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# An Equivalence of Dual Brunn-Minkowski Inequality for Volume Differences

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In this paper, we establish an equivalence form of the dual Brunn-Minkowski inequality for volume differences (which was established in *Geom. Dedicata* 145 (2010) 169–180).

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## 1. Introduction

The well-known classical Brunn-Minkowski inequality can be stated.

If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then (see for example [1])

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},$$

with equality if and only if $K$ and $L$ are homothetic, where $+$ is usual Minkowski sum.

Let $K$ and $L$ are star bodies in $\mathbb{R}^n$, then the dual Brunn-Minkowski inequality can be stated.

$$\tilde{V}(K\tilde{+}L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n},$$

with equality if and only if $K$ and $L$ are dilates (see [2]), where $\tilde{+}$ is radial sum.

A vector addition on $\mathbb{R}^n$, which we call radial addition as follows. If $x, y \in \mathbb{R}^n$, then $x\tilde{+}y$ is defined to the usual vector sum of $x, y$, provided $x, y$ all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise. If $K, L$ are star

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bodies and $\lambda, \mu \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda K \tilde{+} \mu L$, is defined by $\lambda K \tilde{+} \mu L = \{\lambda x \tilde{+} \mu y : x \in K, y \in L\}$. The addition $K \tilde{+} L$ be called radial sum of star bodies $K$ and $L$ (see [3], [4]).

In 2004, the volume difference function of convex bodies $K$ and $L$ was defined in [5].

$$Dv(K, D) = V(K) - V(D), \quad D \subseteq K.$$

The following Brunn-Minkowski inequality for volume difference function was also established.

**Theorem A.** *If $K, L$, and $D$ are compact domains, $D \subseteq K$, $D' \subseteq L$, $D'$ is a homothetic copy of $D$, then*

$$(V(K+L) - V(D+D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}, \quad (1)$$

*with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.*

In 2007, an equivalence form of (1) was established in [6].

It is very interesting that DUAL *volume differences function* for star bodies was introduced and the dual Brunn-Minkowski inequality for dual volume difference was established in [7].

**Theorem B.** *If $K, D$ and $D'$ are star bodies in $\mathbb{R}^n$, and $D \subseteq K, D' \subseteq L$, $L$ is a dilation of $K$. Then*

$$(V(K \tilde{+} L) - (V(D \tilde{+} D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}, \quad (2)$$

*with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\tilde{+}$ is radial sum, $\mu$ is a constant.*

If $K, L$ are star bodies and $\lambda, \mu \geq 0$, then the radial Blaschke liner combination $\lambda \cdot K \breve{+} \mu \cdot L$, as the star body whose radial function is given by (see for example [4])

$$\rho(\lambda \cdot K \breve{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot).$$

It is well known the dual Kneser-Süss inequality can be stated.

**Theorem C.** *If $K, L$ are star bodies in $\mathbb{R}^n$, then*

$$V(K \breve{+} L)^{(n-1)/n} \leq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \quad (3)$$

*with equality if and only if $K$ and $L$ are dilates (see [4]).*

The aim of this paper is to extend the dual Kneser-Süss inequality (Theorem C) to the context of volume differences, which is in turn proved to be equivalent to Lv's result (Theorem B).

## 2. Definitions and preliminaries

The setting of this paper is $n$-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter $u$ for unit vectors and the letter $B$ for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. We denote by $V(K)$ the $n$-dimensional volume of a convex body $K$. Let $h_K : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$, i.e., $h_K(u) = \mathrm{Max}\{u{\cdot}x : x \in K\}, u \in S^{n-1}$, where $u \cdot x$ denotes the usual inner product of $u$ and $x$ in $\mathbb{R}^n$.

Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = \mathrm{Max}\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, $K$ will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$ on $S^{n-1}$.

### 2.1. Dual mixed volume

If $K_i \in \mathcal{S}^n (i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$ are nonnegative real numbers, then the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous $n$-th degree polynomial in $\lambda_i$ given by

$$\tilde{V}(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} \tilde{V}_{i_1 \ldots i_n}, \tag{4}$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $\tilde{V}_{i_1 \ldots i_n}$, which is called the *dual mixed volume* of $K_i, \ldots, K_{i_n}$, depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (4). If $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $\tilde{V}(K_1 \ldots K_n)$ is usually written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume $\tilde{V}(K, B)$ is written as $\tilde{W}_i(K)$ and is called the *i-th dual Quermassintegral* of $K$.

If $K, L$ are star bodies and $\lambda, \mu \geq 0$, then the radial Minkoski liner combination $\lambda K \tilde{+} \mu L$, as the star body whose radial function is given by

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot). \tag{5}$$

By using the polar coordinate formula for volume and (5), one obtain from (4) the following integral representation of dual mixed volumes.

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \tag{6}$$

If $K, L, M \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$, then

$$\tilde{V}_1(M, \lambda K \tilde{+} \mu L) = \lambda \tilde{V}_1(M, K) + \mu \tilde{V}_1(M, L). \tag{7}$$

If $K, L \in \mathcal{S}^n$, then from (4) it follows immediately that

$$\lim_{\varepsilon \to 0} \frac{V(K \tilde{+} \varepsilon \cdot L) - V(K)}{\varepsilon} = n\tilde{V}_1(K, L). \tag{8}$$

### 2.2. Radial Blaschke addition

If $K, L$ are star bodies and $\lambda, \mu \geq 0$, then the radial Blaschke liner combination $\lambda \cdot K \breve{+} \mu \cdot L$, as the star body whose radial function is given by

$$\rho(\lambda \cdot K \breve{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \tag{9}$$

If $K, L, M \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$, then

$$\tilde{V}_1(\lambda \cdot K \breve{+} \mu \cdot L, M) = \lambda \tilde{V}_1(K, M) + \mu \tilde{V}_1(L, M). \tag{10}$$

This an immediate consequence of the definition of a radial Blaschke linear combination (9) and the integral representation of dual mixed volumes (6).

As an aside, we note that one can easily establish the counterpart of (8) (see [4]): If $K, L \in \mathcal{S}^n$

$$\lim_{\varepsilon \to 0} \frac{V(K \breve{+} \varepsilon \cdot L) - V(K)}{\varepsilon} = \frac{n}{n-1} \tilde{V}_1(K, L). \tag{11}$$

To prove this one need only use (4), obtain a series expansion for $\rho(K \breve{+} \varepsilon \cdot L, \cdot)^n$, in $\varepsilon$, and then use the polar coordinate formula for volume.

## 3. Main results

**Theorem 3.1.** *The dual Brunn-Minkowski inequality for volume differences and the dual Knesser-Süss inequality for volume differences are equivalence. Namely: If $D, D'$, and $K$ are star bodies in $\mathbb{R}^n$, $D \subseteq K$, and $D' \subseteq L$, and let $L$ is a dilates copy of $K$, then*

$$[V(K \tilde{+} L) - V(D \tilde{+} D')]^{1/n} \geq [V(K) - V(D)]^{1/n} + [V(L) - V(D')]^{1/n}, \tag{12}$$

*with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.*

$$\begin{aligned} &\Longleftrightarrow [V(K \breve{+} L) - V(D \breve{+} D')]^{(n-1)/n} \\ &\geq [V(K) - V(D)]^{(n-1)/n} + [V(L) - V(D')]^{(n-1)/n}, \end{aligned}$$

*with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.*

We need Lemma 3.2 to prove Theorem 3.1.

**Lemma 3.2.** *If $D, D'$, and $K$ are star bodies in $\mathbb{R}^n$, $D \subseteq K$, and $D' \subseteq L$, and $L$ is a dilates copy of $K$, then*

$$[V(K\breve{+}L) - V(D\breve{+}D')]^{(n-1)/n} \\ \geq [V(K) - V(D)]^{(n-1)/n} + [V(L) - V(D')]^{(n-1)/n}, \tag{13}$$

*with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.*

We need Lemma 3.3 to prove Lemma 3.2.

**Lemma 3.3.** *Let $a = \{a_1, \ldots, a_n\}$ and $b = \{b_1, \ldots, b_n\}$ be two series of positive real numbers and $p > 1$ such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p}, \tag{14}$$

*with equality if and only if $a = \upsilon b$ where $\upsilon$ is a constant (see [8]).*

**Proof of Lemma 3.2.** Applying the dual Knesser-Süss inequality, we obtain for $D, D' \in \mathcal{S}^n$

$$V(D\breve{+}D')^{(n-1)/n} \leq V(D)^{(n-1)/n} + V(D')^{(n-1)/n}, \tag{15}$$

with equality if and only if $D$ and $D'$ are dilates, and

$$V(K\breve{+}L)^{(n-1)/n} = V(K)^{(n-1)/n} + V(L)^{(n-1)/n}. \tag{16}$$

From (15) and (16), we obtain

$$[V(K\breve{+}L) - V(D\breve{+}D')]^{(n-1)/n} \\ \geq \left\{[V(K)^{(n-1)/n} + V(L)^{(n-1)/n}]^{n/(n-1)} \right. \\ \left. - [V(D)^{(n-1)/n} + V(D')^{(n-1)/n}]^{n/(n-1)}\right\}^{(n-1)/n}. \tag{17}$$

From (14) and (17), then

$$[V(K\breve{+}L) - V(D\breve{+}D')]^{(n-1)/n} \\ \geq (V(K) - V(D))^{(n-1)/n} + (V(L) - V(D'))^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

This completes the proof.

**Proof of Theorem 3.1.** $\Longleftarrow$ Suppose the dual Knesser-Süss inequality for volume differences hold, namely:

If $D, D'$, and $K$ are star bodies in $\mathbb{R}^n$, $D \subseteq K$, and $D' \subseteq L$, and let $L$ is a dilates copy of $K$, then

$$Dv(K\breve{+}L, D\breve{+}D')^{(n-1)/n} \geq Dv(K, D)^{(n-1)/n} + Dv(L, D')^{(n-1)/n}, \tag{18}$$

with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

From (11) and (18), we obtain

$$\begin{aligned}
&\frac{n}{n-1}(\tilde{V}_1(K,L) - \tilde{V}_1(D,D')) = \lim_{\varepsilon\to 0}\frac{Dv(L\breve{+}\varepsilon K, D'\breve{+}\varepsilon D) + Dv(D',L)}{\varepsilon}\\
&\geq \lim_{\varepsilon\to 0}\frac{(Dv(L,D')^{(n-1)/n} + \varepsilon\cdot Dv(K,D)^{(n-1)/n})^{n/(n-1)} + Dv(D',L)}{\varepsilon}, \qquad (19)
\end{aligned}$$

with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

On the other hand, from (19) and in view of L'Hôpital's rule, we have

$$\begin{aligned}
&\tilde{V}_1(K,L) - \tilde{V}_1(D,D')\\
&\geq \lim_{\varepsilon\to 0}(Dv(L,D')^{(n-1)/n} + \varepsilon Dv(K,D)^{(n-1)/n})^{1/(n-1)} Dv(K,D)^{(n-1)/n}\\
&= Dv(L,D')^{1/n} Dv(K,D)^{(n-1)/n}. \qquad (20)
\end{aligned}$$

Suppose that $M, N \in \mathcal{S}^n$ and $N \subseteq M$, from (7) and (20), it follows that

$$\begin{aligned}
&\tilde{V}_1(M, K\tilde{+}L) - \tilde{V}_1(N, D\tilde{+}D')\\
&= (\tilde{V}_1(M,K) - \tilde{V}_1(N,D)) + (\tilde{V}_1(M,L) - \tilde{V}_1(N,D'))\\
&\geq (Dv(K,D)^{1/n} + Dv(L,D')^{1/n}) Dv(M,N)^{(n-1)/n}. \qquad (21)
\end{aligned}$$

If we take $M = K\tilde{+}L$ and $N = D\tilde{+}D'$ in (21), in view of $\tilde{V}(K,\ldots,K) = V(K)$, we have

$$Dv(K\tilde{+}L, D\tilde{+}D')^{1/n} \geq Dv(K,D)^{1/n} + Dv(K,D)^{1/n},$$

with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

This is just the dual Brunn-Minkowski inequality for volume differences was established in [7].

$\Longrightarrow$ Suppose the dual Brunn-Minkowski inequality for volume differences hold, namely:

If $D, D'$, and $K$ are star bodies in $\mathbb{R}^n$, $D \subseteq K$,and $D' \subseteq L$, and let $L$ is a dilates copy of $K$, then

$$Dv(K\tilde{+}L, D\tilde{+}D')^{1/n} \geq Dv(K,D)^{1/n} + Dv(L,D')^{1/n},$$

with equality if and only if $D$ and $D'$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

From (8), we have

$$\begin{aligned} & n(\tilde{V}_1(K,L) - \tilde{V}_1(D,D')) \\ = & \lim_{\varepsilon\to 0} \frac{Dv(K\tilde{+}\varepsilon L, D\tilde{+}\varepsilon D') + Dv(D,K)}{\varepsilon} \\ \geq & \lim_{\varepsilon\to 0} \frac{(Dv(K,D)^{1/n} + \varepsilon Dv(L,D')^{1/n})^n + Dv(D,K)}{\varepsilon}, \end{aligned} \tag{22}$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

On the other hand, from (22) and in view of L'Hôpital's rule, we have

$$\begin{aligned} \tilde{V}_1(K,L) - \tilde{V}_1(D,D') &\geq \lim_{\varepsilon\to 0} (Dv(K,D)^{1/n} + \varepsilon Dv(L,D')^{1/n})^{n-1} Dv(L,D')^{1/n} \\ &= Dv(K,D)^{(n-1)/n} Dv(L,D')^{1/n}. \end{aligned} \tag{23}$$

From (10) and (23), for any $M, N \in \mathcal{S}^n$ and $N \subseteq M$, we have

$$\begin{aligned} & V_1(K\breve{+}L, M) - V_1(D\breve{+}D', N) \\ = & (V_1(K,M) - V_1(D,N)) + (V_1(L,M) - V_1(D',N)) \\ \geq & (Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n}) Dv(M,N)^{1/n}. \end{aligned} \tag{24}$$

If we take $M = K\breve{+}L$ and $N = D\breve{+}D'$ in (24), we obtain

$$Dv(K\breve{+}L, D\breve{+}D')^{(n-1)/n} \geq Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n},$$

with equality if and only if $D$ and $D'$ are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

This is just the dual Knesser-Süss inequality for volume differences.

This proof is completed.

## References

[1] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge Univ. Press, Cambridge (1993).

[2] E. Lutwak: Dual mixed volumes, Pacific J. Math. 58 (1975) 531–538.

[3] R. J. Gardner: Geometric Tomography, Cambridge Univ. Press, Cambridge (1995).

[4] E. Lutwak: Intersection bodies and dual mixed volumes, Adv. Math. 71 (1988) 232–261.

[5] G. S. Leng: The Brunn-Minkowski inequality for volume differences, Adv. Appl. Math. 32 (2004) 615–624.

[6] C. Zhao, W. Cheung: An equivalence form of the Brunn-Minkowski inequality for volume differences, J. Korean Math. Soc. 44 (2007) 1373–1381.

[7] S. Lv: Dual Brunn-Minkowski inequality for volume differences, Geom. Dedicata 145 (2010) 169–180.

[8] E. F. Beckenbach, R. Bellman: Inequalities, 2nd Ed., Springer, Berlin (1965).