Journal of Convex Analysis
Volume 20 (2013), No. 2, 501–529

On Well Posed Best Approximation Problems for a Nonsymmetric Seminorm

Grigorii E. Ivanov*
Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutski str. 9, Dolgoprudny, Moscow region, Russia 141700
givanov@mail.mipt.ru

Received: January 30, 2012
Revised manuscript received: August 6, 2012

Let M be a closed convex (generally unbounded) subset of a Banach space E with 0 being an interior point of M, A be a closed subset of E. Let $T_M(A)$ be the set of all $x_0 \in E$ such that the problem $\min_{a\in A} \mu_M(x_0 - a)$ is well posed, where μ_M is the Minkowski functional of M, so μ_M is a nonsymmetric seminorm. We obtain some asymptotic properties (appearance far from the origin) of M which are necessary and/or sufficient for $S_M^{\mathrm{int}}(A) \setminus T_M(A)$ to be a meagre or a σ-porous subset of $S_M^{\mathrm{int}}(A) = \{x_0 \in E \mid 0 < \varrho_M(x_0, A) < \sup_{x\in E} \varrho_M(x, A)\}$, where $\varrho_M(x, A) = \inf_{a\in A} \mu_M(x - a)$.

Keywords: Best approximation, Minkowski functional, residual set, σ-porous set

2010 Mathematics Subject Classification: 41A50, 41A65, 52A21

1. Introduction

Let E be a real normed vector space. For a set $A \subset E$ by $\operatorname{int} A$, $\overline{A}$, and ∂A we denote the interior, the closure, and the boundary of A, respectively. We use $\langle p, x\rangle$ to denote the value of the functional $p \in E^*$ at the vector $x \in E$. For $R > 0$ and $c \in E$ we denote by $\mathfrak{B}_R(c)$ the closed ball with center c and radius R, in particular, we put $\mathfrak{B}_R = \mathfrak{B}_R(0)$.

We say that the subset M of E is a *quasiball* if M is closed convex and $0 \in \operatorname{int} M$.

Let M be a quasiball. Recall that the *Minkowski functional* $\mu_M : E \to \mathbb{R}$ of M is defined by

$$\mu_M(x) = \inf\{t > 0 \mid x \in tM\}, \quad \forall x \in E. \tag{1}$$

A functional $\mu : E \to \mathbb{R}$ is called a *nonsymmetric seminorm* if it is *positively homogeneous*:

$$\mu(tx) = t\mu(x), \quad \forall t \geq 0,\ \forall x \in E$$

and *subadditive*:

$$\mu(x + y) \leq \mu(x) + \mu(y), \quad \forall x, y \in E.$$

*Partially supported by the Russian Foundation for Basic Research, grant 13-01-00295-a.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Remark 1.1. A functional $\mu : E \to [0, +\infty)$ is a Lipschitz continuous nonsymmetric seminorm iff it is the Minkowski functional of a quasiball.

For a closed set $A \subset E$ and a point $x_0 \in E$ we consider the problem to approximate x_0 by a point $a \in A$ best in the sense of the nonsymmetric seminorm:

$$\min_{a \in A} \mu_M(x_0 - a). \tag{2}$$

Through $\varrho_M(x_0, A)$ we denote the *M-distance* from the point x_0 to the set A:

$$\varrho_M(x_0, A) = \inf_{a \in A} \mu_M(x_0 - a). \tag{3}$$

A sequence $\{a_k\} \subset A$ is called a *minimizing sequence* of the problem (2) if

$$\lim_{k \to \infty} \mu_M(x_0 - a_k) = \varrho_M(x_0, A).$$

The problem (2) is said to be *well posed* if every minimizing sequence of the problem (2) converges. We use $T_M(A)$ to denote the set of all $x_0 \in E$ such that the problem (2) is well posed.

Recall that a set $M \subset E$ is called *strictly convex* if for every different points $x, y \in \overline{M}$ we have $\frac{x+y}{2} \in \operatorname{int} M$. A set $M \subset E$ is called *Kadec* if for any sequence $\{x_k\} \subset \partial M$ weak convergence $x_k \to x_0 \in \partial M$ implies strong convergence of $\{x_k\}$.

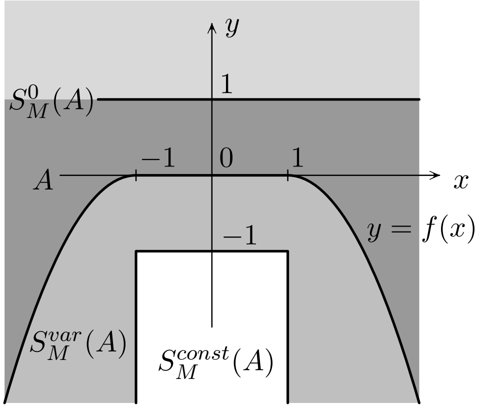
A subset S of a topological space X is called a *meagre set* or a *set of first category* if $S = \bigcup_{n \in \mathbb{N}} S_n$ where S_n are nowhere dense in X. The complement of a meager subset is called a *residual set*.

A subset S of a metric space X is said to be *porous* in X if there exist $r_0 > 0$ and $\vartheta \in (0, 1]$ such that

$$\forall y \in X,\ \forall r \in (0, r_0],\ \exists x \in X : \quad \mathfrak{B}_{\vartheta r}(x) \subset \mathfrak{B}_r(y) \setminus S.$$

A subset S of a metric space X is said to be *σ-porous* in X if $S = \bigcup_{n \in \mathbb{N}} S_n$, where S_n are porous in X. The notion of porosity was introduced by Dolzhenko [2] and has been investigated by many authors (see the survey [9]).

Stechkin [8], Lau [5], Konjagin [4], Borwein and Fitzpatrick [1], and others considered the problem (2) for $M = \mathfrak{B}_1$. It was proved that if the unit ball M is strictly convex and Kadec, a set A is closed, then $T_M(A)$ is a residual subset of E. Instead of the unit ball Li [6] considered a bounded quasiball and obtained a similar result for this more general case. In this paper we refuse from the assumption that the quasiball M is bounded and investigate regions $S \subset E$ such that $S \setminus T_M(A)$ are nowhere dense, meagre or σ-porous in S provided M has some asymptotic properties.



2. Main results

Let $M \subset E$ be a quasiball and a set $A \subset E$ be closed. We split E into three regions:

$$
\begin{aligned}
S_M^0(A) &= \{x_0 \in E \mid \exists \delta > 0 : \ \varrho_M(x, A) = 0, \ \forall x \in \mathfrak{B}_\delta(x_0)\}, \\
S_M^{\text{const}}(A) &= \{x_0 \in E \mid \exists \delta > 0 : \ 0 < \varrho_M(x_0, A) = \varrho_M(x, A), \ \forall x \in \mathfrak{B}_\delta(x_0)\}, \\
S_M^{\text{var}}(A) &= \{x_0 \in E \mid \forall \delta > 0 \ \exists x \in \mathfrak{B}_\delta(x_0) : \ \varrho_M(x, A) \neq \varrho_M(x_0, A)\}. \qquad (4)
\end{aligned}
$$

Note that sets $S_M^0(A)$, $S_M^{\text{const}}(A)$, and $S_M^{\text{var}}(A)$ are disjoint and

$$E = S_M^0(A) \bigcup S_M^{\text{const}}(A) \bigcup S_M^{\text{var}}(A).$$

Obviously sets $S_M^0(A)$ and $S_M^{\text{const}}(A)$ are open and $S_M^{\text{var}}(A)$ is closed. If M is bounded, then $S_M^0(A) = \operatorname{int} A$, $S_M^{\text{const}}(A) = \emptyset$, $S_M^{\text{var}}(A) = \overline{E \setminus A}$.

Example 2.1. Let $E = \mathbb{R}^2$. Consider the quasiball

$$M = \left\{(x, y) \mid x \in \mathbb{R}, \ y \geq x^2 - 1\right\},$$

and the closed set

$$A = \{(x, y) \mid x \in \mathbb{R}, \ f(x) \leq y \leq 1\},$$

where

$$f(x) = \begin{cases} -(x+1)^2, & x < -1, \\ 0, & -1 \leq x \leq 1, \\ -(x-1)^2, & x > 1. \end{cases}$$

Then,

$$
\begin{aligned}
S_M^0(A) &= \{(x, y) \mid x \in \mathbb{R}, \ y > f(x)\}, \\
S_M^{\text{const}}(A) &= \{(x, y) \mid -1 < x < 1, \ y < -1\}, \\
S_M^{\text{var}}(A) &= E \setminus \left(S_M^0(A) \bigcup S_M^{\text{const}}(A)\right), \\
T_M(A) &= E \setminus \left(S_M^0(A) \bigcup \overline{S_M^{\text{const}}(A)}\right).
\end{aligned}
$$

Lemma 2.2. *Let $M \subset E$ be a quasiball and a set $A \subset E$ be closed. Then $S_M^{\mathrm{const}}(A) \bigcap T_M(A) = \emptyset$.*

Lemma 2.2 implies that $S_M^{\mathrm{const}}(A) \bigcap T_M(A)$ can not be dense in $S_M^{\mathrm{const}}(A)$. Example 2.1 shows that $S_M^0(A) \bigcap T_M(A)$ may not be dense in $S_M^0(A)$. We shall consider geometric and asymptotic properties of M that are necessary and sufficient for $S_M^{\mathrm{var}}(A) \bigcap T_M(A)$ to be a dense or residual subset of $S_M^{\mathrm{var}}(A)$.

Consider the barrier cone of M:

$$b(M) = \left\{ p \in E^* \mid \sup_{x \in M} \langle p, x \rangle < +\infty \right\}$$

and the cone

$$b_1(M) = \left\{ p \in E^* \mid \exists x_0 \in M : \langle p, x_0 \rangle = \sup_{x \in M} \langle p, x \rangle \right\}.$$

Theorem 2.3. *For a quasiball M in a Banach space E the following statements are equivalent:*

(*i*) *E is reflexive, M is strictly convex and Kadec, $b_1(M) = b(M)$;*
(*ii*) *for any closed set $A \subset E$ the inclusion $S_M^{\mathrm{var}}(A) \subset \overline{T_M(A)}$ is valid;*
(*iii*) *for any closed set $A \subset E$ the set $S_M^{\mathrm{var}}(A) \bigcap T_M(A)$ is a residual subset of $S_M^{\mathrm{var}}(A)$.*

Remark that investigation of the region $S_M^{\mathrm{var}}(A)$ is somewhat complicated in general. Now we shall introduce the notion of equidistableness. Then we shall see that $S_M^{\mathrm{var}}(A)$ has a simple structure provided that M is not equidistable.

First, we recall that the *Minkowski sum* of sets $A, B \subset E$ is defined as

$$A + B = \{x + y \mid x \in A,\ y \in B\}.$$

We say that a set $M \subset E$ is *equidistable* if there exist $\lambda > 1$, $\delta > 0$, and $d \in E$ such that

$$\forall w \in \mathfrak{B}_\delta,\ \forall \varepsilon > 0,\ \exists x \in M : \quad (1+\varepsilon)x - w \notin \operatorname{int}\left((M + \mathfrak{B}_\delta) \bigcup (\lambda M - d) \right). \quad (5)$$

It follows from (5) that $\forall \varepsilon > 0\ \exists x \in M : (1+\varepsilon)x \notin \operatorname{int} \mathfrak{B}_\delta(x)$ and, therefore, $\|x\| \geq \frac{\delta}{\varepsilon}$. Hence, a bounded set M can not be equidistable. Thus, equidistableness depends on the appearance of the set far from the origin, in this sense this property is asymptotic. An example of an equidistable quasiball in $\mathbb{R}^3$ (Example 7.1) and other examples will be considered in Section 7.

Define *the intermediate region* of the problem (2) as the set of all $x_0 \in E$ such that the value $\varrho_M(x_0, A)$ is not extremal:

$$S_M^{\mathrm{int}}(A) = \left\{ x_0 \in E \mid 0 < \varrho_M(x_0, A) < \sup_{x \in E} \varrho_M(x, A) \right\}. \quad (6)$$

Note that $S_M^{\rm int}(A)$ is an open and rather simple set. The next lemma gives the condition which provides a simple structure of $S_M^{\rm var}(A)$.

Lemma 2.4. *For a quasiball $M \subset E$ the following statements are equivalent:*

(i) *M is not equidistable;*

(ii) *for any closed set $A \subset E$ the inclusion $S_M^{\rm int}(A) \subset S_M^{\rm var}(A)$ is valid;*

(iii) *for any closed set $A \subset E$ the equality $S_M^{\rm var}(A) = \overline{S_M^{\rm int}(A)}$ holds.*

The equivalence of the statements (i) and (ii) in Lemma 2.4 means that a quasiball $M \subset E$ is equidistable iff there exist a closed set $A \subset E$ and a point $x_0 \in S_M^{\rm int}(A)$ such that the distance function $\varrho_M(\cdot, A)$ is constant in some neighbourhood of x_0. This explains the term "equidistable". Using Theorem 2.3 and Lemma 2.4, we deduce the following theorem.

Theorem 2.5. *For a quasiball M in a Banach space E the following statements are equivalent:*

(i) *E is reflexive, M is strictly convex, Kadec and not equidistable, $b_1(M) = b(M)$;*

(ii) *for any closed set $A \subset E$ the inclusion $S_M^{\rm int}(A) \subset \overline{T_M(A)}$ is valid;*

(iii) *for any closed set $A \subset E$ the set $S_M^{\rm int}(A) \bigcap T_M(A)$ is a residual subset of $S_M^{\rm int}(A)$.*

Remark that equidistableness is a rather complicated notion. The following substantially more simple notion provides a sufficient condition for $S_M^{\rm int}(A) \bigcap T_M(A)$ to be a residual subset of $S_M^{\rm int}(A)$, which is also necessary in finite-dimensional spaces.

We say that a set $M \subset E$ is *parabolic* if it is closed convex and for every $d \in E$ the set $M \setminus (2M - d)$ is bounded.

The term "parabolic" is due to the observation that the epigraph of the parabola $y = x^2$ is parabolic while the epigraph of the hyperbola $y = \frac{1}{x}$, $x > 0$ is not parabolic. Since parabolicity depends on the appearance of the set far from the origin, this property is asymptotic.

Lemma 2.6. *If M is a parabolic quasiball in a reflexive Banach space E, then M is not equidistable and $b_1(M) = b(M)$.*

Example 7.3 shows that implication converse to Lemma 2.6 is generally wrong. However the following lemma shows that it is true in a finite-dimensional space.

Lemma 2.7. *A strictly convex quasiball M in a finite-dimensional normed space is parabolic iff $b_1(M) = b(M)$.*

Lemmata 2.6 and 2.7 imply that in a finite-dimensional space there is no strictly convex equidistable quasiball M such that $b_1(M) = b(M)$. Example 7.2 exhibits such a quasiball in ℓ_2. Combining Theorem 2.5 with Lemmata 2.6 and 2.7, we get the following theorem.

Theorem 2.8. *Let M be a quasiball M in a reflexive Banach space E. Then the statement*

(*i*) *M is strictly convex, Kadec and parabolic*

implies that

(*ii*) *for any closed set $A \subset E$ the set $S_M^{\text{int}}(A) \bigcap T_M(A)$ is a residual subset of $S_M^{\text{int}}(A)$.*

If additionally E is finite-dimensional, statements (i) and (ii) are equivalent.

The following two lemmata provide sufficient conditions for the epigraph

$$\operatorname{epi} f = \{(x, y) : \ x \in E, \ y \geq f(x)\}$$

of a function $f : E \to \mathbb{R}$ to be parabolic.

Lemma 2.9. *If a function $f : E \to \mathbb{R}$ is convex, continuous, and coercive, i.e., $\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty$, then its epigraph is parabolic.*

Lemma 2.10. *Let a function $f : E \to \mathbb{R}$ be convex, Lipschitz continuous and the limit*

$$\lim_{t\to+\infty} \frac{f(tx)}{t} = \Lambda_f(x) \in \mathbb{R} \tag{7}$$

exists uniformly on the unit sphere, i.e.,

$$\lim_{t\to+\infty} \sup_{x\in\partial\mathfrak{B}_1} \left| \frac{f(tx)}{t} - \Lambda_f(x) \right| = 0. \tag{8}$$

Suppose also that

$$\lim_{\|x\|\to+\infty} (f(x) - \Lambda_f(x)) = -\infty. \tag{9}$$

Then $\operatorname{epi} f$ *is parabolic.*

Now we consider sufficient conditions for $S_M^{\text{int}}(A) \backslash T_M(A)$ to be σ-porous in $S_M^{\text{int}}(A)$.

A set $M \subset E$ is called *boundedly uniformly convex* if

$$\delta_M^R(\varepsilon) > 0, \quad \forall R > 0, \ \forall \varepsilon > 0,$$

where

$$\delta_M^R(\varepsilon) = \sup \left\{ \delta \in \left(0, \frac{\varepsilon}{2}\right] \ \middle| \ \mathfrak{B}_\delta \left(\frac{x+y}{2}\right) \subset M, \ \forall x, y \in M \cap \mathfrak{B}_R : \ \|x - y\| \geq \varepsilon \right\} \tag{10}$$

is an analogue of the modulus of convexity.

Theorem 2.11. *Let M be a boundedly uniformly convex and parabolic quasiball in a reflexive Banach space E, a set $A \subset E$ be closed. Then $S_M^{\text{int}}(A) \setminus T_M(A)$ is σ-porous in $S_M^{\text{int}}(A)$.*

3. Proofs of Lemma 2.2 and Theorem 2.3

Recall a few more notions.

For a closed set $A \subset E$ and a point $x_0 \in E$ we use $P_M(x_0, A)$ to denote the set of all solutions of the problem (2), i.e., *M-projection* of x_0 on A:

$$P_M(x_0, A) = \{a \in A \mid \mu_M(x_0 - a) \le \varrho_M(x_0, A)\}. \tag{11}$$

For every $\varepsilon > 0$ we also denote

$$P_M^{\varepsilon}(x_0, A) = \{a \in A \mid \mu_M(x_0 - a) \le \varrho_M(x_0, A) + \varepsilon\}. \tag{12}$$

The following remark results from the definition of well posedness.

Remark 3.1. Let $M \subset E$ be a quasiball, a set $A \subset E$ be closed, and $x_0 \in T_M(A)$. Then $P_M(x_0, A)$ is a singleton.

The *diameter* of a set $A \subset E$ is defined as

$$\operatorname{diam} A = \sup_{x,\, y \in A} \|x - y\|.$$

For any functional $p \in E^*$ through $s(p, A)$ we denote the value of the *support function* of a set $A \subset E$:

$$s(p, A) = \sup_{x \in A} \langle p, x\rangle.$$

A functional $p \in E^*$ is called the *Fréchet derivative* of a function $f : E \to \mathbb{R}$ at $x_0 \in E$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} = 0.$$

Hereafter we shall use the following two relations which are valid for any set $A \subset E$ and any quasiball $M \subset E$:

$$\varrho_M(x, A) = \inf\{t > 0 \mid x \in A + tM\}, \quad \forall x \in E, \tag{13}$$

$$\varrho_M(x_1, A) - \varrho_M(x_2, A) \le \mu_M(x_1 - x_2), \quad \forall x_1, x_2 \in E. \tag{14}$$

The equality (13) results from (1) and (3); (14) follows from (3) and the subadditivity of the Minkowski functional.

Proof of Lemma 2.2. Assume the contrary: there exists $x_0 \in S_M^{\mathrm{const}}(A) \bigcap T_M(A)$. According to Remark 3.1, there is $a \in P_M(x_0, A)$. Therefore, whenever $t \in (0, 1)$, $x_t = (1 - t)x_0 + ta$, we have $\varrho_M(x_t, A) \le \mu_M(x_t - a) = (1 - t)\mu_M(x_0 - a) = (1 - t)\varrho_M(x_0, A) < \varrho_M(x_0, A)$. Consequently, for every $\delta > 0$ there exists $x_t \in \mathfrak{B}_\delta(x_0)$ such that $\varrho_M(x_t, A) < \varrho_M(x_0, A)$. This contradicts the assumption that $x_0 \in S_M^{\mathrm{const}}(A)$. $\square$

Lemma 3.2. *Let $M \subset E$ be a quasiball and A be a subset of E. Suppose that $p \in E^*$ is the Fréchet derivative of $\varrho_M(\cdot, A)$ at $x_0 \in E$. Then $s(p, M) \leq 1$.*

Proof. According to the definfition of the Fréchet derivative, we have $\langle p, x\rangle = \lim_{t\to+0} \frac{\varrho_M(x_0+tx,A)-\varrho_M(x_0,A)}{t}$ for all $x \in E$. This and (14) imply that $\langle p, x\rangle \leq \mu_M(x)$ for all $x \in E$. In particular, $\langle p, x\rangle \leq 1$ for all $x \in M$. □

Lemma 3.3. *Let E be a reflexive Banach space and a set $M \subset E$ be strictly convex and Kadec. Suppose that $z_0 \in M$ and $p \in E^* \setminus \{0\}$ satisfy $\langle p, z_0\rangle = s(p, M)$. Suppose a bounded sequence $\{z_k\} \subset M$ is such that $\lim_{k\to\infty}\langle p, z_k\rangle = s(p, M)$. Then $\lim_{k\to\infty} \|z_k - z_0\| = 0$.*

Proof. Assume the contrary. Extracting a proper subsequence from $\{z_k\}$ and denoting it again as $\{z_k\}$, we have $\inf_{k\in\mathbb{N}} \|z_k - z_0\| = \varepsilon > 0$. Since $p \neq 0$, there exists $d \in E$ such that $\langle p, d\rangle = 1$. For every $k \in \mathbb{N}$ denote

$$w_k = z_k + (s(p, M) - \langle p, z_k\rangle + 2^{-k})d.$$

Then, $\langle p, w_k\rangle = s(p, M) + 2^{-k} > s(p, M)$. Consequently, $w_k \notin M$. Therefore for every $k \in \mathbb{N}$ there is $y_k \in [z_k, w_k] \cap \partial M$. Since $\{z_k\}$ is bounded and $\|y_k - z_k\| \leq \|w_k - z_k\| \to 0$ as $k \to \infty$, it follows that $\{y_k\}$ is bounded. Due to the reflexivity of E, there exists a subsequence $\{y_{k_j}\}$, which weakly converges to some $y_0 \in E$. Since M is closed convex, we have $y_0 \in M$. Moreover, $\langle p, y_0\rangle = \lim_{j\to\infty}\langle p, y_{k_j}\rangle = \lim_{k\to\infty}\langle p, z_k\rangle = s(p, M) = \langle p, z_0\rangle$. Using the strict convexity of M, we get $y_0 = z_0 \in \partial M$. The Kadec property of M implies that $\lim_{j\to\infty} \|y_{k_j} - z_0\| = 0$. Therefore, $\varepsilon \leq \lim_{j\to\infty} \|z_{k_j} - z_0\| = 0$. Contradiction. □

Lemma 3.4. *Let E be a reflexive Banach space, $M \subset E$ be a strictly convex and Kadec quasiball, and $A \subset E$ be a closed set. Suppose that $p \in E^* \setminus \{0\}$ is the Fréchet derivative of $\varrho_M(\cdot, A)$ at $x_0 \in E$ and $\varrho_M(x_0, A) > 0$. Suppose there exists $z_0 \in M$ such that $\langle p, z_0\rangle = s(p, M)$. Then $x_0 \in T_M(A)$.*

Proof. Let $\{a_k\}$ be a minimizing sequence of the problem (2). Denote

$$\varrho_0 = \varrho_M(x_0, A), \qquad \varrho_k = \mu_M(x_0 - a_k).$$

Then,

$$\lim_{k\to\infty} \varrho_k = \varrho_0 > 0. \tag{15}$$

Without loss of generality assume that $\varrho_k > 0$ for all $k \in \mathbb{N}$. Hence, $\inf_{k\in\mathbb{N}} \varrho_k > 0$ and, therefore, $\inf_{k\in\mathbb{N}} \|a_k - x_0\| > 0$. Since the sequence

$$t_k = \frac{2^{-k} + \sqrt{\varrho_k - \varrho_0}}{\|a_k - x_0\|}$$

tends to zero, we may suppose that $t_k < 1$ for all $k \in \mathbb{N}$. Observe that the sequence

$$x_k = x_0 + t_k(a_k - x_0)$$

converges to x_0. Since p is the Fréchet derivative of $\varrho_M(\cdot, A)$ at x_0, we get

$$\lim_{k\to\infty} \frac{\varrho_M(x_k, A) - \varrho_M(x_0, A) - \langle p, x_k - x_0\rangle}{\|x_k - x_0\|} = 0.$$

Hence, taking into account that $\varrho_M(x_k, A) \le \mu_M(x_k - a_k) = (1-t_k)\mu_M(x_0 - a_k) = (1-t_k)\varrho_k$, we obtain

$$\liminf_{k\to\infty} \frac{(1-t_k)\varrho_k - \varrho_0 - t_k\langle p, a_k - x_0\rangle}{t_k\|a_k - x_0\|} \ge 0.$$

Thus, thanks to $\frac{\varrho_k - \varrho_0}{t_k\|a_k - x_0\|} = \frac{\varrho_k-\varrho_0}{2^{-k}+\sqrt{\varrho_k-\varrho_0}} \to 0$ (as $k \to \infty$), we have

$$\liminf_{k\to\infty} \frac{\langle p, x_0 - a_k\rangle - \varrho_0}{\|a_k - x_0\|} \ge 0. \tag{16}$$

Observe that the points

$$y_k = \frac{x_0 - a_k}{\varrho_k} \tag{17}$$

belong to M for all $k \in \mathbb{N}$.

Suppose that $\{y_k\}$ is unbounded. Without loss of generality assume that $\lim_{k\to\infty}\|y_k\| = +\infty$ and $\|y_k - z_0\| > 1$ for all $k \in \mathbb{N}$. Then, (16) and (17) imply that $\liminf_{k\to\infty}\frac{\langle p, y_k\rangle}{\|y_k\|} \ge 0$. Using the convexity of M and the inclusions $z_0 \in M$, $y_k \in M$ we deduce that the points

$$z_k = z_0 + \frac{y_k - z_0}{\|y_k - z_0\|}$$

belong to M as well. Therefore, $s(p, M) \ge \langle p, z_k\rangle$ and $\liminf_{k\to\infty}\langle p, z_k\rangle = \langle p, z_0\rangle + \liminf_{k\to\infty}\frac{\langle p, y_k\rangle}{\|y_k\|} \ge \langle p, z_0\rangle = s(p, M)$. Thus, $\lim_{k\to\infty}\langle p, z_k\rangle = s(p, M)$ and Lemma 3.3 implies that $\lim_{k\to\infty}\|z_k - z_0\| = 0$. However, $\|z_k - z_0\| = 1$ for all $k \in \mathbb{N}$. The contradiction shows that $\{y_k\}$ is bounded. This and (15)–(17) imply that

$$\liminf_{k\to\infty}\langle p, y_k\rangle \ge 1.$$

Using Lemma 3.2 and the inclusion $y_k \in M$ we get $\lim_{k\to\infty}\langle p, y_k\rangle = s(p, M) = 1$. Thus, according to Lemma 3.3, we deduce the convergence of $\{y_k\}$ and, therefore, the convergence of $\{a_k\}$. Consequently, $x_0 \in T_M(A)$. □

Theorem 1.7 in [7] implies the following lemma.

Lemma 3.5. *Let E be a reflexive Banach space, $x_0 \in E$, $\delta > 0$ and let $f : \operatorname{int}\mathfrak{B}_\delta(x_0) \to \mathbb{R}$ be a Lipschitz continuous and not constant function. Then f has a nonzero Fréchet derivative at some point $x \in \operatorname{int}\mathfrak{B}_\delta(x_0)$.*

Lemma 3.6. *Let $M \subset E$ be a quasiball and $p \in b(M) \setminus b_1(M)$. Then for the half-space*

$$A = \{x \in E \mid \langle p, x\rangle \le 0\}$$

we have

$$S_M^{\mathrm{var}}(A) = \{x \in E \mid \langle p, x\rangle \ge 0\} \tag{18}$$

and

$$T_M(A) \subset A. \tag{19}$$

Proof. We claim that

$$\varrho_M(x, A) = \frac{\langle p, x\rangle}{s(p, M)}, \quad \forall x \in E \setminus A. \tag{20}$$

Indeed, fix any $x \in E \setminus A$. If $x \in A + tM$ for some $t > 0$, then we get $\langle p, x\rangle \le s(p, A) + ts(p, M) = ts(p, M)$. Therefore, according to (13), we deduce that $\varrho_M(x, A) \ge \frac{\langle p,x\rangle}{s(p,M)}$. Assume that $\varrho_M(x, A) > \frac{\langle p,x\rangle}{s(p,M)}$. In this case, by virtue of (13), one can find $t_1 > \frac{\langle p,x\rangle}{s(p,M)}$ such that $x \notin A + t_1 M$. Since $s(p, M) > \frac{\langle p,x\rangle}{t_1}$, there exists $z \in M$ with $\langle p, z\rangle > \frac{\langle p,x\rangle}{t_1}$, that is, $\langle p, x - t_1 z\rangle < 0$. Consequently, $x - t_1 z \in A$, which contradicts the assumption $x \notin A + t_1 M$. This proves (20). The equality (18) follows immediately from (4) and (20).

In order to deduce (19), assume the contrary: there exists $x \in T_M(A) \setminus A$. Then, according to Remark 3.1, we have $P_M(x, A) = \{a\}$ for some $a \in E$. Since $x \in E\setminus A$, using (20), we get $\varrho_M(x, A) = \frac{\langle p,x\rangle}{s(p,M)} > 0$. Consequently, using (11), for $z = \frac{x-a}{\varrho_M(x,A)}$ we have $\mu_M(z) \le 1$ and, therefore, $z \in M$. Thus, $\langle p, z\rangle = \frac{\langle p,x\rangle - \langle p,a\rangle}{\varrho_M(x,A)} \ge \frac{\langle p,x\rangle}{\varrho_M(x,A)} = s(p, M)$, which contradicts the assumption $p \notin b_1(M)$. □

Lemma 3.7. *Let $\varepsilon_0 > \varepsilon > 0$, $\alpha > 0$, $x_0, x \in E$ and a quasiball $M \subset E$ be such that $\mathfrak{B}_\alpha \subset M$ and $2\|x - x_0\| \le (\varepsilon_0 - \varepsilon)\alpha$. Then $P_M^\varepsilon(x, A) \subset P_M^{\varepsilon_0}(x_0, A)$ for any set $A \subset E$.*

Proof. Since $\pm(x - x_0) \in \frac{\varepsilon_0-\varepsilon}{2}\mathfrak{B}_\alpha \subset \frac{\varepsilon_0-\varepsilon}{2}M$, it follows that $\mu_M(x - x_0) \le \frac{\varepsilon_0-\varepsilon}{2}$ and $\mu_M(x_0 - x) \le \frac{\varepsilon_0-\varepsilon}{2}$. This and (14) imply that $\varrho_M(x, A) \le \varrho_M(x_0, A) + \frac{\varepsilon_0-\varepsilon}{2}$. Therefore, using (12), for every $y \in P_M^\varepsilon(x, A)$ we have $\mu_M(x - y) \le \varrho_M(x, A) + \varepsilon \le \varrho_M(x_0, A) + \frac{\varepsilon_0+\varepsilon}{2}$. Consequently, $\mu_M(x_0 - y) \le \mu_M(x - y) + \mu_M(x_0 - x) \le \varrho_M(x_0, A) + \varepsilon_0$. Thus, $y \in P_M^{\varepsilon_0}(x_0, A)$. □

Recall that a subset S of a topological space X is called a G_δ *set*, if $S = \bigcap_{n\in\mathbb{N}} S_n$ where S_n are open in X.

Lemma 3.8. *Let M be a quasiball in a Banach space E and $A \subset E$ be a closed set. Then*

(*i*) $T_M(A) = \{x \in E \mid \lim_{\varepsilon\to+0} \operatorname{diam} P_M^\varepsilon(x, A) = 0\}$;

(*ii*) $T_M(A)$ *is a G_δ subset of E.*

Proof. (i) Denote $T = \{x \in E \mid \lim_{\varepsilon\to+0} \operatorname{diam} P_M^\varepsilon(x,A) = 0\}$. Take $x_0 \in T$. Since $\{P_M^{1/k}(x_0,A)\}_{k\in\mathbb{N}}$ is a sequence of nested closed sets with diameters tending to zero, we deduce that the sequence has some point

$$a_0 \in \bigcap_{k\in\mathbb{N}} P_M^{1/k}(x_0,A).$$

Let $\{a_k\} \subset A$ be a minimizing sequence of the problem (2). Then the sequence $\varepsilon_k = \mu_M(x_0 - a_k) - \varrho_M(x_0,A)$ tends to zero. Hence, taking into account that $a_k \in P_M^{\varepsilon_k}(x_0,A)$ and $x_0 \in T$, we obtain $\lim_{k\to\infty} \|a_k - a_0\| = 0$. Thus, $x_0 \in T_M(A)$ and, consequently, we get $T \subset T_M(A)$.

To prove the reverse inclusion, fix any $x_0 \in T_M(A)$. Let $\{\varepsilon_k\}$ be an infinitesimal sequence of positive numbers. Using the well posedness we deduce that every sequence $a_k \in P_M^{\varepsilon_k}(x_0,A)$ converges. Consequently, $\lim_{\varepsilon\to+0} \operatorname{diam} P_M^\varepsilon(x,A) = 0$, that is, $x_0 \in T$. Thus, $T_M(A) = T$.

(ii) For every $k \in \mathbb{N}$ we denote

$$X_k = \left\{x \in E \mid \exists\varepsilon > 0 : \operatorname{diam} P_M^\varepsilon(x,A) < \frac{1}{k}\right\}.$$

We claim that the sets X_k are open. Indeed, fix any $k \in \mathbb{N}$ and $x_0 \in X_k$. Then, there exists $\varepsilon_0 > 0$ such that $\operatorname{diam} P_M^{\varepsilon_0}(x_0,A) < \frac{1}{k}$. Take $\varepsilon \in (0,\varepsilon_0)$. It follows from Lemma 3.7 that there is $\gamma > 0$ such that $P_M^\varepsilon(x,A) \subset P_M^{\varepsilon_0}(x_0,A)$ whenever $x \in \mathfrak{B}_\gamma(x_0)$. Consequently, $\mathfrak{B}_\gamma(x_0) \subset X_k$, which proves that X_k is open. To complete the proof, it suffices to observe that (i) implies

$$T_M(A) = \bigcap_{k\in\mathbb{N}} X_k. \qquad \square$$

Proof of Theorem 2.3. $(i) \Rightarrow (ii)$. Suppose that (i) holds. Take any closed set $A \subset E$, $x_0 \in S_M^{\text{var}}(A)$, and $\delta > 0$. Lemma 3.5 and the continuity of $\varrho_M(\cdot, M)$ imply that there exists $x \in \mathfrak{B}_\delta(x_0)$ such that $\varrho_M(x,A) > 0$ and the function $\varrho_M(\cdot,A)$ has the Fréchet derivative $p \in E^* \setminus \{0\}$ at the point x. Lemma 3.2 and (i) imply that $p \in b(M) = b_1(M)$. Hence, using Lemma 3.4 we deduce that $x \in T_M(A)$. Thus, for every $\delta > 0$ we have $\mathfrak{B}_\delta(x_0) \cap T_M(A) \neq \emptyset$. Hence, $x_0 \in \overline{T_M(A)}$ and, therefore, $S_M^{\text{var}}(A) \subset \overline{T_M(A)}$.

$(ii) \Rightarrow (i)$. Suppose that (ii) holds. Lemma 3.6 implies that $b_1(M) = b(M)$. Theorem 3.3 from [6] shows that E is reflexive, M is strictly convex and Kadec. Though in [6] boundedness of the quasiball M (in [6] it is denoted through C) was assumed, the proof of this fact remains valid for an unbounded quasiball.

$(ii) \Rightarrow (iii)$. Suppose that (ii) holds and $A \subset E$ is a closed set. Using Lemma 3.8(ii) we deduce that $S_M^{\text{var}}(A) \bigcap T_M(A)$ is a dense G_δ subset of the topological space $S_M^{\text{var}}(A)$, that is a topological subspace of E. Therefore $S_M^{\text{var}}(A) \bigcap T_M(A)$ is a residual subset of $S_M^{\text{var}}(A)$.

$(iii) \Rightarrow (ii)$ follows from the Baire category theorem. $\square$

4. Proofs of Lemma 2.4 and Theorem 2.5

Proof of Lemma 2.4. Let $M \subset E$ be a quasiball. Fix $\lambda > 1$, $\delta > 0$, and $d \in E$. Denote

$$A_0 = -E \setminus \operatorname{int}\left((M + \mathfrak{B}_\delta) \bigcup (\lambda M - d) \right). \tag{21}$$

Since (5) may be equivalently rewritten as

$$w \in A_0 + (1+\varepsilon)M, \quad \forall w \in \mathfrak{B}_\delta, \ \forall \varepsilon > 0,$$

using (13), we deduce that (5) is equivalent to

$$\varrho_M(w, A_0) \leq 1, \quad \forall w \in \mathfrak{B}_\delta. \tag{22}$$

It follows from (13) and (21) that

$$\varrho_M(w, A_0) \geq 1, \quad \forall w \in \mathfrak{B}_\delta$$

and

$$\varrho_M(d, A_0) \geq \lambda.$$

Hence, if M is equidistable, there exists the closed set A_0, defined by (21) such that

$$\sup_{x \in E} \varrho_M(x, A_0) \geq \varrho_M(d, A_0) \geq \lambda > 1 = \varrho_M(w, A_0), \quad \forall w \in \mathfrak{B}_\delta.$$

In this case $0 \in S_M^{\mathrm{int}}(A_0) \cap S_M^{\mathrm{const}}(A_0)$. Therefore $0 \notin S_M^{\mathrm{var}}(A_0)$ and the inclusion $S_M^{\mathrm{int}}(A_0) \subset S_M^{\mathrm{var}}(A_0)$ is false. Thus, $(ii) \Rightarrow (i)$ is proved.

For proving $(i) \Rightarrow (ii)$ assume that (ii) is false, i.e., there exist a closed set $A \subset E$ and a point

$$x_0 \in S_M^{\mathrm{int}}(A) \setminus S_M^{\mathrm{var}}(A).$$

Denote $\varrho_0 = \varrho_M(x_0, A)$. Since $x_0 \in S_M^{\mathrm{int}}(A)$, we have $0 < \varrho_0 < \sup_{x \in E} \varrho_M(x, A)$. Consider

$$A_1 = \frac{1}{\varrho_0}(A - x_0).$$

Thus, $1 = \varrho_M(0, A_1) < \sup_{x \in E} \varrho_M(x, A_1)$. Hence, there exists $d \in E$ such that $1 < \varrho_M(d, A_1)$. Fix $\lambda \in (1, \varrho_M(d, A_1))$. Using (13), we deduce that

$$d \notin A_1 + \lambda M. \tag{23}$$

Since $x_0 \notin S_M^{\mathrm{var}}(A)$, we have $0 \notin S_M^{\mathrm{var}}(A_1)$. Therefore, (4) implies that there exists $\delta > 0$ such that

$$\varrho_M(x, A_1) = 1, \quad \forall x \in \mathfrak{B}_{2\delta}. \tag{24}$$

To prove that

$$\mathfrak{B}_\delta \bigcap (A_1 + M) = \emptyset, \tag{25}$$

we assume the contrary: there exists $x \in \mathfrak{B}_\delta \cap (A_1 + M)$. Then there is $a \in A_1$ with $x - a \in M$. Since $x \in \mathfrak{B}_\delta$, one can find $t \in (0,1)$ such that $x_t = (1-t)x + ta$ satisfies the inclusion $x_t \in \mathfrak{B}_{2\delta}$. Thus, $\varrho_M(x_t, A_1) \le \mu_M(x_t - a) = (1-t)\mu_M(x-a) \le 1 - t < 1$, which contradicts (24). This proves (25). It follows from (23) and (25) that

$$-A_1 \subset E \setminus (\lambda M - d), \qquad -A_1 \subset E \setminus (M + \mathfrak{B}_\delta).$$

Therefore the set A_0, defined by (21), satisfies the inclusion $A_1 \subset A_0$. This and (24) imply (22), which is equivalent to (5). Thus, M is equidistable and the proof of $(i) \Rightarrow (ii)$ is completed.

Since $S_M^{\mathrm{var}}(A)$ is closed, the inclusion $S_M^{\mathrm{int}}(A) \subset S_M^{\mathrm{var}}(A)$ is equivalent to the inclusion $\overline{S_M^{\mathrm{int}}(A)} \subset S_M^{\mathrm{var}}(A)$. The reverse inclusion is always true according to (4) and (6). Thus, $(ii) \Leftrightarrow (iii)$. □

Proof of Theorem 2.5. In the case that M is not equidistable Lemma 2.4 implies that each statement of Theorem 2.5 is equivalent to the corresponding statement of Theorem 2.3. Hence, Theorem 2.3 implies that in this case the statements (i), (ii) and (iii) of Theorem 2.5 are equivalent.

Now assume that M is equidistable. To complete the proof it suffices to show that in this case statements (ii) and (iii) of Theorem 2.5 are false. Lemma 2.4 in this case implies that there is a closed set $A \subset E$ such that $S_M^{\mathrm{int}}(A) \not\subset S_M^{\mathrm{var}}(A)$. Hence, there exists $x_0 \in S_M^{\mathrm{int}}(A) \setminus S_M^{\mathrm{var}}(A)$. Consequently, one can find $\delta > 0$ such that $\mathfrak{B}_\delta(x_0) \subset S_M^{\mathrm{int}}(A) \cap S_M^{\mathrm{const}}(A)$. This and Lemma 2.2 imply that $\mathfrak{B}_\delta(x_0) \cap T_M(A) = \emptyset$. Therefore, using the inclusion $\mathfrak{B}_\delta(x_0) \subset S_M^{\mathrm{int}}(A)$, we deduce that in this case the statement (ii) of Theorem 2.5 is false. According to the Baire category theorem, we deduce that in this case the statement (iii) of Theorem 2.5 is false as well. □

5. Proofs of Lemmata 2.6–2.10

Remark 5.1. It follows immediately from the definition of a parabolic set that the parabolic property is invariant with respect to shifting and scaling, i.e. if a set $M \subset E$ is parabolic then $\lambda M + x$ is parabolic for every $x \in E$, $\lambda \in \mathbb{R}$.

Lemma 5.2. *Let $\beta > 1$. A closed convex set $M \subset E$ is parabolic iff $M \setminus (\beta M - y)$ is bounded for all $y \in E$.*

Proof. For every $\gamma > 1$ we consider the condition

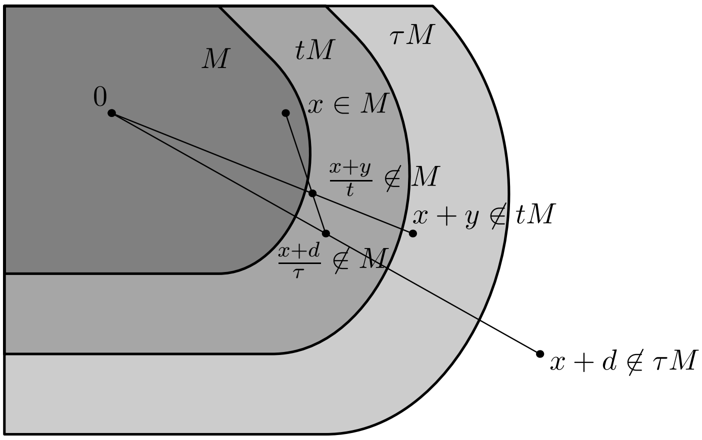
$$\mathfrak{C}(\gamma){:}\quad M \setminus (\gamma M - y) \text{ is bounded for all } y \in E.$$

Let $1 < t < \tau$. Let us prove that $\mathfrak{C}(\tau)$ and $\mathfrak{C}(t)$ are equivalent.

Using Remark 5.1, we assume without loss of generality that $0 \in M$. Then $tM \subset \tau M$. Therefore $\mathfrak{C}(t) \Rightarrow \mathfrak{C}(\tau)$. Conversely, suppose $\mathfrak{C}(\tau)$ and take any $y \in E$. Denote $\lambda = \frac{(t-1)\tau}{t(\tau-1)} \in (0,1)$ and $d = \frac{\tau-1}{t-1} y$.

We claim that

$$M \setminus (tM - y) \subset M \setminus (\tau M - d). \tag{26}$$



Indeed, if (26) is false, there exists $x \in M \setminus (tM - y)$ such that $x \in \tau M - d$. Hence, using the convexity of M, we deduce that $(1-\lambda)x + \frac{\lambda}{\tau}(x+d) \in M$. Taking into account that $(1-\lambda)x + \frac{\lambda}{\tau}(x+d) = \frac{1}{t}(x+y)$, we get a contradiction to the inclusion $x \in M \setminus (tM - y)$, which proves (26). Therefore, $\mathfrak{C}(\tau) \Rightarrow \mathfrak{C}(t)$. Thus, $\mathfrak{C}(\tau) \Leftrightarrow \mathfrak{C}(t)$, in particular, $\mathfrak{C}(\beta)$ is equivalent to $\mathfrak{C}(2)$, that is the parabolicity of M. $\square$

Lemma 5.3. *Let $M \subset E$ be a parabolic set. Then $b(M) \setminus \{0\}$ is open.*

Proof. We assume without loss of generality that $0 \in M$. Fix a functional $p_0 \in b(M) \setminus \{0\}$. To complete the proof it suffices to show that $p_0 \in \operatorname{int} b(M)$. Denote

$$M_1 = \{x \in M \mid \langle p_0, x\rangle \geq -1\}\,. \tag{27}$$

Since $p_0 \neq 0$ and $s(p_0, M) \in \mathbb{R}$, there exists $d \in E$ such that $\langle p_0, d\rangle = 2s(p_0, M) + 2$. We claim that

$$M_1 \subset M \setminus (2M - d). \tag{28}$$

Indeed, if (28) is false, there exists $x \in M_1 \cap (2M - d)$. Therefore $\langle p_0, x\rangle \leq 2s(p_0, M) - \langle p_0, d\rangle = -2$. Hence, using (27), we get a contradiction to the inclusion $x \in M_1$. This proves (28). Since M is parabolic, (28) implies that M_1 is bounded. Therefore, there exists $R > 0$ such that $M_1 \subset \operatorname{int} \mathfrak{B}_R$.

Fix a functional $p \in \mathfrak{B}_{1/R}(p_0)$ and a vector $x \in M$. We claim that

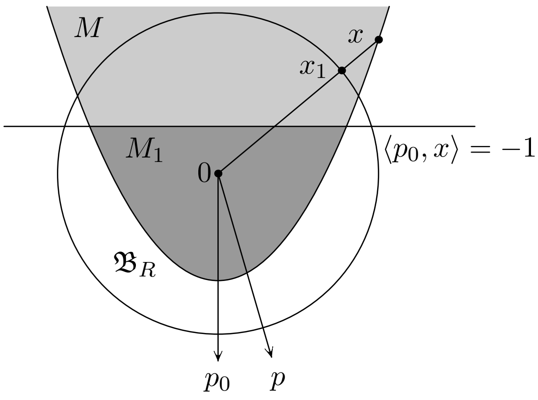
$$\langle p, x\rangle \leq R\|p_0\| + 1. \tag{29}$$

Indeed, in the case that $\|x\| \leq R$ we have

$$\langle p, x\rangle \leq R\|p\| \leq R\left(\|p_0\| + \frac{1}{R}\right) = R\|p_0\| + 1.$$

Therefore, in this case (29) holds.

Now assume that $\|x\| > R$. Denote $x_1 = \frac{R}{\|x\|}x$. Using that $M_1 \subset \operatorname{int} \mathfrak{B}_R$ and



$\|x_1\| = R$, we get $x_1 \notin M_1$. Since $x_1 \in [0, x]$ and since M is convex and contains the points 0 and x, we deduce that $x_1 \in M$. Consequently, $x_1 \in M \setminus M_1$. Hence, (27) implies that $\langle p_0, x_1\rangle < -1$. Thus,

$$\langle p, x_1\rangle \le \langle p_0, x_1\rangle + \|p - p_0\| \cdot \|x_1\| < -1 + \frac{1}{R}R = 0.$$

Using the equality $x_1 = \frac{R}{\|x\|}x$, we get $\langle p, x\rangle < 0$. Hence, (29) holds. Thus, for every $p \in \mathfrak{B}_{1/R}(p_0)$ we have $s(p, M) \le R\|p_0\| + 1 < +\infty$. Therefore, $\mathfrak{B}_{1/R}(p_0) \subset b(M)$ and the inclusion $p_0 \in \operatorname{int} b(M)$ is proved. □

Lemma 5.4. *Let M be a closed convex subset of a reflexive Banach space E such that $b(M) \setminus \{0\}$ is open. Then $b_1(M) = b(M)$.*

Proof. Fix any $p_0 \in b(M) \setminus \{0\}$. Since $b(M) \setminus \{0\}$ is open, we have $p_0 \in \operatorname{int} b(M)$. Using [3, Chapter 1, Proposition 5.2], we deduce that the subdifferential of the support function $s(\cdot, M)$ is not empty at p_0. Taking into account that E is reflexive, we may assume that this subdifferential contains a vector $x_0 \in E$, i.e., $\langle p-p_0, x_0\rangle \le s(p, M) - s(p_0, M)$ for all $p \in E^*$. Hence, $x_0 \in M$ and $\langle p_0, x_0\rangle = s(p_0, M)$. Therefore, we get $p_0 \in b_1(M)$. Consequently, $b(M) \setminus \{0\} \subset b_1(M)$ and we have $b_1(M) = b(M)$. □

Proof of Lemma 2.6. Assume that M is a parabolic quasiball in a reflexive Banach space E. Lemmata 5.3 and 5.4 imply that $b_1(M) = b(M)$. To complete the proof it suffices to show that M is not equidistable. Assume the contrary: there exist $\lambda > 1$, $\delta > 0$ and $d \in E$ such that (5) holds. Hence, for every $k \in \mathbb{N}$ there is

$$x_k \in M: \quad \left(1 + \frac{1}{k}\right) x_k \notin \operatorname{int}\left((M + \mathfrak{B}_\delta) \bigcup (\lambda M - d)\right).$$

Thus, $\left(1 + \frac{1}{k}\right) x_k \notin \operatorname{int} \mathfrak{B}_\delta(x_k)$ and, therefore, $\|x_k\| \ge k\delta \to +\infty$ as $k \to \infty$. Fix β, γ and $k_0 \in \mathbb{N}$ such that $1 + \frac{1}{k_0} < \beta < \gamma < \lambda$. Then,

$$\left(1 + \frac{1}{k}\right) x_k \in \beta M \setminus (\gamma M - d), \quad \forall k \ge k_0.$$

Hence, $M \setminus \left(\frac{\gamma}{\beta}M - \frac{d}{\beta}\right)$ is unbounded. This contradicts Lemma 5.2. $\square$

Proof of Lemma 2.7. Let M be a strictly convex quasiball in a finite-dimensional normed space E. Assume that $b_1(M) = b(M)$. In view of Lemma 2.6 to complete the proof it suffices to show that M is parabolic. Assume the contrary: there exists $d \in E$ such that $M \setminus (2M - d)$ is unbounded. Consequently, there is an infinitely large sequence

$$\{x_k\} \subset M \setminus (2M - d). \tag{30}$$

Assume without loss of generality that $x_k \neq 0$ for all $k \in \mathbb{N}$. Since E is finite-dimensional, by extracting a subsequence, we may also assume that

$$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = y \tag{31}$$

for some $y \in E$.

To prove that the line

$$\ell = \{d + ty \mid t \in \mathbb{R}\}$$

intersects $\operatorname{int} M$, we assume the contrary. In this case, according to the separation theorem, there exists a functional $p \in E^* \setminus \{0\}$ such that

$$\langle p, x\rangle < \langle p, d + ty\rangle, \quad \forall x \in \operatorname{int} M, \ \forall t \in \mathbb{R}.$$

Hence, $p \in b(M)$ and $\langle p, y\rangle = 0$. Since $b_1(M) = b(M)$, we have $p \in b_1(M)$ and, therefore, there exists $x_0 \in M$ such that $\langle p, x_0\rangle = s(p, M)$.

Since $\{x_k\}$ is infinitely large, there exists $k_0 \in \mathbb{N}$ such that $\|x_k - x_0\| > 1$ whenever $k \geq k_0$. Hence, taking into account that M is convex and $x_0, x_k \in M$, we deduce that $x_0 + \frac{x_k - x_0}{\|x_k - x_0\|} \in M$ for all $k \geq k_0$. Since $\{x_k\}$ is infinitely large, (31) implies that $\lim_{k\to\infty} \frac{x_k - x_0}{\|x_k - x_0\|} = y$. Therefore, using the closedness of M, we deduce that $x_0 + y \in M$. Thus, for two different points x_0 and $x_0' = x_0 + y$ of M we have $\langle p, x_0'\rangle = \langle p, x_0\rangle = s(p, M)$, which contradicts the strict convexity of M.

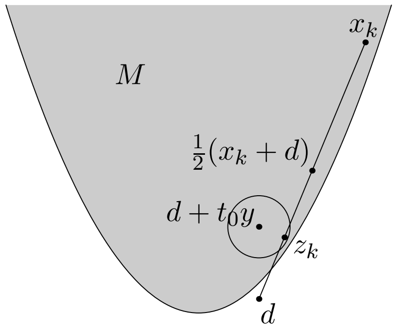
This proves that $\ell \cap \operatorname{int} M \neq \emptyset$ and, therefore, there exists $t_0 \in \mathbb{R}$:

$$d + t_0 y \in \operatorname{int} M.$$

For every $k \in \mathbb{N}$ denote

$$t_k = \frac{t_0}{\|x_k\|}, \qquad z_k = (1 - t_k)d + t_k x_k, \qquad \lambda_k = \frac{1}{2(1 - t_k)}.$$

Since $\{x_k\}$ is infinitely large, we have $\lim_{k\to\infty} t_k = 0$, $\lim_{k\to\infty} \lambda_k = \frac{1}{2}$ and, using (31), we get $\lim_{k\to\infty} z_k = d + t_0 y \in \operatorname{int} M$. Therefore, there exists $k \in \mathbb{N}$ such that $\lambda_k \in [0, 1]$ and $z_k \in M$. Consequently, thanks to the convexity of M, we get $\frac{x_k + d}{2} = (1 - \lambda_k)x_k + \lambda_k z_k \in M$. This contradicts (30). $\square$



Proof of Lemma 2.9. Let a function $f : E \to \mathbb{R}$ be convex, continuous, and coercitive. Fix any $x_0 \in E$ and $y_0 \in \mathbb{R}$. It suffices to prove that the set

$$M_1 = \operatorname{epi} f \setminus (2 \operatorname{epi} f - (x_0, y_0)) \tag{32}$$

is bounded. Since f is continuous at x_0, there exist $\delta > 0$ and $C \in \mathbb{R}$:

$$f(x) \le C, \quad \forall x \in \mathfrak{B}_\delta(x_0). \tag{33}$$

Using the coercitivity of f, one can find $R \ge 2\delta$ such that

$$\frac{f(x)}{\|x - x_0\|} > \frac{C + |y_0|}{\delta}, \quad \forall x \in E \setminus \mathfrak{B}_R(x_0). \tag{34}$$

We claim that

$$\|x - x_0\| \le R, \qquad y \le \frac{R}{\delta}(C + |y_0|), \qquad \forall (x, y) \in M_1. \tag{35}$$

Indeed, fix any $(x, y) \in M_1$. Then, according to (32), we have

$$y \ge f(x), \qquad \frac{y + y_0}{2} < f\left(\frac{x + x_0}{2}\right). \tag{36}$$

Take any $\lambda \in \left[0, \frac{1}{2}\right]$. The convexity of f implies that

$$f\left(\frac{x + x_0}{2}\right) \le \frac{1 - 2\lambda}{2(1 - \lambda)} f(x) + \frac{1}{2(1 - \lambda)} f((1 - \lambda)x_0 + \lambda x).$$

Using (36), we get

$$\lambda f(x) \le \lambda y < f((1 - \lambda)x_0 + \lambda x) - (1 - \lambda)y_0, \quad \forall \lambda \in \left[0, \tfrac{1}{2}\right]. \tag{37}$$

Assume that $\|x - x_0\| > R$. Now for $\lambda = \frac{\delta}{\|x - x_0\|}$ we have $(1 - \lambda)x_0 + \lambda x \in \mathfrak{B}_\delta(x_0)$ and, therefore, (33) implies that $f((1 - \lambda)x_0 + \lambda x) \le C$. Hence, using (37), we get $\frac{\delta}{\|x - x_0\|} f(x) < C - (1 - \lambda)y_0 \le C + |y_0|$, which contradicts (34). This proves the inequality $\|x - x_0\| \leq R$. Hence, for $\lambda_1 = \frac{\delta}{R}$ we have $(1-\lambda_1)x_0 + \lambda_1 x \in \mathfrak{B}_\delta(x_0)$. Therefore, taking into account (33) and (37), we deduce that $\frac{\delta}{R}y < C + |y_0|$. Thus, (35) is proved. The convexity and continuity of f imply that f is bounded from below on any bounded set. Hence, it follows from (35) and (36) that M_1 is bounded. □

Proof of Lemma 2.10. Denote

$$g(x) = f(x) - \Lambda_f(x), \quad \forall x \in E.$$

Rewrite (8) and (9) as follows:

$$\lim_{\|x\| \to +\infty} \frac{|g(x)|}{\|x\|} = 0, \tag{38}$$

$$\lim_{\|x\| \to +\infty} g(x) = -\infty. \tag{39}$$

Fix any $x_0 \in E$ and $y_0 \in \mathbb{R}$. It suffices to prove that the set M_1, defined by (32), is bounded. It follows from (39) that there exists $r > 0$:

$$g(x) + L_f\|x_0\| + |y_0| < -1, \quad \forall x \in E: \ \|x\| \geq r, \tag{40}$$

where L_f is a Lipschitz constant of f. According to (38), one can find $R \geq 2r$ such that

$$\frac{|g(x)|}{\|x\|} < \frac{1}{r}, \quad \forall x \in E: \ \|x\| \geq R. \tag{41}$$

Fix any $(x, y) \in M_1$. As in the proof of Lemma 2.9, we get (37). Hence, using the Lipschitz continuity of f, we have

$$\lambda f(x) \leq \lambda y < f(\lambda x) + L_f\|x_0\| + |y_0|, \quad \forall \lambda \in \left[0, \tfrac{1}{2}\right]. \tag{42}$$

Consequently, because $\Lambda_f(\cdot)$ is positively homogeneous, we deduce that

$$\lambda g(x) < g(\lambda x) + L_f\|x_0\| + |y_0|, \quad \forall \lambda \in \left[0, \tfrac{1}{2}\right]. \tag{43}$$

Assume that $\|x\| \geq R$. Putting $\lambda = \frac{r}{\|x\|}$ in (43), we get

$$\frac{r}{\|x\|} g(x) < g\left(\frac{r}{\|x\|}x\right) + L_f\|x_0\| + |y_0|.$$

This and (40) imply that $\frac{r}{\|x\|}g(x) < -1$, which contradicts (41). Hence, the assumption $\|x\| \geq R$ is false and we have $\|x\| < R$. Thus, using the Lipschitz continuity of f and putting $\lambda = \frac{1}{2}$ in (42), we deduce that

$$\begin{aligned} f(0) - L_f R < f(x) \leq y &< 2f\left(\frac{x}{2}\right) + 2L_f\|x_0\| + 2|y_0| \\ &< 2f(0) + L_f(R + 2\|x_0\|) + 2|y_0|. \end{aligned}$$

Consequently, M_1 is bounded. □

6. Proof of Theorem 2.11

Lemma 6.1. *Let $M \subset E$ be a quasiball, $R > 0$, and $x, y \in E$ satisfy the inequalities $0 < \|x\| \le R\mu_M(x)$ and $0 < \|y\| \le R\mu_M(y)$. Then*

$$\frac{\min\{\mu_M(x), \mu_M(y)\}}{R} \cdot \delta_M^R\left(\left\|\frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)}\right\|\right) \le \mu_M(x) + \mu_M(y) - \mu_M(x+y),$$

where $\delta_M^R(\cdot)$ is defined by (10).

Proof. Denote

$$a = \frac{x}{\mu_M(x)}, \qquad b = \frac{y}{\mu_M(y)}, \qquad c = \frac{a+b}{2}, \qquad \delta = \delta_M^R(\|a-b\|).$$

Since $a, b \in M \cap \mathfrak{B}_R$, we have $\|c\| \le R$ and, using (10), we get $\mathfrak{B}_\delta(c) \subset M$. Therefore, $c\left(1 + \frac{\delta}{R}\right) \in M$ and, consequently, $\mu_M(c) \le \frac{1}{1+\frac{\delta}{R}}$. Taking into account that $\delta = \delta_M^R(\|a-b\|) \le \frac{\|a-b\|}{2} \le R$, we deduce that

$$\mu_M(c) \le 1 - \frac{\delta}{2R}. \tag{44}$$

Denote $\mu_1 = \mu_M(x)$, $\mu_2 = \mu_M(y)$. Assume without loss of generality that $\mu_2 \le \mu_1$. Using the subadditivity of the Minkowski functional and the equality $\mu_M(a) = 1$, we get

$$\begin{aligned} \mu_M(x+y) &= \mu_M((\mu_1 - \mu_2)a + \mu_2(a+b)) \\ &\le (\mu_1 - \mu_2) \cdot \mu_M(a) + \mu_2 \cdot \mu_M(a+b) \\ &= \mu_1 - \mu_2 + 2\mu_2 \cdot \mu_M(c). \end{aligned}$$

This and (44) imply

$$\mu_M(x+y) \le \mu_1 + \mu_2 - \frac{\mu_2 \delta}{R} = \mu_1 + \mu_2 - \frac{\delta}{R}\min\{\mu_1, \mu_2\}. \qquad \square$$

Lemma 6.2. *Let $M \subset E$ be a quasiball, $A \subset E$ be a closed set, $\beta > 0$, $\varepsilon > 0$, $\lambda \in \left(0, \frac{1}{2}\right]$, $R > 0$,*

$$x_0 \in E, \qquad a_0 \in P_M(x_0, A), \qquad x_\lambda = (1-\lambda)x_0 + \lambda a_0,$$

$$\varrho_M(x_0, A) = \varrho_0, \qquad \|x_0\| \le \frac{R\varrho_0}{4}, \qquad \sup_{a \in P_M^{2\varepsilon\lambda}(x_\lambda, A)} \|a\| \le \frac{R\varrho_0}{4}, \qquad 2R\varepsilon < \varrho_0 \delta_M^R(\beta).$$

Then

$$\operatorname{diam} P_M^{2\varepsilon\lambda}(x_\lambda, A) \le 3\varrho_0 \beta.$$

Proof. Fix any $a \in P_M^{2\varepsilon\lambda}(x_\lambda, A)$. Since $\varrho_M(x_\lambda, A) \le \mu_M(x_\lambda - a_0) = (1-\lambda)\mu_M(x_0 - a_0) = (1-\lambda)\varrho_0$, we have

$$\mu_M(x_\lambda - a) \le (1-\lambda)\varrho_0 + 2\varepsilon\lambda. \tag{45}$$

Then, observing that

$$\mu_M(x_0 - a) \ge \varrho_0, \qquad \mu_M(x_0 - x_\lambda) = \lambda\varrho_0, \tag{46}$$

we get

$$\mu_M(x_0 - x_\lambda) + \mu_M(x_\lambda - a) - \mu_M(x_0 - a) \le 2\varepsilon\lambda. \tag{47}$$

It follows from (46) and the subadditivity of the Minkowski functional that

$$\mu_M(x_\lambda - a) \ge \mu_M(x_0 - a) - \mu_M(x_0 - x_\lambda) \ge (1-\lambda)\varrho_0 \ge \frac{\varrho_0}{2}. \tag{48}$$

Since $a_0 \in P_M(x_\lambda, A) \subset P_M^{2\varepsilon\lambda}(x_\lambda, A)$, we have $\|a_0\| \le \frac{R\varrho_0}{4}$. Consequently, $\|x_\lambda\| \le \max\{\|x_0\|, \|a_0\|\} \le \frac{R\varrho_0}{4}$. Thus, according to (48), we deduce that

$$0 < \|x_\lambda - a\| \le \|x_\lambda\| + \|a\| \le \frac{R\varrho_0}{2} \le R\mu_M(x_\lambda - a). \tag{49}$$

Using (46) and (48), we get

$$\|x_0 - x_\lambda\| = \lambda\|x_0 - a_0\| \le \lambda(\|x_0\| + \|a_0\|) \le \lambda R\varrho_0 = R\mu_M(x_0 - x_\lambda), \tag{50}$$

and

$$\min\{\mu_M(x_0 - x_\lambda), \mu_M(x_\lambda - a)\} = \lambda\varrho_0. \tag{51}$$

Lemma 6.1 together with (47), and (49)–(51) imply that

$$\frac{\lambda\varrho_0}{R}\,\delta_M^R\left(\left\|\frac{x_0 - x_\lambda}{\mu_M(x_0 - x_\lambda)} - \frac{x_\lambda - a}{\mu_M(x_\lambda - a)}\right\|\right) \le 2\varepsilon\lambda < \frac{\lambda\varrho_0}{R}\delta_M^R(\beta).$$

Consequently,

$$\left\|\frac{x_\lambda - a_0}{\mu_M(x_\lambda - a_0)} - \frac{x_\lambda - a}{\mu_M(x_\lambda - a)}\right\| = \left\|\frac{x_0 - x_\lambda}{\mu_M(x_0 - x_\lambda)} - \frac{x_\lambda - a}{\mu_M(x_\lambda - a)}\right\| < \beta.$$

Therefore,

$$\begin{aligned}
\|a - a_0\| &= \mu_M(x_\lambda - a_0)\left\|\frac{x_\lambda - a_0}{\mu_M(x_\lambda - a_0)} - \frac{x_\lambda - a}{\mu_M(x_\lambda - a_0)}\right\| \\
&\le \mu_M(x_\lambda - a_0)\left(\beta + \|x_\lambda - a\|\left|\frac{1}{\mu_M(x_\lambda - a)} - \frac{1}{\mu_M(x_\lambda - a_0)}\right|\right) \\
&\le (1-\lambda)\varrho_0\beta + \frac{\|x_\lambda - a\|}{\mu_M(x_\lambda - a)}|\mu_M(x_\lambda - a) - \mu_M(x_\lambda - a_0)| \\
&\le \varrho_0\beta + 2R\varepsilon\lambda < \varrho_0\left(\beta + \delta_M^R(\beta)\right) \le \frac{3}{2}\varrho_0\beta.
\end{aligned}$$

Thus,

$$\operatorname{diam} P_M^{2\varepsilon\lambda}(x_\lambda, A) \le 2 \sup_{a \in P_M^{2\varepsilon\lambda}(x_\lambda, A)} \|a - a_0\| \le 3\varrho_0\beta. \qquad \square$$

Proof of Theorem 2.11. Since M is boundedly uniformly convex, it is strictly convex and Kadec. It follows from Lemma 2.6 that M is not equidistable and $b_1(M) = b(M)$. Hence, Theorem 2.5 implies that

$$S_M^{\text{int}}(A) \subset \overline{T_M(A)}. \tag{52}$$

For every $x \in S_M^{\text{int}}(A)$ we define

$$\eta(x) = \lim_{\varepsilon \to +0} \sup_{a \in P_M^\varepsilon(x,A)} \|a\|. \tag{53}$$

Existence of this limits follows from monotonicity. We claim that

$$\lim_{\varepsilon \to +0} \lim_{\delta \to +0} \sup_{x \in \mathfrak{B}_\delta(x_0)} \sup_{a \in P_M^\varepsilon(x,A)} \|a\| \le \eta(x_0) < +\infty, \quad \forall x_0 \in S_M^{\text{int}}(A). \tag{54}$$

Indeed, fix any $x_0 \in S_M^{\text{int}}(A)$. Since, according to (6), we have $\varrho_M(x_0, A) < \sup_{x \in E} \varrho_M(x, A)$, then there exist $\varepsilon > 0$ and $x_1 \in E$ such that $\varrho_M(x_0, A) + \varepsilon < \varrho_M(x_1, A)$. Thus,

$$P_M^\varepsilon(x_0, A) \subset (x_0 - (\varrho_M(x_0, A) + \varepsilon)M) \setminus (x_1 - \varrho_M(x_1, A) \operatorname{int} M)$$

and Lemma 5.2 implies that $P_M^\varepsilon(x_0, A)$ is bounded. Therefore $\eta(x_0) \le \sup_{a \in P_M^\varepsilon(x_0,A)} \|a\| < +\infty$. The second inequality in (54) is proved. Now let us prove the first one. Fix any number $\eta_0 > \eta(x_0)$. Using (53), one can find $\varepsilon_0 > 0$ such that $\sup_{a \in P_M^{\varepsilon_0}(x_0,A)} \|a\| < \eta_0$. For $\varepsilon_1 = \frac{\varepsilon_0}{2}$ Lemma 3.7 implies that there exists $\delta_1 > 0$ such that $P_M^{\varepsilon_1}(x, A) \subset P_M^{\varepsilon_0}(x_0, A)$ whenever $x \in \mathfrak{B}_{\delta_1}(x_0)$. Then,

$$\lim_{\varepsilon \to +0} \lim_{\delta \to +0} \sup_{x \in \mathfrak{B}_\delta(x_0)} \sup_{a \in P_M^\varepsilon(x,A)} \|a\| \le \sup_{x \in \mathfrak{B}_{\delta_1}(x_0)} \sup_{a \in P_M^{\varepsilon_1}(x,A)} \|a\| \le \sup_{a \in P_M^{\varepsilon_0}(x_0,A)} \|a\| < \eta_0.$$

This completes the proof of (54). It follows from (54) that the function $\eta(\cdot)$ is upper semicontinuous. Therefore for every $n \in \mathbb{N}$ the set

$$S_n = \left\{ x \in S_M^{\text{int}}(A) \mid \|x\| < n,\ \eta(x) < n \right\} \tag{55}$$

is open.

According to Remark 3.1, for every $x \in T_M(A)$ there exists $\pi(x)$ such that $P_M(x, A) = \{\pi(x)\}$.

We claim that for every $n, k \in \mathbb{N}$ the set Y_k^n is a porous subset of $S_M^{\text{int}}(A)$, where $Y_k^n = S_n \setminus X_k^n$,

$$X_k^n = \bigcup_{x_0 \in S_n \bigcap T_M(A)} \ \bigcup_{\lambda \in \left(0, \frac{1}{2}\right]} \mathfrak{B}_{\lambda/k}\left((1-\lambda)x_0 + \lambda\pi(x_0)\right). \tag{56}$$

Indeed, fix any $n, k \in \mathbb{N}$ and denote $\vartheta = \frac{1}{5kn}$. Take any $y \in S_M^{\rm int}(A)$ and $r \in (0,1]$. At first, assume that

$$\mathfrak{B}_{\vartheta r}(y) \cap S_n \neq \emptyset, \tag{57}$$

i.e., there exists $y_1 \in \mathfrak{B}_{\vartheta r}(y) \cap S_n$. According to (52) and (55), we have $y_1 \in S_n \subset S_M^{\rm int}(A) \subset \overline{T_M(A)}$. Since S_n is open, one can find

$$x_0 \in \mathfrak{B}_{\vartheta r}(y_1) \bigcap S_n \bigcap T_M(A). \tag{58}$$

Denote $\lambda = \frac{r}{5n}$, $x_\lambda = (1-\lambda)x_0 + \lambda \pi(x_0)$. Using (56), we have

$$\mathfrak{B}_{\vartheta r}(x_\lambda) = \mathfrak{B}_{\lambda/k}(x_\lambda) \subset X_k^n. \tag{59}$$

It follows from (55) that $\|x_0 - \pi(x_0)\| < 2n$. Therefore,

$$\begin{aligned} \|x_\lambda - y\| &\leq \|x_\lambda - x_0\| + \|x_0 - y_1\| + \|y_1 - y\| \\ &\leq \lambda\|x_0 - \pi(x_0)\| + 2\vartheta r < 2\lambda n + 2\vartheta r \leq r - \vartheta r. \end{aligned}$$

Hence, $\mathfrak{B}_{\vartheta r}(x_\lambda) \subset \mathfrak{B}_r(y)$. Thus, we have proved that in the case (57) there exists $x = x_\lambda \in S_M^{\rm int}(A)$ such that

$$\mathfrak{B}_{\vartheta r}(x) \subset \mathfrak{B}_r(y) \setminus Y_k^n. \tag{60}$$

If (57) is not the case, for $x = y$ we get (60) again. This proves that Y_k^n is a porous subset of $S_M^{\rm int}(A)$. Therefore,

$$Y = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} Y_k^n \tag{61}$$

is a σ-porous subset of $S_M^{\rm int}(A)$. To complete the proof, it suffices to show that

$$S_M^{\rm int}(A) \setminus T_M(A) \subset Y.$$

Assume the contrary: there exists $z \in S_M^{\rm int}(A) \setminus T_M(A)$, $z \notin Y$. Take $n \in \mathbb{N}$ such that $n > \max\{\|z\|, \eta(z)\}$. Then, using (55), we have $z \in S_n$. It follows from (54) that there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that

$$\sup_{x\in\mathfrak{B}_{\delta_0}(z)} \sup_{a\in P_M^{\varepsilon_0}(x,A)} \|a\| < n. \tag{62}$$

Since $z \notin T_M(A)$, Lemma 3.8(i) implies that $\lim_{\varepsilon\to+0} \operatorname{diam} P_M^\varepsilon(z,A) > 0$. Therefore, there exists $\beta > 0$ such that

$$9\varrho_M(z,A)\beta < \lim_{\varepsilon\to+0} \operatorname{diam} P_M^\varepsilon(z,A). \tag{63}$$

Because M is a quasiball, there exists $\alpha > 0$ such that $\mathfrak{B}_\alpha \subset M$. Denote

$$R = \frac{8n}{\varrho_M(z,A)}, \qquad \varepsilon_1 = \min\left\{\varepsilon_0,\ \frac{\delta_0}{\alpha},\ \frac{\varrho_M(z,A)}{5R}\delta_M^R(\beta),\ \varrho_M(z,A)\right\}. \tag{64}$$

Since M is boundedly uniformly convex, we have $\varepsilon_1 > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\alpha\varepsilon_1}{2}$. Recalling that $z \in S_n$, $z \notin Y$ and using (61), we get $z \notin Y_k^n = S_n \setminus X_k^n$ and, therefore, $z \in X_k^n$. Hence, taking into account (56), we deduce that there exists $x_0 \in S_n \bigcap T_M(A)$ and $\lambda \in \left(0, \frac{1}{2}\right]$ such that $z \in \mathfrak{B}_{\lambda/k}(x_\lambda) \subset \mathfrak{B}_{\alpha\varepsilon_1\lambda/2}(x_\lambda)$, where $x_\lambda = (1-\lambda)x_0 + \lambda\pi(x_0)$. Denote $\varrho_0 = \varrho_M(x_0, A)$. Since $\pm(z - x_\lambda) \in \mathfrak{B}_{\alpha\varepsilon_1\lambda/2} \subset \varepsilon_1\lambda M$, using (14) and (64), we get

$$\begin{aligned}|\varrho_M(z,A) - \varrho_M(x_\lambda, A)| \le\ & \max\{\mu_M(z - x_\lambda), \mu_M(x_\lambda - z)\} \\ & \le \varepsilon_1\lambda \le \frac{\varepsilon_1}{2} \le \frac{\varrho_M(z,A)}{2}.\end{aligned}$$

Hence, taking into account that $\varrho_M(x_\lambda, A) = (1-\lambda)\varrho_0 \in \left[\frac{\varrho_0}{2}, \varrho_0\right]$, we have

$$\frac{\varrho_0}{3} \le \varrho_M(z, A) \le 2\varrho_0. \tag{65}$$

Since $x_\lambda \in \mathfrak{B}_{\alpha\varepsilon_1\lambda/2}(z) \subset \mathfrak{B}_{\delta_0}(z)$, it follows from (62), (64), (65) that

$$\sup_{a \in P_M^{2\varepsilon_1\lambda}(x_\lambda, A)} \|a\| \le \sup_{a \in P_M^{\varepsilon_0}(x_\lambda, A)} \|a\| < n = \frac{R\varrho_M(z,A)}{8} \le \frac{R\varrho_0}{4}, \qquad \|x_0\| < n \le \frac{R\varrho_0}{4},$$

$$2R\varepsilon_1 \le \frac{2\varrho_M(z,A)}{5}\delta_M^R(\beta) < \varrho_0\delta_M^R(\beta).$$

Therefore, Lemma 6.2 implies that $\operatorname{diam} P_M^{2\varepsilon_1\lambda}(x_\lambda, A) \le 3\varrho_0\beta$. According to Lemma 3.7 and the inclusion $z \in \mathfrak{B}_{\alpha\varepsilon_1\lambda/2}(x_\lambda)$, we get $P_M^{\varepsilon_1\lambda}(z, A) \subset P_M^{2\varepsilon_1\lambda}(x_\lambda, A)$. Thus, using (65), we have

$$\lim_{\varepsilon \to +0} \operatorname{diam} P_M^{\varepsilon}(z, A) \le \operatorname{diam} P_M^{\varepsilon_1\lambda}(z, A) \le 3\varrho_0\beta \le 9\varrho_M(z, A)\beta.$$

This contradicts (63). □

7. Examples

The following lemma is useful to prove that a set is equidistable.

Lemma 7.1. *Let $M \subset E$ be a quasiball for which there exists an infinitesimal sequence $\{\varepsilon_k\}$ of positive numbers, $\{c_k\} \subset E$, $\lambda > 1$, $\delta > 0$, $d \in E$, and $k_0 \in \mathbb{N}$ such that*

$$\mathfrak{B}_\delta(c_k) \subset (1+\varepsilon_k)M, \qquad c_k \notin M + \mathfrak{B}_\delta, \qquad c_k \notin \lambda M - d, \qquad \forall k \ge k_0.$$

Then M is equidistable.

Proof. Fix any $\varepsilon > 0$ and $w \in \mathfrak{B}_\delta$. Since $\lim_{k\to\infty} \varepsilon_k = 0$, there exists $k \ge k_0$ such that $\varepsilon_k < \varepsilon$. Using the inclusions $\mathfrak{B}_\delta(c_k) \subset (1+\varepsilon_k)M \subset (1+\varepsilon)M$, one can find $x \in M$ such that $c_k + w = (1+\varepsilon)x$. Thus, $(1+\varepsilon)x - w \notin (M + \mathfrak{B}_\delta) \bigcup (\lambda M - d)$ and we get (5). □

Example 7.2. A boundedly uniformly convex and equidistable quasiball in $\mathbb{R}^3$. Define

$$M = \left\{ \vec{x} = (x, y, z) \,\middle|\, -2 < y < 2,\ z + 1 \geq x^2 + \frac{1}{4 - y^2} \right\},$$

$$\vec{c}_k = (k+1, 0, k^2 - 2), \quad \lambda = 2, \quad \vec{d} = (0, 4, 0), \quad \delta = 1, \quad \varepsilon_k = \frac{6}{k}, \quad \forall k \in \mathbb{N}.$$

Since

$$\lim_{k \to \infty} \left(\frac{k+2}{1 + \frac{6}{k}} - k \right) = -4, \qquad \lim_{k \to \infty} \left(\frac{k^2 - 2}{1 + \frac{6}{k}} - \left(k - \frac{7}{2} \right)^2 \right) = +\infty,$$

there exists some k_0 such that

$$\frac{k+2}{1 + \frac{6}{k}} < k - \frac{7}{2}, \qquad \frac{k^2 - 2}{1 + \frac{6}{k}} > \left(k - \frac{7}{2} \right)^2 + \frac{1}{3}, \quad \forall k \geq k_0.$$

Therefore, for all $k \geq k_0$ and $\vec{x} = (x, y, z) \in \mathfrak{B}_\delta(\vec{c}_k)$ we have

$$\left| \frac{x}{1 + \varepsilon_k} \right| \leq \frac{k+2}{1 + \frac{6}{k}} < k - \frac{7}{2}, \qquad \left| \frac{y}{1 + \varepsilon_k} \right| < |y| \leq 1,$$

$$\frac{z}{1 + \varepsilon_k} + 1 > \frac{k^2 - 2}{1 + \frac{6}{k}} > \left(k - \frac{7}{2} \right)^2 + \frac{1}{3} > \left(\frac{x}{1 + \varepsilon_k} \right)^2 + \frac{1}{4 - \left(\frac{y}{1 + \varepsilon_k} \right)^2}.$$

Consequently, $\frac{\vec{x}}{1+\varepsilon_k} \in M$. Therefore, $\mathfrak{B}_\delta(\vec{c}_k) \subset (1 + \varepsilon_k)M$ for all $k \geq k_0$.

If $\vec{x} = (x, y, z) \in \mathfrak{B}_\delta(\vec{c}_k)$, then $z + 1 < k^2 \leq x^2$, so $\vec{x} \notin M$. Therefore $\vec{c}_k \notin M + \mathfrak{B}_\delta$. Besides that, $\vec{c}_k + \vec{d} \notin 2M$. Lemma 7.1 implies that M is equidistable. □

Example 7.3. A boundedly uniformly convex and equidistable quasiball $M \subset \ell_2$ such that $b_1(M) = b(M)$. Define

$$M = \{ \vec{x} = (x_0, x_1, x_2, \ldots) \in \ell_2 \mid f(\vec{x}) \leq 1 \},$$

where

$$f(\vec{x}) = \sum_{k=0}^{\infty} \varphi_k(x_k), \quad \forall \vec{x} = (x_0, x_1, \ldots) \in \ell_2,$$

$$\varphi_0(t) = -t, \qquad \varphi_k(t) = \begin{cases} \sqrt{1 + t^2} - 1, & t \leq k, \\ \sqrt{1 + t^2} - 1 + (|t| - k)^2, & |t| > k, \end{cases} \quad \forall k \in \mathbb{N},\ t \in \mathbb{R}. \tag{66}$$

A functional $\vec{p} \in \ell_2^*$ such that $\langle \vec{p}, \vec{x} \rangle = \sum_{k=0}^{\infty} p_k x_k$ for all $\vec{x} = (x_0, x_1, \ldots) \in \ell_2$ is written in the form $\vec{p} = (p_0, p_1, \ldots)$. Observe that

$$b(M) \setminus \{0\} = \{ \vec{p} = (p_0, p_1, \ldots) \in \ell_2^* \mid p_0 < 0 \}$$

is an open set. Consequently, Lemma 5.4 implies that $b_1(M) = b(M)$.

For any index $k \in \mathbb{N} \cup \{0\}$ denote $\vec{e}_k = (e_{k,0}, e_{k,1}, e_{k,2}, \dots) \in \ell_2$, where $e_{k,j} = 0$ if $k \neq j$ and $e_{k,j} = 1$ if $k = j$. Put $\vec{d} = -4\vec{e}_0$, $\lambda = 2$, $\delta = \frac{1}{3}$. For any index $k \in \mathbb{N}$ denote

$$\vec{c}_k = (k+2)(\vec{e}_0 + \vec{e}_k), \quad \varepsilon_k = \frac{3}{k}.$$

Take any $\vec{x} = (x_0, x_1, \dots) \in \frac{1}{1+\varepsilon_k}\mathfrak{B}_\delta(\vec{c}_k)$. Then, for every $j \in \mathbb{N} \setminus \{k\}$ we have $\varphi_j(x_j) = \sqrt{1+x_j^2} - 1 \le x_j^2$. Therefore $\sum_{j\in\mathbb{N}\setminus\{k\}} \varphi_j(x_j) \le \delta^2 = \frac{1}{9}$. Using the inequalities $0 < x_k \le \frac{k+2+\frac{1}{3}}{1+\varepsilon_k} < k$, we deduce that $\varphi_k(x_k) = \sqrt{1+x_k^2} - 1 < x_k - \frac{1}{2}$. Consequently, $f(\vec{x}) \le -x_0 + \varphi_k(x_k) + \frac{1}{9} \le -x_0 + x_k - \frac{1}{2} + \frac{1}{9} \le 1$. Therefore $\vec{x} \in M$. Thus,

$$\mathfrak{B}_\delta(\vec{c}_k) \subset (1+\varepsilon_k)M, \quad \forall k \in \mathbb{N}. \tag{67}$$

Now take any $\vec{x} = (x_0, x_1, \dots) \in \mathfrak{B}_\delta(\vec{c}_k)$. Then,

$$f(\vec{x}) \ge \varphi_k(x_k) - x_0 \ge x_k - 1 + (x_k - k)^2 - x_0 \ge \left(\frac{5}{3}\right)^2 - 1 - \frac{2}{3} > 1.$$

Hence, $\vec{x} \notin M$ and

$$\vec{c}_k \notin M + \mathfrak{B}_\delta. \tag{68}$$

Since $f\left(\frac{\vec{c}_k+\vec{d}}{2}\right) = -\frac{k-2}{2} + \varphi_k\left(\frac{k+2}{2}\right) > -\frac{k-2}{2} + \frac{k+2}{2} - 1 = 1$, we have

$$\vec{c}_k \notin \lambda M - \vec{d}. \tag{69}$$

Consequently, according to (67)–(69) and Lemma 7.1, we deduce that M is equidistable. $\square$

Example 7.4. A strictly convex set $M \subset \ell_2$ such that $b_1(M) = b(M)$ and M is neither equidistable nor parabolic quasiball. Define

$$M = \left\{ \vec{x} = (x_0, x_1, \dots) \in \ell_2 \mid x_0 \ge \sqrt{\sum_{k=1}^{\infty} x_k^2} - \sum_{k=1}^{\infty} \frac{\sqrt{1+|x_k|}}{k^2} \right\}.$$

For every $\vec{x} = (x_0, x_1, \dots) \in \ell_2$ we introduce the following notation

$$\varphi(\vec{x}) = \sum_{k=1}^{\infty} \frac{\sqrt{1+|x_k|}}{k^2}, \qquad \psi(\vec{x}) = \sqrt{\sum_{k=1}^{\infty} x_k^2}, \qquad f(\vec{x}) = \psi(\vec{x}) - \varphi(\vec{x}). \tag{70}$$

Then,

$$M = \{\vec{x} = (x_0, x_1, \dots) \in \ell_2 \mid x_0 \ge f(\vec{x})\}. \tag{71}$$

Assume that M is equidistable. Hence, there are $\lambda > 1$, $\delta > 0$ and $\vec{d} = (d_0, d_1, \ldots) \in \ell_2$ such that (5) holds. Denote

$$\mu = \frac{1+\lambda}{2}, \qquad C_0 = \frac{1}{\sqrt{\lambda} - \sqrt{\mu}} \left(\sum_{k=1}^{\infty} \frac{|d_k|}{2k^2} + \psi\left(\vec{d}\right) - d_0 \right), \qquad \varepsilon = \min\left\{ \frac{\delta}{2C_0}, \mu - 1 \right\}. \tag{72}$$

It follows from (5) that there exists $\vec{x} = (x_0, x_1, \ldots) \in \ell_2$ such that

$$\vec{x} \in ((1+\varepsilon)M) \setminus \operatorname{int}\left((M + \mathfrak{B}_\delta) \bigcup \left(\lambda M - \vec{d} \right) \right). \tag{73}$$

Then, using the inequality $1 + \varepsilon \le \mu$, we get

$$\vec{x} \in (\mu M) \setminus \operatorname{int}\left(\lambda M - \vec{d} \right),$$

that is, according to (71), we have

$$\frac{x_0}{\mu} \ge f\left(\frac{\vec{x}}{\mu} \right), \qquad \frac{x_0 + d_0}{\lambda} \le f\left(\frac{\vec{x} + \vec{d}}{\lambda} \right).$$

Hence,

$$\mu f\left(\frac{\vec{x}}{\mu} \right) - \lambda f\left(\frac{\vec{x} + \vec{d}}{\lambda} \right) + d_0 \le 0.$$

Therefore, according to (70), we deduce that

$$\psi(\vec{x}) - \psi(\vec{x} + \vec{d}) + d_0 \le \mu\varphi\left(\frac{\vec{x}}{\mu} \right) - \lambda\varphi\left(\frac{\vec{x} + \vec{d}}{\lambda} \right)$$

and, consequently,

$$d_0 - \psi\left(\vec{d}\right) \le \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\mu\sqrt{1 + \frac{|x_k|}{\mu}} - \lambda\sqrt{1 + \frac{|x_k + d_k|}{\lambda}} \right). \tag{74}$$

Since

$$\sqrt{|u|} - \sqrt{|v|} \le \frac{|u - v|}{\sqrt{|u|} + \sqrt{|v|}}, \quad \forall u, v \in \mathbb{R},$$

we get

$$\begin{aligned} \mu\sqrt{1 + \frac{|x_k|}{\mu}} - \lambda\sqrt{1 + \frac{|x_k + d_k|}{\lambda}} &= \sqrt{\mu}\,\sqrt{\mu + |x_k|} - \sqrt{\lambda}\,\sqrt{\lambda + |x_k + d_k|} \\ &\le \sqrt{\mu}\,\sqrt{\mu + |x_k|} - \sqrt{\lambda}\left(\sqrt{\lambda + |x_k|} - \frac{|d_k|}{2\sqrt{\lambda}} \right) \\ &\le \frac{|d_k|}{2} - (\sqrt{\lambda} - \sqrt{\mu})\sqrt{\mu + |x_k|}. \end{aligned}$$

This and (74) imply that

$$d_0 - \psi\left(\vec{d}\right) \leq \sum_{k=1}^{\infty} \frac{|d_k|}{2k^2} - \left(\sqrt{\lambda} - \sqrt{\mu}\right) \sum_{k=1}^{\infty} \frac{\sqrt{1+|x_k|}}{k^2}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{\sqrt{1+|x_k|}}{k^2} \leq C_0, \tag{75}$$

where C_0 is defined in (72).

The inclusion (73) implies that $\vec{x} \in ((1+\varepsilon)M) \setminus \operatorname{int}(M + \mathfrak{B}_\delta)$. Therefore,

$$\frac{x_0}{1+\varepsilon} \geq f\left(\frac{\vec{x}}{1+\varepsilon}\right), \qquad x_0 + \delta \leq f(\vec{x}).$$

Consequently,

$$(1+\varepsilon) f\left(\frac{\vec{x}}{1+\varepsilon}\right) - f(\vec{x}) + \delta \leq 0,$$

that is,

$$(1+\varepsilon)\varphi\left(\frac{\vec{x}}{1+\varepsilon}\right) - \varphi(\vec{x}) \geq \delta. \tag{76}$$

Since

$$(1+\varepsilon)\sqrt{1+\frac{|x_k|}{1+\varepsilon}} - \sqrt{1+|x_k|} \leq \varepsilon\sqrt{1+|x_k|},$$

the inequality (76) implies that

$$\varepsilon \sum_{k=1}^{\infty} \frac{\sqrt{1+|x_k|}}{k^2} \geq \delta.$$

Hence, using (75), we get the inequality $\varepsilon C_0 \geq \delta$, which contradicts (72). The contradiction proves that M is not equidistable.

As in the previous example, a functional $\vec{p} \in \ell_2^*$ such that $\langle \vec{p}, \vec{x} \rangle = \sum_{k=0}^{\infty} p_k x_k$ for all $\vec{x} = (x_0, x_1, \ldots) \in \ell_2$, is written in the form $\vec{p} = (p_0, p_1, \ldots)$. We claim that the set

$$B = \left\{ \vec{p} = (p_0, p_1, \ldots) \in \ell_2^* \;\middle|\; p_0 + \sqrt{\sum_{k=1}^{\infty} p_k^2} < 0 \right\}. \tag{77}$$

satisfies the equality

$$b(M) \setminus \{0\} = B. \tag{78}$$

Indeed, take $\vec{p} = (p_0, p_1, \ldots) \in b(M) \setminus \{0\}$. Fix an arbitrarily $t > 0$ and consider $\vec{x} = (x_0, x_1, \ldots) \in \ell_2$, where

$$x_k = t p_k, \quad \forall k \in \mathbb{N}, \quad x_0 = \sqrt{\sum_{k=1}^{\infty} x_k^2} - \sum_{k=1}^{\infty} \frac{\sqrt{1+|x_k|}}{k^2}.$$

Then $\vec{x} \in M$ and

$$\langle \vec{p}, \vec{x} \rangle = tp_0 \sqrt{\sum_{k=1}^{\infty} p_k^2} - p_0 \sum_{k=1}^{\infty} \frac{\sqrt{1 + t|p_k|}}{k^2} + t \sum_{k=1}^{\infty} p_k^2.$$

Since $\vec{p} \in b(M)$, we have

$$\lim_{t \to +\infty} \left(t \left(p_0 + \sqrt{\sum_{k=1}^{\infty} p_k^2} \right) \sqrt{\sum_{k=1}^{\infty} p_k^2} \; - \; p_0 \sum_{k=1}^{\infty} \frac{\sqrt{1 + t|p_k|}}{k^2} \right) < +\infty.$$

Therefore $\vec{p} \in B$. Thus, the inclusion $b(M) \setminus \{0\} \subset B$ is proved.

Let us prove the reverse inclusion. Take $\vec{p} = (p_0, p_1, \ldots) \in B$. Using the equality $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and the Cauchy–Schwarz inequality we have

$$\sum_{k=1}^{\infty} \frac{|x_k|}{k} \leq \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{\infty} x_k^2} = \frac{\pi}{\sqrt{6}} \psi(\vec{x}).$$

Consequently,

$$\begin{aligned} \varphi(\vec{x}) &= \sum_{k=1}^{\infty} \frac{\sqrt{1 + |x_k|}}{k^2} \leq \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{\infty} \frac{1 + |x_k|}{k^2}} \\ &\leq \frac{\pi}{\sqrt{6}} \sqrt{\frac{\pi}{\sqrt{6}} + \frac{\pi}{\sqrt{6}} \psi(\vec{x})} < 2\sqrt{1 + \psi(\vec{x})}. \end{aligned}$$

Therefore, for any $\vec{x} = (x_0, x_1, \ldots) \in M$ we get

$$x_0 \geq \psi(\vec{x}) - \varphi(\vec{x}) \geq \psi(\vec{x}) - 2\sqrt{1 + \psi(\vec{x})}.$$

Hence,

$$s(\vec{p}, M) = \sup_{\vec{x} \in M} \langle \vec{p}, \vec{x} \rangle \leq \sup_{\substack{\vec{x} = (x_0, x_1, \ldots) \in \ell_2 : \\ x_0 \geq \psi(\vec{x}) - 2\sqrt{1 + \psi(\vec{x})}}} \left(p_0 x_0 + \sum_{k=1}^{\infty} p_k x_k \right).$$

Thus, using the inequality $\sum_{k=1}^{\infty} p_k x_k \leq \psi(\vec{p}) \psi(\vec{x})$, we deduce

$$\begin{aligned} s(\vec{p}, M) &\leq \sup_{x \in \ell_2} \left(p_0 \psi(\vec{x}) - 2p_0 \sqrt{1 + \psi(\vec{x})} + \psi(\vec{p}) \psi(\vec{x}) \right) \\ &\leq \sup_{t \geq 0} \left(t(p_0 + \psi(\vec{p})) - 2p_0 \sqrt{1 + t} \right). \end{aligned}$$

Since $\vec{p} \in B$, we have $p_0 + \psi(\vec{p}) < 0$ and, therefore, $s(\vec{p}, M) < +\infty$. So, the inclusion $B \subset b(M) \setminus \{0\}$ is proved as well. Using (77) and (78), we deduce that $b(M) \setminus \{0\}$ is open. Therefore, from Lemma 5.4 it results that $b_1(M) = b(M)$.

Finally, let us prove that M is not parabolic. For any $k \in \mathbb{N} \cup \{0\}$ denote $\vec{e}_k$ as in Example 7.3. Consider $\vec{d} = -3\vec{e}_0$ and $\vec{x}_k = k(\vec{e}_0 + \vec{e}_k)$. Then $\vec{x}_k \in M \setminus (2M - \vec{d})$ for all $k \in \mathbb{N}$. Thus, $M \setminus (2M - \vec{d})$ is unbounded and, therefore, M is not parabolic.

Acknowledgements. The author thanks to Mariana S. Lopushanski and Grigory M. Ivanov for useful discussions of the questions considered here and is also grateful to the referee for very helpful comments.

References

[1] J. M. Borwein, S. Fitzpatrick: Existence of nearest points in Banach spaces, Canad. J. Math. 41 (1989) 707–720.

[2] E. P. Dolzhenko: Boundary properties of arbitrary functions, Izv. Akad. Nauk SSSR, Ser. Mat. 31 (1967) 3–14 (in Russian).

[3] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, North-Holland, Amsterdam (1976).

[4] S. V. Konjagin: On approximation properties of closed sets in Banach spaces, Sov. Math., Dokl. 21 (1980) 418–422.

[5] K. S. Lau: Almost Chebyshev subsets in reflexive Banach spaces, Indiana Univ. Math. J. 27 (1978) 791–795.

[6] C. Li: On well posed generalized best approximation problems, J. Approximation Theory 107 (2000) 96–108.

[7] J. Lindenstrauss, D. Preiss, J. Tiser: Frechet differentiability of Lipschitz maps and porous sets in Banach spaces, in: Banach Spaces and their Applications in Analysis (Oxford, 2006), B. Randrianantoanina et al. (ed.), de Gruyter, Berlin (2007) 111–123.

[8] S. B. Stechkin: Approximate properties of sets in linear normed spaces, Acad. Républ. Popul. Roum., Rev. Math. Pur. Appl. 8 (1963) 5–18 (in Russian).

[9] L. Zajíček: Porosity and σ-porosity, Real Anal. Exch. 13 (1987–1988) 314–350.