. The rest of the proof follows from simple limit calculation. □

Proof of Theorem 1.2. Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function and let $t > 0$. As mentioned before, we first prove Theorem 1.2 with respect to $f_{[t]}$, then from the following facts

$$F(f_{[t]}, \lambda) \xrightarrow{t \rightarrow \infty} F(f, \lambda) \quad \text{and} \quad S(f_{[t]}, d) \xrightarrow{t \rightarrow \infty} S(f, d),$$

we will reach the proof.

For $s \in \mathbb{N}$ consider $f_{[t],s}(x) = (1 + \frac{\log f_{[t]}(x)}{s})_+^s$. From Lemma 4.3 we know

$$\left\{ a \in \mathbb{R}^n : \int f_{[t],s} \int \mathcal{L}_s(f_{[t],s,a}) \leq (2\pi)^n d \right\} \xrightarrow{s \rightarrow \infty} S(f_{[t]}, d). \quad (9)$$

We claim

$$F(f_{[t],s}, \lambda) \subseteq \left\{ a \in \mathbb{R}^n : \int f_{[t],s} \int \mathcal{L}_s(f_{[t],s,a}) \leq (2\pi)^n d \right\}. \quad (10)$$

Indeed, Lemma 4.11 implies $\left[\frac{\text{Vol}(B_2^s)^2}{\text{Vol}(B_2^{s+n})^2} \left(\frac{2\pi}{s} \right)^n \right] \xrightarrow{s \rightarrow \infty} 1$, and $\left[\frac{\text{Vol}(B_2^s)^2}{\text{Vol}(B_2^{s+n})^2} \left(\frac{2\pi}{s} \right)^n \right] \geq 1$.

Hence

$$\begin{aligned} & \left\{ a \in \mathbb{R}^n : \int f_{[t],s} \int \mathcal{L}_s(f_{[t],s,a}) \leq (2\pi)^n d \right\} \\ &= \left\{ a \in \mathbb{R}^n : \text{Vol}(K_s(f_{[t],s})) \text{Vol}(K_s(\mathcal{L}_s(f_{[t],s,a}))) \right. \\ & \quad \leq \left[\frac{\text{Vol}(B_2^s)^2}{\text{Vol}(B_2^{s+n})^2} \left(\frac{2\pi}{s} \right)^n \right] \text{Vol}(B_2^{s+n})^2 d \Big\} \\ & \supseteq \{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}(K_s(\mathcal{L}_s(f_{[t],s,a}))) \leq \text{Vol}(B_2^{s+n})^2 d \} \\ &= \{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}(K_s(f_{[t],s,a})^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \}, \end{aligned}$$

Since $K_s(f_{[t],s,a}) = K_s(f_{[t],s}) + \frac{1}{\sqrt{s}}(a, 0)$, So we get:

$$\begin{aligned} & \{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}(K_s(f_{[t],s,a})^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \} \\ &= \left\{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol} \left(\left(K_s(f_{[t],s}) + \frac{1}{\sqrt{s}}(a, 0) \right)^\circ \right) \right. \\ & \quad \left. \leq \text{Vol}(B_2^{s+n})^2 d \right\} \\ &= \sqrt{s} \{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}((K_s(f_{[t],s}) + (a, 0))^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \}. \end{aligned}$$

Since $K_s(f_{[t],s})$ is a centrally symmetric convex body, from Lemma 4.9 we conclude

$$\text{Vol}((K_s(f_{[t],s}) + (a, 0))^\circ) = \text{Vol}((K_s(f_{[t],s}) - (a, 0))^\circ).$$

From Corollary 4.8, assumption of Theorem 1.2 and Lemma 4.6

$$\begin{aligned} & \sqrt{s} \{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}((K_s(f_{[t],s}) - (a, 0))^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \} \\ &= \sqrt{s} P_{\mathbb{R}^n}[S(K_s(f_{[t],s}), d)] \supseteq \sqrt{s} P_{\mathbb{R}^n}[F(K_s(f_{[t],s}), \lambda)] \\ &= F(f_{[t],s}, \lambda). \end{aligned}$$

From Lemma 4.10 we conclude

$$F(f_{[t],s}, \lambda) \xrightarrow{s \rightarrow \infty} F(f_{[t]}, \lambda).$$

From Lemmas 4.1, 4.2 we complete the proof. $\square$

We turn to the proof of Theorem 1.5.

Proof of Theorem 1.5. We use the same notation as in Theorem 1.2. Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even log-concave function and $t > 0$. First we prove Theorem 1.5 for $f_{[t]}$, then from the facts $F(f_{[t]}, \lambda) \xrightarrow{t \rightarrow \infty} F(f, \lambda)$ and $S(f_{[t]}, d) \xrightarrow{t \rightarrow \infty} S(f, d)$ we will reach the proof.

As before, for $s \in \mathbb{N}$ we consider $f_{[t],s}(x) = \left(1 + \frac{\log f_{[t]}(x)}{s}\right)_+^s$. We define

$$\begin{aligned} L_s &:= \left\{ a \in \mathbb{R}^n : \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int f_{[t],s} dx \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int \mathcal{L}_s(f_{[t],s,a}) dx \leq \text{Vol}(B_2^{n+2})^2 d \right\} \\ &= \left\{ a \in \mathbb{R}^n : \int f_{[t],s} dx \int \mathcal{L}_s(f_{[t],s,a}) dx \leq \left[\frac{\text{Vol}(B_2^{n+s})^2}{\text{Vol}(B_2^s)^2} \left(\frac{s}{2\pi}\right)^n \right] (2\pi)^n d \right\}. \end{aligned}$$

In what follows we will show

$$L_s \xrightarrow{s \rightarrow \infty} S(f_{[t]}, d). \quad (11)$$

$$L_s \subseteq F(f_{[t],s}, \lambda). \quad (12)$$

Before we continue, since $F(f_{[t],s}, \lambda) \xrightarrow{s \rightarrow \infty} F(f_{[t]}, \lambda)$ (Lemma 4.10), then from (11) and (12) we complete the proof.

We begin with (11). From Lemma 4.11 we know $\left[\frac{\text{Vol}(B_2^{n+s})^2}{\text{Vol}(B_2^s)^2} \left(\frac{s}{2\pi}\right)^n \right] \xrightarrow{s \rightarrow \infty} 1$. Hence Lemma 4.3 implies that $L_s \xrightarrow{s \rightarrow \infty} S(f_{[t]}, d)$.

We continue with (12). Simple calculation yields

$$\begin{aligned} \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int f_{[t],s} dx &= \text{Vol}(K_s(f_{[t],s})) \\ \frac{\text{Vol}(B_2^s)}{s^{\frac{n}{2}}} \int \mathcal{L}_s(f_{[t],s,a}) dx &= \text{Vol}(K_s(\mathcal{L}_s(f_{[t],s,a}))). \end{aligned}$$

Thus, we conclude

$$L_s = \{a \in \mathbb{R}^n : \text{Vol}(K_s(f_{[t],s})) \text{Vol}(K_s(\mathcal{L}_s(f_{[t],s,a}))) \leq \text{Vol}(B_2^{s+n})^2 d\}.$$

Moreover, from Lemma 2.8

$$\begin{aligned} &\{(a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}(K_s(\mathcal{L}_s(f_{[t],s,a}))) \leq \text{Vol}(B_2^{s+n})^2 d\} \\ &= \{(a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}((K_s(f_{[t],s,a}))^\circ) \leq \text{Vol}(B_2^{s+n})^2 d\}. \end{aligned}$$

As before, we know $K_s(f_{[t],s,a}) = K_s(f_{[t],s}) + \frac{1}{\sqrt{s}}(a, 0)$, which leads to

$$\begin{aligned} & \left\{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}((K_s(f_{[t],s,a}))^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \right\} \\ = & \left\{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol} \left(\left(K_s(f_{[t],s}) + \frac{1}{\sqrt{s}}(a, 0) \right)^\circ \right) \right. \\ & \left. \leq \text{Vol}(B_2^{s+n})^2 d \right\} \\ = & \sqrt{s} \left\{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}((K_s(f_{[t],s}) + (a, 0))^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \right\}. \end{aligned}$$

Since $K_s((f_{[t]})_s)$ is a centrally symmetric convex body, from Lemma 4.9 we conclude

$$\text{Vol}((K_s(f_{[t],s}) + (a, 0))^\circ) = \text{Vol}((K_s(f_{[t],s}) - (a, 0))^\circ).$$

Thus, from Corollary 4.8, assumption of Theorem 1.5 and Lemma 4.6

$$\begin{aligned} & \sqrt{s} \left\{ (a, 0) \in \mathbb{R}^n \times \mathbb{R}^s : \text{Vol}(K_s(f_{[t],s})) \text{Vol}((K_s(f_{[t],s}) - (a, 0))^\circ) \leq \text{Vol}(B_2^{s+n})^2 d \right\} \\ = & \sqrt{s} P_{\mathbb{R}^n}[S(K_s(f_{[t],s}), d)] \subseteq \sqrt{s} P_{\mathbb{R}^n}[F(K_s(f_{[t],s}), \lambda)] \\ = & F(f_{[t],s}, \lambda), \end{aligned}$$

which completely shows (12). □

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