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An Evolutionary Structure of Convex Quadrilaterals. Part II

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We solve explicitly the generalized Gauss problem for convex quadrilaterals in the two dimensional Euclidean Space. By introducing the variable $c = c_G + \frac{|B_1 - B_4| + |B_2 - B_3|}{2}$, where $c_G = \frac{1}{2}$ is the Gauss constant and B_i are positive real variables, such that $\sum_{i=1}^{4} B_i = 1$, we derive some new evolutionary structures of convex quadrilaterals, and we give the definition of the degree of plasticity of convex quadrilaterals which could be extended to the degree of plasticity of convex polygons with respect to the topology of weighted Steiner minimal trees. Finally, the solution of the weighted Steiner tree problem for convex quadrilaterals gives a second property, which is the translation between the two Fermat-Torricelli points.

Keywords: Weighted Fermat-Torricelli problem, Steiner minimal tree, convex quadrilaterals

1991 Mathematics Subject Classification: 51E12, 52A10, 52A55, 51E10

1. Introduction

We mention the following two problems raised by P. Fermat and C. F. Gauss, respectively:

In 1643, P. Fermat raised the problem:

Given three points in the Euclidean plane, find the point that the sum of distances to these three points is minimized.

In 1836, C. F. Gauss posed the problem to the astronomer Schumacher which may be considered as a generalization of the problem of P. Fermat (see [1], page 326, [3], Chapter 2): How to find a railway network of minimal total length which connects the four cities Bremen, Harburg (today part of the city of Hamburg), Hannover, and Braunschweig.

In 1879, K. Bopp (see [2], [1], page 327) gave a complete solution to Gauss problem for any four given points in the two-dimensional Euclidean Space and gave a description of an experimental solution by applying the property of soapsuds to

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span a minimal surface between the given points.

A formulation of the Steiner problem as an extension of Gauss problem has been given in [4], page 360: "Given n points $A_1A_2...A_n$ to find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system."

We need the following result which was proved in ([5] (see also [1], page 328–329):

A solution of the Steiner problem is a Steiner tree with at most $n-2$ Fermat-Torricelli (or Steiner) points which are vertices of the polygonal tree which do not belong in $\{A_1A_2...A_n\}$, where each Fermat-Torricelli point has valency 3, and the angle between any two edges incident with a Fermat-Torricelli point is of $120°$.

Concerning cases where a variant of Steiner problem is uniquely determined and for the definition of topologies of Steiner trees on the two dimensional Euclidean Space, we refer to [5].

In this paper, we obtain explicitly the weighted (full) Steiner tree for convex quadrilaterals in the two-dimensional Euclidean Space by applying a method of cyclical differentiation of the objective function with respect to four variable angles which are formulated between four line segments and four given sides of the quadrilateral (see Theorem 1 in [7]), respectively, and by introducing as a new variable the angle which is formulated between the line defined by two given vertices A_1 and A_2 and the line which passes from A_1 and it is parallel to the line defined by the two weighted Fermat-Torricelli points A_0 and A_0', which are located at the interior of the convex quadrilateral (see Theorem 2.2).

By introducing the variable $c = c_G + \frac{|B_1-B_4|+|B_2-B_3|}{2}$, where $c_G = \frac{1}{2}$ is the Gauss constant and B_i are positive real variables, such that $\sum_i^4 B_i = 1$ and B_0, $B_{0'}$ are the variable weights that corresponds to the weighted Fermat-Torricelli points A_0 and A_0', respectively, we derive some new evolutionary structures of convex quadrilaterals in $\mathbb{R}^2$ (Proposition 2.5).

By performing a parallel translation of the triangle $\triangle A_{0'}A_2A_3$ to $\triangle A_0A_2'A_3'$ (see Fig. 2.3), we derive that for given values of the weights B_i, the plasticity equations of $A_1A_2'A_3'A_4$ (see [6], Proposition 4.4) with respect to the weighted Fermat-Torricelli point A_0 of $A_1A_2'A_3'A_4$ are satisfied but depend also on the value of c (Proposition 2.6).

Finally, as a consequence of Proposition 2.6 and Remark 2.7, we give the definition of the degree of plasticity ($Degplasticity\{A_1A_2A_3A_4\}$) of the weighted Steiner tree of a convex quadrilateral $A_1A_2A_3A_4$ in $\mathbb{R}^2$, with respect to the number of the weighted Fermat-Torricelli points which are located at the interior of $A_1A_2A_3A_4$.

2. A weighted Steiner minimal tree for convex quadrilaterals in the two-dimensional Euclidean Space.

Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$. Suppose that a positive number B_i'(weight) corresponds to each vertex A_i, for $i = 1, 2, 3, 4$, respectively. We denote

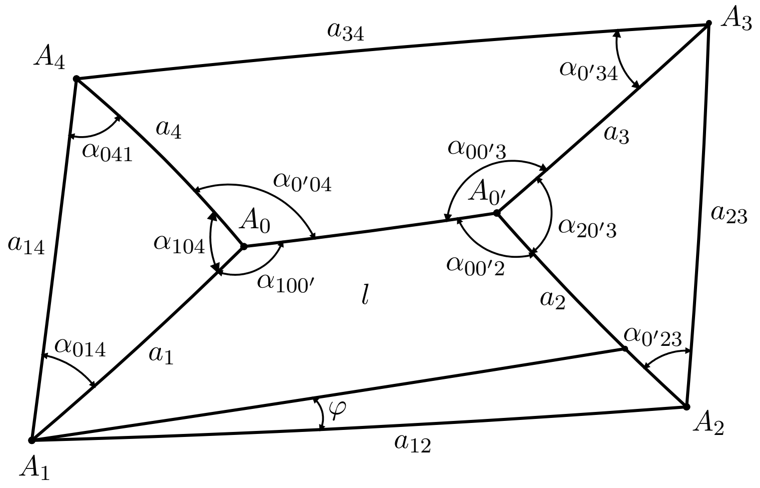

Figure 2.1.

by l the length of the line segment that connects A_0 with $A_{0'}$, a_{ij} the length of the line segment that connects the vertex A_i with A_j, and α_{ijk}, the angle that is formulated between the line segment A_iA_j and A_jA_k, for $i, j, k \in \{0, 0', 1, 2, 3, 4\}$ and $i \neq j \neq k$. Furthermore, we denote by $a_{10} = a_1$, $a_{40} = a_4$, $a_{20'} = a_2$, and $a_{30'} = a_3$ (see Figure 2.1). We mention the following theorem which has been proved in [7] Theorem 1.

Theorem 2.1. *A weighted (full) Steiner minimal tree of $A_1A_2A_3A_4$ consists of two (weighted) Fermat-Torricelli points A_0, A_0' which are located at the interior convex domain with corresponding weights B_0, $B_{0'}$ and minimizes the objective function:*

$$B_1'a_1 + B_2'a_2 + B_3'a_3 + B_4'a_4 + \frac{c(\sum_{i=1}^{4} B_i')}{2} l = minimum, \tag{1}$$

such that:

$$|B_i' - B_j'| < B_k' < B_i' + B_j', \tag{2}$$

$$|B_l' - B_m'| < B_n' < B_l' + B_m' \tag{3}$$

for $i, j, k \in \{1, 4, 5\}$, $l, m, n \in \{2, 3, 5\}$, $i \neq j \neq k$, $l \neq m \neq n$, and $c = \frac{B_0 + B_{0'}}{2}$.

By applying the method of cyclical differentiation of the objective function with respect to four variable angles and four variable line segments (see the proof of Theorem 1 in [7]), we shall find an explicit solution of the generalized Gauss problem (full weighted Steiner minimal tree) of a convex quadrilateral $A_1A_2A_3A_4$ in $\mathbb{R}^2$.

Theorem 2.2. *The location of A_0 and A_0' is given by the relations:*

$$\cot\varphi = \frac{\frac{c}{2}a_{12} + B_4 a_{14}\cos(\alpha_{214} - \alpha_{400'}) + B_3 a_{23}\cos(\alpha_{123} - \alpha_{30'0})}{B_4 a_{14}\sin(\alpha_{214} - \alpha_{400'}) - B_3 a_{23}\sin(\alpha_{123} - \alpha_{30'0})}, \tag{4}$$

$$a_1 = \frac{a_{14}\sin(\alpha_{214} - \varphi - \alpha_{400'})}{\sin(\alpha_{100'} + \alpha_{400'})} \tag{5}$$

and

$$a_2 = \frac{a_{23}\sin(\alpha_{123} + \varphi - \alpha_{300'})}{\sin(\alpha_{20'0} + \alpha_{30'0})} \tag{6}$$

where φ is the angle which is formulated between the line defined by A_1 and A_2 and the line which passes from A_1 and it is parallel to the line defined by A_0 and A_0'.

Proof of Theorem 2.2. Without loss of generality, for all $i = 1, 2, 3, 4$, we set

$$B_i = \frac{B_i'}{\sum_{i=1}^4 B_i'},$$

so

$$\sum_{i=1}^4 B_i = 1.$$

Dividing (1) by $\sum_{i=1}^4 B_i'$, we get:

$$B_1 a_1 + B_2 a_2 + B_3 a_3 + B_4 a_4 + \frac{c}{2} l = \textit{minimum}.$$

By expressing the length of the line segments a_3, a_4, l as functions of a_1, a_2, α_{014}, $\alpha_{120'}$, we derive (see also the proof of Theorem 1 in [7]):

$$\frac{B_1}{\sin\alpha_{0'04}} = \frac{B_4}{\sin\alpha_{100'}} = \frac{c}{2\sin\alpha_{104}}. \tag{7}$$

The following angular relation is satisfied in $\mathbb{R}^2$:

$$\alpha_{0'04} + \alpha_{100'} + \alpha_{104} = 2\pi, \tag{8}$$

or

$$\sin(\alpha_{0'04}) = -\sin(\alpha_{100'} + \alpha_{104}), \tag{9}$$

or

$$\sin(\alpha_{0'04}) = -\sin(\alpha_{100'})\cos(\alpha_{104}) - \cos(\alpha_{100'})\sin(\alpha_{104}). \tag{10}$$

By multiplying both members of (10) with $\frac{c}{2}$ and by taking into account (7), we get:

$$B_1\sin(\alpha_{104}) = \frac{c}{2}\sin(\alpha_{0'04}) = -\frac{c}{2}\sin(\alpha_{104})\cos(\alpha_{100'}) - \frac{c}{2}\sin(\alpha_{100'})\cos(\alpha_{104}). \tag{11}$$

or

$$B_1\sin(\alpha_{104}) = -\frac{c}{2}\sin(\alpha_{104})\cos(\alpha_{100'}) - (\frac{c}{2}\sin(\alpha_{100'}))\cos(\alpha_{104}).$$

or

$$B_1\sin(\alpha_{104}) = -\frac{c}{2}\sin(\alpha_{104})\cos(\alpha_{100'}) - (B_4\sin(\alpha_{104}))\cos(\alpha_{104})$$

or

$$\frac{c}{2}\cos(\alpha_{100'}) + B_4\cos(\alpha_{104}) = -B_1. \tag{12}$$

By following the same process, we derive the following two equations:

$$B_1\cos(\alpha_{100'}) + B_4\cos(\alpha_{0'04}) = -\frac{c}{2} \tag{13}$$

and

$$B_1\cos(\alpha_{104}) + \frac{c}{2}\cos(\alpha_{0'04}) = -B_4. \tag{14}$$

The solution of (12), (13) and (14) with respect to the variables $\cos\alpha_{100'}$, $\cos\alpha_{0'04}$ and $\cos\alpha_{104}$ is:

$$\cos\alpha_{100'} = \frac{B_4^2 - B_1^2 - (\frac{c}{2})^2}{2B_1(\frac{c}{2})}, \tag{15}$$

$$\cos\alpha_{0'04} = \frac{B_1^2 - B_4^2 - (\frac{c}{2})^2}{2B_4(\frac{c}{2})}, \tag{16}$$

$$\cos\alpha_{104} = \frac{(\frac{c}{2})^2 - B_1^2 - B_4^2}{2B_1B_4}. \tag{17}$$

Similarly, by expressing a_1, a_2, l as a function of a_3, a_4, α_{041} and $\alpha_{0'34}$, we derive that:

$$\frac{B_2}{\sin\alpha_{00'3}} = \frac{B_3}{\sin\alpha_{00'2}} = \frac{c}{2\sin\alpha_{20'3}}. \tag{18}$$

The relations (15), (16) and (17) show that A_0 is the weighted Fermat-Torricelli point of $\nabla A_1A_{0'}A_4$ which minimizes the function $f_1 = B_1a_1 + \frac{c}{2}l + B_4a_4$.

$$\cos\alpha_{00'3} = \frac{B_2^2 - B_3^2 - (\frac{c}{2})^2}{2B_3(\frac{c}{2})}, \tag{19}$$

$$\cos\alpha_{00'2} = \frac{B_3^2 - (\frac{c}{2})^2 - B_2^2}{2(\frac{c}{2})B_2}, \tag{20}$$

$$\cos\alpha_{20'3} = \frac{(\frac{c}{2})^2 - B_2^2 - B_3^2}{2B_2B_3}. \tag{21}$$

The relations (19), (20) and (21) show that $A_{0'}$ is the weighted Fermat-Torricelli point of $\nabla A_0A_2A_3$ which minimizes the function $f_2 = \frac{c}{2}l + B_2a_2 + B_3a_3$.

By taking into account the distance of A_1 and A_2 from the line defined by A_0 and A_0', and the distance of A_2 from the line which passes from A_1 and it is parallel to the line defined by A_0 and A_0', we get:

$$l - a_1 \cos(\alpha_{100'}) - a_2 \cos(\alpha_{20'0}) = a_{12} \cos(\varphi) \tag{22}$$

and

$$a_2 \sin(\alpha_{20'0}) - a_1 \sin(\alpha_{100'}) = a_{12} \sin(\varphi). \tag{23}$$

By taking into account the distance of A_1 and A_4 from the line defined by A_0 and A_0', and the distance of A_4 from the line which passes from A_1 and it is parallel to the line defined by A_0 and A_0', we get:

$$a_1 \sin(\alpha_{100'}) + a_4 \sin(\alpha_{400'}) = a_{14} \sin(\alpha_{214} - \varphi) \tag{24}$$

and

$$-a_1 \cos(\alpha_{100'}) + a_4 \cos(\alpha_{400'}) = a_{14} \cos(\alpha_{214} - \varphi). \tag{25}$$

By taking into account the distance of A_2 and A_3 from the line defined by A_0 and A_0', and the distance of A_2 from the line which passes from A_3 and it is parallel to the line defined by A_0 and A_0', we get:

$$a_2 \sin(\alpha_{20'0}) + a_3 \sin(\alpha_{30'0}) = a_{23} \sin(\alpha_{123} + \varphi) \tag{26}$$

and

$$-a_3 \cos(\alpha_{30'0}) + a_2 \cos(\alpha_{20'0}) = -a_{23} \cos(\alpha_{123} + \varphi). \tag{27}$$

By solving (24) and (25) with respect to a_1 and a_4, we obtain (5).

By solving (26) and (27) with respect to a_2 and a_3, we obtain (6).

By replacing (5) and (6) in (23), we derive (4).

The location of A_0 and $A_{0'}$ are given by (4), (5) and (6) taking into account (15), (16), (17), (19), (20) and (21).

The distance l is derived by (22). □

Corollary 2.3. *If $B_i = 0.25$, for $i = 1, 2, 3, 4$ and $c = 0.5$, then*

$$\alpha_{100'} = \alpha_{0'04} = \alpha_{104} = \alpha_{00'2} = \alpha_{00'3} = \alpha_{203} = 120^\circ.$$

We call $c = c_G = 0.5$ the Gauss constant, which characterizes the unweighted full Steiner trees for convex quadrilaterals in $\mathbb{R}^2$.

Proof of Corollary 2.3. By replacing $B_i = \frac{c_G}{2} = 0.25$, in (7) and (18), we obtain

$$\alpha_{100'} = \alpha_{0'04} = \alpha_{104} = \alpha_{00'2} = \alpha_{00'3} = \alpha_{203} = 120^\circ. \qquad \square$$

Corollary 2.4. *If A_0 and A_0' are the corresponding weighted Fermat-Torricelli points of the convex quadrilateral $A_1A_2A_3A_4$ in $\mathbb{R}^2$, A_c belongs to the circle that*

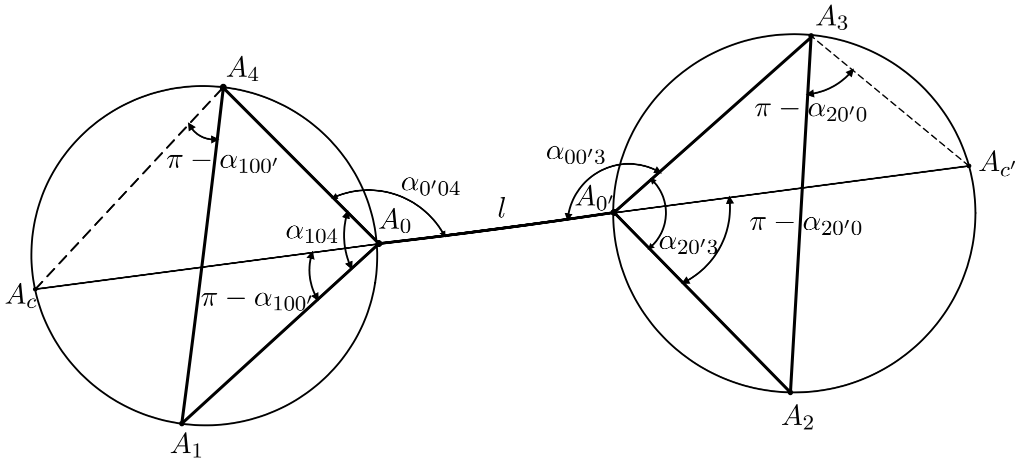

Figure 2.2.

passes from A_0, A_1 *and* A_4 *and* $A_{c'}$ *belongs to the circle that passes from* $A_{0'}$, A_2 *and* A_3 *such that:*

$$\alpha_{14c} = \angle A_1 A_4 A_c = \pi - \alpha_{100'}$$

and

$$\alpha_{23c'} = \angle A_2 A_3 A_{c'} = \pi - \alpha_{20'0}$$

then A_c, A_0, $A_{0'}$, $A_{c'}$ *belong to the same line (see Fig. 2.2).*

Proof of Corollary 2.4. By taking into account that

$$\alpha_{14c} = \angle A_1 A_4 A_c = \pi - \alpha_{100'}$$

and

$$\alpha_{23c'} = \angle A_2 A_3 A_{c'} = \pi - \alpha_{20'0},$$

we have $\alpha_{14c} = \alpha_{10c}$ and $\alpha_{23c'} = \alpha_{20'c'}$, because they correspond to the same arc $A_1 A_c$ and $A_2 A_{c'}$, respectively. $\square$

Proposition 2.5. *If* $c = c_G + \frac{|B_1 - B_4| + |B_2 - B_3|)}{2}$, *where*

$$B_i = \frac{B_i'}{\sum_{i=1}^{4} B_i'},$$

for $i = 1, 2, 3, 4$, *and* $c_G = 0.5$ *is the Gauss constant, then*

(*I*) $\frac{1}{2} \leq c < 1$ *and*

(*II*) *if* $B_1 + B_4 > B_2 + B_{3,}$ *and* $c < 1$, *then* A_0' *does not coincide with* A_0 *and for* $c = 1$, A_0' *coincides with* A_0.

Proof of (I)**, Proposition 2.5.** The weights B_i are positive numbers such that:

$$B_i = \frac{B_i'}{\sum_{i=1}^{4} B_i'},$$

$$0 < B_i < 1,$$
$$\sum_{i=1}^{n} B_i = 1.$$

For $B_1 = B_4$ and $B_2 = B_3$ we get:

$$c = c_G + \frac{|B_1 - B_4| + |B_2 - B_3|)}{2} = c_G = 0.5$$

and

$$\begin{aligned} c &= c_G + \frac{|B_1 - B_4| + |B_2 - B_3|)}{2} \\ &= \frac{1}{2}(1 + |B_1 - B_4| + |B_2 - B_3|) \\ &< \frac{1}{2}\left(1 + \sum_{i=1}^{4} B_i\right) < 1. \end{aligned}$$ □

Proof of (II)**, Proposition 2.5.** Taking into account Theorem 2.2, the location of A_0 and A_0' depend on the value of c. From Theorem (2.1) in [6], we obtain that A_0 is the weighted Fermat-Torricelli point of $\nabla A_1 A_{0'} A_4$ which minimizes the function $f_1 = B_1 a_1 + \frac{c}{2} l + B_4 a_4$ and $A_{0'}$ is the weighted Fermat-Torricelli point of $\nabla A_0 A_2 A_3$ which minimizes the function $f_2 = \frac{c}{2} l + B_2 a_2 + B_3 a_3$. For $c = 1$, we have that $B_2 + B_3 < \frac{c}{2} = 0.5$, because $B_1 + B_4 > B_2 + B_3$, and the function f_2 is minimized only when A_0' equals to A_0 because A_0 is the absorbed equilibrium point of $\triangle A_0 A_2 A_3$. □

Proposition 2.6. *By performing a parallel translation of $\triangle A_2 A_{0'} A_3$ to $A_2' A_0 A_3'$, (see Fig. 2.3, $A_2' A_0$ is parallel to $A_2 A_{0'}$ and $A_3' A_0$ is parallel to $A_3 A_{0'}$), the following equations are satisfied:*

$$-B_2 \sin(\alpha_{100'} + \alpha_{20'0} - \pi) + B_3 \sin(\alpha_{100'} + \alpha_{20'0} - \pi) + B_4 \sin(\alpha_{104}) = 0, \quad (28)$$

$$-B_1 \sin(\alpha_{400'} + \alpha_{30'0} - \pi + \alpha_{104}) + B_2 \sin(\alpha_{20'3}) - B_4 \sin(\alpha_{400'} + \alpha_{30'0} - \pi) = 0, \quad (29)$$

$$-B_1 \sin(\alpha_{100'} + \alpha_{20'0} - \pi) + B_3 \sin(\alpha_{20'3}) + B_4 \sin(\alpha_{400'} + \alpha_{30'0} + \alpha_{20'3} - \pi) = 0. \quad (30)$$

Proof of Proposition 2.6. By taking into account the parallel translation of $\triangle A_2 A_{0'} A_3$ to $A_{2'} A_0 A_{3'}$, (see Fig. 2.3, we consider that A_0' approaches A_0 and $l \to 0$. By replacing in (1) $l = 0$ and $B_i = \frac{B_i'}{\sum_{i=1}^{4} B_i'}$, we get:

$$B_1 a_1 + B_2 a_2' + B_3 a_3' + B_4 a_4 = minimum, \quad (31)$$

By differentiating (31) with respect to the variables $\angle A_0 A_2' A_3' = \alpha_{02'3'}$ and a_2', we have:

$$B_1 \frac{\partial a_1}{\partial \alpha_{02'3'}} + B_3 \frac{\partial a_3'}{\partial \alpha_{02'3'}} + B_4 \frac{\partial a_4}{\partial \alpha_{02'3'}} = 0,$$

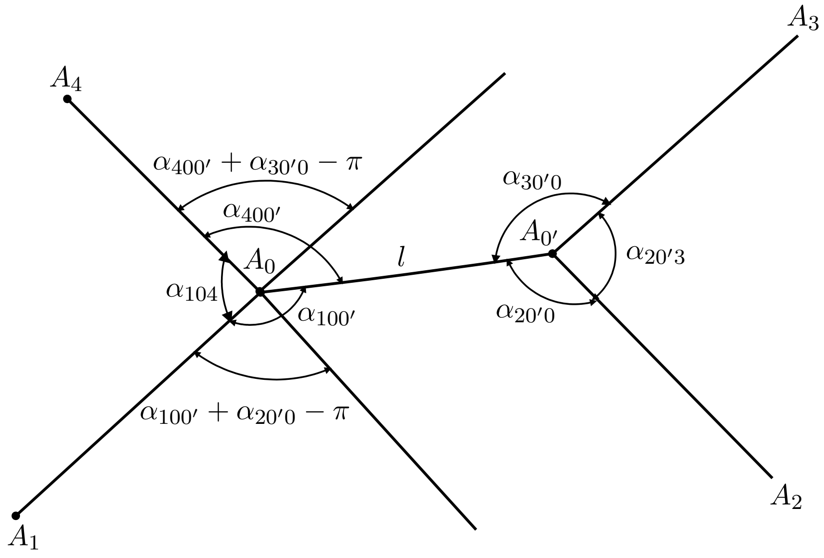

Figure 2.3.

$$B_1\frac{\partial a_1}{\partial a_2'} + B_2 + B_3\frac{\partial a_3'}{\partial a_2'} + B_4\frac{\partial a_4}{\partial a_2'} = 0.$$

By calculating $\frac{\partial a_1}{\partial \alpha_{02'3'}}$, $\frac{\partial a_3'}{\partial \alpha_{02'3'}}$, $\frac{\partial a_4}{\partial \alpha_{02'3'}}$ and by applying the "cosine law" and the "sine law" to the corresponding triangles $\nabla A_1A_2'A_0$, $\nabla A_3'A_2'A_0$, $\nabla A_4A_2'A_0$, we obtain:

$$-B_1\sin(\alpha_{102'}) + B_3\sin(\alpha_{2'03'}) + B_4\sin(\alpha_{3'04'} + \alpha_{2'03'}) = 0. \tag{32}$$

Similarly, by differentiating the objective function (31) with respect to the variables $(\angle A_0A_3'A_2' = \alpha_{03'2'})$, $(\angle A_0A_1A_3' = \alpha_{013'})$, respectively, (see Fig. 2.3) we obtain

$$-B_1\sin(\alpha_{3'04} + \alpha_{104}) + B_2\sin(\alpha_{2'03'}) - B_4\sin(\alpha_{3'04}) = 0 \tag{33}$$

and

$$-B_2\sin(\alpha_{102'}) + B_3\sin(\alpha_{104} + \alpha_{3'04}) + B_4\sin(\alpha_{104}) = 0 \tag{34}$$

such that:

$$\alpha_{102'} + \alpha_{2'03'} + \alpha_{3'04} + \alpha_{104} = 2\pi. \tag{35}$$

where

$$\alpha_{3'04} + \alpha_{104} = \alpha_{400'} + \alpha_{30'0} - \pi + \alpha_{104},$$
$$\alpha_{2'03'} = \alpha_{20'3},$$
$$\alpha_{3'04} = \alpha_{400'} + \alpha_{30'0} - \pi,$$
$$\alpha_{102'} = \alpha_{100'} + \alpha_{20'0} - \pi,$$
$$\alpha_{3'04'} + \alpha_{2'03'} = \alpha_{400'} + \alpha_{30'0} + \alpha_{20'3} - \pi. \qquad \square$$

Remark 2.7. For $l = 0$ and from (29) and (30), we obtain the plasticity equations (see in [6], Proposition 4.4, page 417) with respect to variable weights B_i, such that:

$$B_1 + B_2 + B_3 + B_4 = 1.$$

We may also consider this case as a limiting case of the full weighted Steiner minimal tree such that A_0 and A_0' approach $A_{0,1234}$, where $A_{0,1234}$ is the corresponding weighted Fermat-Torricelli point of $A_1A_2A_3A_4$ and the limiting angle of $\varphi + \pi - \alpha_{100'}$ will be the angle $\alpha_{213} - \alpha_{013}$, where the angle α_{013} is calculated in [6], page 413, formulas (7) and (8).

For $l \neq 0$, we cannot derive the plasticity of the full weighted Steiner minimal tree with respect to the convex quadrilateral $A_1A_2A_3A_4$, because from (4), we derive that φ depends not only on the values of B_i for $i = 1, 2, 3, 4$ but also on the value of c (see Theorems 2.1 and 2.2). Furthermore, the conditions (7) and (18) shows the dependence of c with the weights B_i, for $i = 1, 2, 3, 4$.

Taking into account Proposition 2.6 and Remark 2.7, we introduce the following two definitions:

Definition 2.8. The degree of plasticity ($Degplasticity\{A_1A_2A_3A_4\}$) of the weighted (full) Steiner minimal tree of a convex quadrilateral in $\mathbb{R}^2$ which consists of exactly two weighted Fermat-Torricelli points at the interior of the quadrilateral is zero.

Definition 2.9. The degree of plasticity ($Degplasticity\{A_1A_2A_3A_4\}$) of the weighted Steiner minimal tree of a convex quadrilateral in $\mathbb{R}^2$ which consists of exactly one weighted Fermat-Torricelli point at the interior of the quadrilateral is one.

By introducing the classes of (full) weighted Steiner minimal trees having $n - 2$ weighted Fermat-Torricelli points at the interior of the convex polygon $A_1A_2...A_n$ with degree of plasticity zero, we can define various classes of evolutionary steiner minimal trees having less than $n - 2$ weighted Fermat-Torricelli points at the interior of the convex polygon $A_1A_2...A_n$ with degree of plasticity

$$0 < Degplasticity\{A_1A_2...A_n\} \leq n - 3.$$

The distorted network which has exactly one weighted Fermat-Torricelli point at the interior of $A_1A_2...A_n$ will be defined to have $Degplasticity\{A_1A_2...A_n\} = n-3$.

A future work will be to derive various types of evolution of the topology of weighted Steiner minimal trees and obtain generalizations of Feynman diagrams.

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