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# Normal Cones and Continuity of Vector-Valued Convex Functions

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It is known that convex functions with values in an ordered Banach space $Y$, whose positive cone $Y_+$ is normal, have the same continuity properties as real-valued convex functions. We show that normality of $Y_+$ is not only sufficient, but also necessary for this.

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In this paper we consider mappings from locally convex topological vector spaces (TVS) to ordered normed linear spaces. Recall that an *ordered linear space* is a couple $(Y, \leq)$, where $Y$ is a vector space over the real field, and $\leq$ is an order relation on the set $Y$ such that, for every $u$, $v$ in $Y$, $u \leq v$ implies both $u + w \leq v + w$ for all $w$ in $Y$, and $\lambda u \leq \lambda v$ for every non-negative scalar $\lambda$. In the following, we write simply $Y$ instead of $(Y, \leq)$.
It is an easy known fact that every vector space ordering is completely determined by the convex cone of non-negative elements $Y_+$, the *positive cone*, thus we may regard a certain property as a feature of the order relation or of the cone indifferently. Throughout this article we refer to [2] and [3] for the theory of ordered linear spaces.
The order relation allows us to define the notion of convexity for maps valued in such a vector space: if $f\colon C \to Y$ is a function from a convex subset $C$ of a vector space and $Y$ is as above, we say that $f$ is *convex* if, taking $x$ and $y$ in $C$ and $\lambda$ in $[0, 1]$,

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

holds. It is not difficult to show that this definition of convexity is equivalent, in the case when $C \subset \mathbf{R}$, to the fact that taking $x$, $y$ and $z$ in $C$ such that $x < y < z$,

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one of the following (vector) inequalities holds:

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}$$

(the validity of a single inequality implies the validity of each other one).

Without any relation between the structures of ordered space and topological space of $Y$, one should not expect that a purely "algebraic" condition like the one in the definition above would imply any continuity property about convex functions. An effective way to estabilish this relation is given by considering *normal* orderings, for the general theory of which we may refer to [3], Chap. 2, §1. We only recall one of the properties that are equivalent to normality in the case where $Y$ is a normed space, and we put it as a definition:

**Definition 1.** Let $Y$ be an ordered normed linear space. The positive cone $Y_+$ is *nomal* if there exists a positive constant $\delta$ such that, for every $u$, $v$ satisfying $0 \leq u \leq v$, the inequality $\|u\| \leq \delta\|v\|$ holds.

Using the normality property of the positive cone, regularity results similar to the ones that hold for real-valued convex functions can be obtained. We focus in particular on the following proposition, that is a simple consequence of Proposition 2.3 in [1].

**Theorem 2.** *Let $X$ be a locally convex TVS, $C \subset X$ an open and convex set, $Y$ an ordered normed linear space and $f\colon C \to Y$ a convex function. Suppose that $Y_+$ is normal. The following assertions are equivalent:*

- *$f$ is bounded above on some neighbourhood $V$ of a point $x_0 \in C$ by a function $g\colon V \to Y$ that is continuous at $x_0$;*
- *$f$ is continuous at $x_0$.*

In this article we want to prove that normality is also necessary to obtain continuity for convex functions, when the target space $Y$ satisfies certain properties.

We begin with a possibly known fact, for which we have not found any reference.

**Lemma 3.** *Let $Y$ be an ordered Banach space, whose positive cone $Y_+$ is closed but not normal. Then there exists a norm-unbounded order interval, of the form $[0, w]$.*

**Proof.** Being the cone not normal, it is possible to find two sequences $\{x_n\}_{n\in\mathbf{N}}$, $\{y_n\}_{n\in\mathbf{N}}$ such that $0 \leq x_n \leq y_n$, $\|y_n\| = 1$ and $\|x_n\| = 3^n$. Define now $w$ as

$$w = \sum_{n=1}^{\infty} 2^{-n} y_n \ ;$$

it is well defined because $Y$ is complete, and is also positive because $Y_+$ is closed.

Consider the sequence

$$a_n = w - \sum_{k=1}^{n-1} 2^{-k} y_k - 2^{-n} x_n.$$

It is easy to observe that, for sufficiently large $n$,

$$\|a_n\| \geq \left(\frac{3}{2}\right)^n - \left\| w - \sum_{k=1}^{n-1} 2^{-k} y_k \right\|$$

and therefore, being the second quantity bounded, $\{a_n\}_{n\in\mathbf{N}}$ turns out to be an unbounded sequence. Moreover, for every $n$,

$$0 \leq w - \sum_{k=1}^{n} 2^{-k} y_k \leq a_n \leq w$$

and so $[0, w]$ is unbounded. □

**Proposition 4.** *Let $Y$ be as in Lemma* 3*. Then there exists a convex function $f\colon \mathbf{R} \to Y$ which is locally order bounded, but norm unbounded in every neighbourhood of* 0*.*

**Proof.** Let $[0, w]$ be the norm unbounded interval constructed in Lemma 3. It is clear that also every interval of the form $[\alpha w, \beta w]$, where $0 \leq \alpha < \beta$, is unbounded and contained in the positive cone. Let $\lambda \in (0, 1)$; since $1 - \lambda + \lambda^2 > \lambda$, there exists $\rho \in \left(1, \frac{1-\lambda+\lambda^2}{\lambda}\right)$. For every $n \in \mathbf{N} \cup \{0\}$ choose $w_n \in [\lambda^{2n} w, \rho\lambda^{2n} w]$ so that $\|w_n\| > n$. Let the function $f\colon \mathbf{R} \to Y$ be such that $f = 0$ on $(-\infty, 0]$, $f(\lambda^n) = w_n$ for every integer $n$ and $f$ is affine on each interval $[\lambda^{n+1}, \lambda^n]$; that is, for $t \in [\lambda^{n+1}, \lambda^n]$,

$$f(t) = \frac{\lambda^n}{\lambda^n - \lambda^{n+1}} w_{n+1} - \frac{\lambda^{n+1}}{\lambda^n - \lambda^{n+1}} w_n + \mu_n t,$$

where we put $\mu_n = \frac{w_n - w_{n+1}}{\lambda^n - \lambda^{n+1}}$. The sequence $\{\mu_n\}_{n\in\mathbf{N}}$ is decreasing, in fact for every $n \geq 0$ we have

$$\begin{aligned}
&\frac{w_n - w_{n+1}}{\lambda^n - \lambda^{n+1}} - \frac{w_{n+1} - w_{n+2}}{\lambda^{n+1} - \lambda^{n+2}} \\
&\geq \frac{\lambda^{2n} w - \rho\lambda^{2n+2} w}{\lambda^n - \lambda^{n+1}} - \frac{\rho\lambda^{2n+2} w - \lambda^{2n+4} w}{\lambda^{n+1} - \lambda^{n+2}} \\
&= \frac{\lambda^n w}{1-\lambda} \left(1 - \rho\lambda^2 - \rho\lambda + \lambda^3\right) \\
&\geq \frac{\lambda^n w}{1-\lambda} \left(1 - \lambda\left(1 - \lambda + \lambda^2\right) - \left(1 - \lambda + \lambda^2\right) + \lambda^3\right) = 0.
\end{aligned}$$

In order to prove the convexity of $f$ it is sufficient to show that, for every choice of $x$, $y$, $z$ in $\mathbf{R}$ with $x < y < z$,

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}. \tag{1}$$

Let us distinguish several cases.
If $y \leq 0$, the inequality (1) is trivially satisfied since the left-hand side is zero, and the right-hand side belongs to $Y_+$.
If $x \leq 0 < y$, suppose that $y \in (\lambda^{m+1}, \lambda^m]$ and $z \in (\lambda^{n+1}, \lambda^n]$; assume for the moment that $m$ is strictly greater than $n$, and evaluate the right-hand side of (1):

$$\begin{aligned}
&\frac{f(z)-f(y)}{z-y} \\
&= \frac{1}{z-y}\left(f(y) + (\lambda^m - y)\mu_m + \sum_{j=n+1}^{m-1} (\lambda^j - \lambda^{j+1})\mu_j + (z-\lambda^{n+1})\mu_n - f(y)\right).
\end{aligned}$$

Being $\mu_j \geq \mu_m$ we obtain the following estimate:

$$\frac{f(z)-f(y)}{z-y} \geq \frac{1}{z-y}\left(\lambda^m - y + \sum_{j=n+1}^{m-1} (\lambda^j - \lambda^{j+1}) + z - \lambda^{n+1}\right)\mu_m = \mu_m.$$

Notice that the final result is valid also if $m = n$. Since $f(x) = 0$ we have

$$\frac{f(y)-f(x)}{y-x} - \frac{f(z)-f(y)}{z-y} \leq \frac{f(y)}{y-x} - \mu_m. \tag{2}$$

Since $x \leq 0$, replacing $f(y)$ in the right-hand side of (2) with its explicit expression we obtain:

$$\begin{aligned}
&\frac{1}{y-x}\left(\frac{\lambda^m}{\lambda^m - \lambda^{m+1}} w_{m+1} - \frac{\lambda^{m+1}}{\lambda^m - \lambda^{m+1}} w_m\right) + \frac{y}{y-x}\mu_m - \mu_m \\
&\leq \frac{1}{y-x}\left(\frac{\lambda^m}{\lambda^m - \lambda^{m+1}} w_{m+1} - \frac{\lambda^{m+1}}{\lambda^m - \lambda^{m+1}} w_m\right) \\
&= \frac{1}{y-x}\frac{\lambda^m}{\lambda^m - \lambda^{m+1}}\left(w_{m+1} - \lambda w_m\right) \\
&\leq \frac{1}{y-x}\frac{\lambda^m}{\lambda^m - \lambda^{m+1}}\left(\rho\lambda^{2m+2} - \lambda^{2m+1}\right) w \\
&= \frac{1}{y-x}\frac{\lambda^{m+2m+1}}{\lambda^m(1-\lambda)}\left(\rho\lambda - 1\right) w \\
&\leq \frac{1}{y-x}\frac{\lambda^{3m+1}}{\lambda^m(1-\lambda)}\left(-\lambda + \lambda^2\right) w \\
&= -\frac{\lambda^{2m+2}}{y-x} w \leq 0.
\end{aligned}$$

This proves (1) for $z \le \lambda$. It is easy to verify in a similar way that the same holds also if $y$ or $z$ is greater than $\lambda$.
In the case when $x > 0$ we can proceed in a similar way as above; let us only sketch the calculation. Take $x \in \left(\lambda^{l+1}, \lambda^{l}\right]$, $y \in (\lambda^{m+1}, \lambda^{m}]$, $z \in (\lambda^{n+1}, \lambda^{n}]$, and assume that $l \ge m \ge n$. Now, both the estimates $\frac{f(z)-f(y)}{z-y} \ge \mu_m$, $\frac{f(y)-f(x)}{y-x} \le \mu_m$ come as a consequence of the already used equality

$$f(z) - f(y) = f(y) + (\lambda^m - y)\mu_m + \sum_{j=n+1}^{m-1} (\lambda^j - \lambda^{j+1})\mu_j + (z - \lambda^{n+1})\mu_n - f(y)$$

(with obvious adaptations, where needed). As above, it is easy to extend the considerations to the case that $x$, $y$ or $z$ lie in $(\lambda, +\infty)$. The relation (1) is hence satisfied also in this case, and the proof is complete. $\square$

The construction above can be exploited in order to obtain a more general result.

**Corollary 5.** *Let $X$ be a locally convex TVS and $Y$ as in Lemma* 3*. Then there exists a function $F\colon X \to Y$ that is convex, bounded above on some neighbourhood of* 0*, and norm unbounded on every neighbourhood of* 0*.*

**Proof.** Let $f$ be the function constructed in Proposition 4. Let $v$ be an element of $X \setminus \{0\}$, and consider an element $x^*$ of the topological dual $X^*$ of $X$ such that $x^*(v) = 1$; this functional exists since $X$ is locally convex.[1] The function $F$ defined as

$$F(x) = f(x^*(x))$$

is convex, unbounded with respect to the norm near 0 and is bounded above on every set of the form $\{x \in X\colon x^*(x) \in (-\varepsilon, \varepsilon)\}$, where $\varepsilon > 0$. $\square$

Finally, it is possible to conclude with the following result:

**Theorem 6.** *Let $I \subset \mathbf{R}$ be an open interval, $X$ a (nontrivial) locally convex TVS, $C \subset X$ an open, convex set, $Y$ an ordered Banach space whose positive cone $Y_+$ is closed. The following assertions are equivalent:*

- $(i)$ *$Y_+$ is normal;*
- $(ii)$ *every convex function $f\colon I \to Y$ is continuous;*
- $(iii)$ *every convex function $f\colon I \to Y$ is locally norm bounded;*
- $(iv)$ *every convex function $F\colon C \to Y$, which is bounded above on some open subset of $C$, is continuous;*
- $(v)$ *every convex function $F\colon C \to Y$, which is bounded above on some nonempty open subset of $C$, is locally norm bounded.*

**Proof.** The equivalence of $(i)$, $(ii)$ and $(iii)$ follows from Proposition 4; the implication "$(i) \Rightarrow (iv)$" is a consequence of Theorem 2, the implication "$(iv) \Rightarrow (v)$" is evident, and "$(v) \Rightarrow (i)$" follows directly from Corollary 5. $\square$

[1]This is a consequence of the Hahn-Banach Theorem, see [4] for details.

## References

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