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Strongly Midquasiconvex Functions

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Let V be a nonempty convex subset of a normed space X and let $\varepsilon > 0$ and $p > 0$ be given. We call a function $f : V \to \mathbb{R}$ (ε, p)-*strongly midquasiconvex* if

$$f\left(\frac{x+y}{2}\right) \leq \max[f(x), f(y)] - \varepsilon\left(\frac{\|x-y\|}{2}\right)^p \quad \text{for } x, y \in V.$$

We call f p-strongly midquasiconvex if it is (ε, p)-strongly midquasiconvex with a certain $\varepsilon > 0$.

We show that if either $p < 1$ and $\dim V = 1$ or $p < 2$ and $\dim V > 1$ then there are no p-strongly midquasiconvex functions defined on V. On the other hand if X is an inner product space with $\dim X \geq 2$, $p \geq 2$, then there exists an $(1, p)$-strongly midquasiconvex function defined on an arbitrary ball in X.

Consequently, the case when $p = 1$ and $\dim V = 1$ is of a special interest. Under this assumptions we characterize lower semicontinuous 1-strongly midquasiconvex functions.

Keywords: Quasiconvexity, midquasiconvex function, strongly midquasiconvex function

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1. Introduction

Let V be a nonempty convex subset of a linear space X. A function $f : V \to \mathbb{R}$ is called *quasiconvex*, if

$$f(tx + (1-t)y) \leq \max[f(x), f(y)] \quad \text{for } x, y \in V, t \in [0, 1].$$

If the above inequality holds for $x, y \in V$, $t = \frac{1}{2}$ then f is called *midquasiconvex*. Quasiconvex functions play an important role in the theory of convex functions

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and applications [3, 6, 7, 14].

In the whole paper by I we denote a subinterval of $\mathbb{R}$. By $\mathbb{N}_0$ we denote the set of nonnegative integers. For $a \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ we put

$$I_{\leq a} := \{x \in I : x \leq a\}, \qquad I_{\geq a} := \{x \in I : x \geq a\}.$$

Lower semicontinuous quasiconvex functions on a subinterval of $\mathbb{R}$ have a nice characterization.

Theorem CM ([3, Theorem 2.5.2]). *Let $f : I \to \mathbb{R}$ be a lower semicontinuous function defined on the interval $I \subset \mathbb{R}$. Then f is quasiconvex if and only if there exists $a \in \operatorname{cl} I$ in $\overline{\mathbb{R}}$ such that $f|_{I \leq a}$ is decreasing and $f|_{I \geq a}$ is increasing.*

Inspired by the optimization theory there appeared the notions of strong convexity and strong quasiconvexity [11, 13, 8]. A function $f : V \to \mathbb{R}$ is *strongly convex* if for a certain $\varepsilon > 0$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \varepsilon t(1-t)\|x-y\|^2 \text{ for } x, y \in V, t \in [0,1],$$

and *strongly quasiconvex* if for a certain $\varepsilon > 0$

$$f(tx + (1-t)y) \leq \max[f(x), f(y)] - \varepsilon t(1-t)\|x-y\|^2 \text{ for } x, y \in V, t \in [0,1].$$

A function f is *strongly midconvex* if for a certain $\varepsilon > 0$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \varepsilon \left(\frac{\|x-y\|}{2}\right)^2 \quad \text{for } x, y \in V,$$

and *strongly midquasiconvex* if for a certain $\varepsilon > 0$

$$f\left(\frac{x+y}{2}\right) \leq \max[f(x), f(y)] - \varepsilon \left(\frac{\|x-y\|}{2}\right)^2 \quad \text{for } x, y \in V.$$

For some recent results concerning strongly (mid)convex functions on inner product spaces we refer the reader to [10].

In general it is not easy to verify if a given function is convex or strongly (quasi)convex. It occurs that in many cases to check convexity it is sufficient (under some weak regularity assumptions) to verify the respective mid-convexity condition, see for example the classical Theorem of Bernstein-Doetsch [2] (for similar result for quasiconvexity see [5]). A typical example of such result is given by the following proposition.

Proposition 1.1. *Let $f : V \to \mathbb{R}$ be a midquasiconvex function. Then*

$$f(qx + (1-q)y) \leq \max[f(x), f(y)] \quad \text{for } x, y \in V, q \in [0,1]_D, \tag{1}$$

where $[0,1]_D$ denotes the set of dyadic numbers in $[0,1]$.

Moreover, if f is lower-semicontinuous then f is quasiconvex.

Although the proof is standard we present it for the sake of completeness of our considerations.

Proof. By D_n we denote the set of dyadic numbers of order n, that is $D_n := \mathbb{Z}/2^n$. The proof of (1) goes by the induction over the dyadic order of $q \in [0,1]$. It clearly holds for $q = 0$ and $q = 1$.

Suppose that it holds for all $q \in [0,1]$, $q \in D_n$. Let $\bar{q} = (2k+1)/2^{n+1} \in D_{n+1} \backslash D_n$ be arbitrary. We put $q_0 = k/2^n \in D_n$, $q_1 = (k+1)/2^n \in D_n$. Clearly $\bar{q} = (q_0 + q_1)/2$. By the midquasiconvexity of f we get

$$f(\bar{q}x + (1-\bar{q})y) = f\left(\frac{(q_0x + (1-q_0)y) + (q_1x + (1-q_1)y)}{2}\right)$$
$$\leq \max[f(q_0x + (1-q_0)y), f(q_1x + (1-q_1)y)] \leq \max[f(x), f(y)].$$

Now suppose that f is lower semicontinous. Take an arbitrary $q \in [0,1]$ and choose a sequence of dyadic numbers $q_n \in [0,1]$ such that $q_n \to q$. Then by the lower-semicontinuity of f we get

$$f(qx + (1-q)y) \leq \liminf_{n\to\infty} f(q_nx + (1-q_n)y) \leq \max(f(x), f(y)). \qquad \square$$

In this paper we consider a generalization of the strong quasiconvexity in the spirit of [1].

Definition 1.2. Let X be a normed space, V a convex subset of X and $\varepsilon > 0$, $p > 0$ be given. A function $f : V \to \mathbb{R}$ will be called (ε, p)-*strongly midquasiconvex* if

$$f\left(\frac{x+y}{2}\right) \leq \max[f(x), f(y)] - \varepsilon\left(\frac{\|x-y\|}{2}\right)^p \quad \text{for } x, y \in V.$$

A function $f : V \to \mathbb{R}$ will be called *p-strongly midquasiconvex* if it is (ε, p)-strongly midquasiconvex with a certain $\varepsilon > 0$.

We are interested in the study of p-strongly midquasiconvex functions. We show that

- there are no p-strongly midquasiconvex functions for $p < 1$ in the case when the domain of a function is one-dimensional;
- there are no p-strongly midquasiconvex functions for $p < 2$ in the case when the domain of a function is at least two-dimensional.

Next we investigate the special and most interesting case when $p = 1$ on subintervals of $\mathbb{R}$. We show that in this case a version of Theorem CM holds (see Theorem 3.8).

Theorem. *Let $f : I \to \mathbb{R}$ be a lower semicontinuous function defined on the interval $I \subset \mathbb{R}$ and let $\varepsilon > 0$ be given. Then f is ε-strongly midquasiconvex if and only if there exists $a \in \operatorname{cl} I$ in $\overline{\mathbb{R}}$ such that $I_{\leq a} \ni x \to f(x) + \varepsilon x$ is decreasing and $I_{\geq a} \ni x \to f(x) - \varepsilon x$ is increasing.*

2. Nonexistence

Since there is a difference in the one and more dimensional situations, we divide this section into two parts. We first consider the case when the domain of a function is one-dimensional.

Let us first observe that there exist p-strongly midquasiconvex functions on subintervals of $\mathbb{R}$ for every $p \geq 1$.

Example 2.1. Consider arbitrary $a, b \in \mathbb{R}$, $a < b$; $p \geq 1$. Then the function $[a, b] \ni x \mapsto \frac{x-a}{b-a}$ is $(\frac{1}{(b-a)^p}, p)$-strongly midquasiconvex.

Theorem 2.2. *Let I be a nonempty subinterval of $\mathbb{R}$ which is not a singleton and let $p < 1$ be given. Then there is no p-strongly midquasiconvex function on I.*

Proof. Let $p < 1$ be given. For an indirect proof assume that there exists a p-strongly midquasiconvex function $f : I \to \mathbb{R}$. Let $M_p([a, b])$ denote the set of all $(1, p)$-strongly midquasiconvex functions on $[a, b]$. We put

$$R := \sup \left\{ r \geq 0 \,|\, f\left(\frac{a+b}{2}\right) \leq \max[f(a), f(b)] - r\left(\frac{|a-b|}{2}\right)^p : \right. \\ \left. a, b \in \mathbb{R},\ a < b,\ f \in M_p([a, b]) \right\}. \tag{2}$$

Directly from the definition of R we obtain that $R \geq 1$. We will show that $R = \infty$. Suppose that $R < \infty$.

Take an arbitrary $f \in M_p([a, b])$ and consider the functions $f_0 = f|_{[a,(a+b)/2]}$, $f_1 = f|_{[(a+b)/2,b]}$. Since $f((a+b)/2) \leq \max[f(a), f(b)]$ we obtain by (2) that

$$\begin{aligned} f\left(\frac{3a+b}{4}\right) = f_0\left(\frac{3a+b}{4}\right) &\leq \max[f_0(a), f_0((a+b)/2)] - R\left(\frac{|a-b|}{4}\right)^p \\ &\leq \max[f(a), f(b)] - R\left(\frac{|a-b|}{4}\right)^p, \end{aligned}$$

and

$$f\left(\frac{a+3b}{4}\right) \leq \max[f(a), f(b)] - R\left(\frac{|a-b|}{4}\right)^p.$$

By applying (2) to $f|_{[(3a+b)/4,(a+3b)/4]}$ and the above inequalities we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \max[f((3a+b)/4), f((a+3b)/4)] - R\left(\frac{|a-b|}{4}\right)^p \\ &\leq \left(\max[f(a), f(b)] - R\left(\frac{|a-b|}{4}\right)^p\right) - R\left(\frac{|a-b|}{4}\right)^p \\ &= \max[f(a), f(b)] - \frac{2}{2^p} R\left(\frac{|a-b|}{2}\right)^p. \end{aligned}$$

Since f was arbitrarily chosen, we obtain that $R \geq 2^{1-p}R$, and consequently that $R \leq 0$, a contradiction. $\square$

Now we proceed to discuss the case when the domain of f is at least two dimensional. We begin with the example which shows that there exist p-strongly midquasiconvex functions on convex subsets for every $p \geq 2$.

Example 2.3. Let H be an inner product space and let $V = B(0,1)$. Let us consider the function $f : V \ni x \to \|x\|^2 \in \mathbb{R}$. Then f is $(1,p)$-strongly midquasiconvex for every $p \geq 2$, since for $x, y \in B(0,1)$ we have

$$\left\|\frac{x+y}{2}\right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} - \left(\frac{\|x-y\|}{2}\right)^2 \leq \max(\|x\|^2, \|y\|^2) - \left(\frac{\|x-y\|}{2}\right)^p.$$

To show that for $p \in (0,2)$ there are no p-strongly midquasiconvex functions defined on a convex subset with dimension greater than one, we first prove such a result for the square $[0,1]^2 \subset \mathbb{R}^2$.

We begin with some auxiliary lemmas. We denote by D the set of dyadic numbers, i.e. the set of elements of the form $\frac{k}{2^n}$; $k \in \mathbb{Z}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2.4. *Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$ be a midquasiconvex function and let*

$$M := \max\{f(-1,1), f(-1,1), f(1,-1), f(1,1)\}.$$

Then

$$f(q_1, q_2) \leq M \quad \textit{for } q_1, q_2 \in [-1,1] \cap D.$$

Proof. By Proposition 1.1

$$f(qx + (1-q)y) \leq \max[f(x), f(y)] \tag{3}$$

for $x, y \in V$, $q \in [-1,1] \cap D$.

Fix arbitrarily $q_1, q_2 \in [-1,1] \cap D$. Then $\frac{1-q_1}{2}, \frac{1-q_2}{2} \in [-1,1] \cap D$. Applying (3) we get

$$\begin{aligned} f(q_1,-1) &= f\left(\frac{1-q_1}{2}(-1,-1) + \frac{1+q_1}{2}(1,-1)\right) \\ &\leq \max[f(-1,-1), f(1,-1)] \leq M, \end{aligned}$$

$$\begin{aligned} f(q_1,1) &= f\left(\frac{1-q_1}{2}(-1,1) + \frac{1+q_1}{2}(1,1)\right) \\ &\leq \max[f(-1,1), f(1,1)] \leq M, \end{aligned}$$

$$\begin{aligned} f(q_1,q_2) &= f\left(\frac{1-q_2}{2}(q_1,-1) + \frac{1+q_2}{2}(q_1,1)\right) \\ &\leq \max[f(q_1,-1), f(q_1,1)] \leq M. \end{aligned}$$

□

For an arbitrary convex polygon $W \subset \mathbb{R}^2$ we put

$$b(W) := \min\{\textit{length of sides of } W\}.$$

We define an operation P which, to an an arbitrary convex polygon W assigns polygon, whose vertices are the midpoints of sides of W. We also put $P^0(W) = W$.

Lemma 2.5. *Let $W \subset \mathbb{R}^2$ be a convex n-polygon with vertices $w_1, ..., w_n$ and let $f : W \to \mathbb{R}$ be an (ε, p)-strongly midquasiconvex function. Assume that*

$$f(w_i) \le M \quad \textit{for } i = 1, \dots, n.$$

Let $q_1, \dots, q_n$ be vertices of $P(W)$. Then

$$f(q_i) \le M - \frac{\varepsilon}{2^p} b^p(W) \quad \textit{for } i = 1, \dots, n.$$

Proof. Let $i \in \{1, \dots, n\}$ be arbitrary. By the definition of $P(W)$ we have that $q_i = \frac{w_j + w_k}{2}$ for some neighbouring vertices w_i, w_j of W. Hence

$$f(q_i) \le \max[f(w_j), f(w_k)] - \varepsilon \left(\frac{\|w_j - w_k\|}{2}\right)^p \le M - \varepsilon \left(\frac{b(W)}{2}\right)^p. \qquad \square$$

Let W be a convex polygon in $\mathbb{R}^2$ and let $p > 0$. We put

$$S_p^k(W) := b^p(W) + b^p(P(W)) + \ldots + b^p(P^{k-1}(W)) \quad \text{for } k \in \mathbb{N}_0.$$

Lemma 2.6. *Let $p \in (0, 2)$ be fixed. Let W_n be the regular n-polygon inscribed in the unit circle in $\mathbb{R}^2$ and let*

$$S_n^k := S_p^k(W_n) \quad \textit{for } n \in \mathbb{N},\ n \ge 3,\ k \in \mathbb{N}_0.$$

Then $\lim_{n\to\infty} \lim_{k\to\infty} S_n^k = \infty$.

Proof. Consider an arbitrary $n \in \mathbb{N}$, $n \ge 3$. Let

$$b_n^i := b(P^i(W_n)) \quad \text{for } i \in \mathbb{N}_0.$$

One can check easily that $b_n^0 = 2\sin(\frac{\pi}{n})$ and that

$$\frac{b_n^{i+1}}{b_n^i} = \cos\left(\frac{\pi}{n}\right) \quad \text{for } i \in \mathbb{N}_0.$$

Whence we obtain that

$$S_n^k = \sum_{i=0}^{k} (b_n^0)^p \left(\cos\frac{\pi}{n}\right)^{pi} = \left(2\sin\frac{\pi}{n}\right)^p \cdot \frac{1 - (\cos\frac{\pi}{n})^{kp}}{1 - \cos^p\frac{\pi}{n}}.$$

Consequently

$$\lim_{n\to\infty}\lim_{k\to\infty} S_n^k = \lim_{n\to\infty}\lim_{k\to\infty}\left(2\sin\frac{\pi}{n}\right)^p \frac{1-(\cos\frac{\pi}{n})^{pk}}{1-\cos^p\frac{\pi}{n}}$$
$$= \lim_{n\to\infty} 2^p \frac{\sin^p\frac{\pi}{n}}{1-\cos^p\frac{\pi}{n}}.$$

By applying the de'Hospital's Lemma we get

$$\lim_{n\to\infty}\frac{\sin^p\frac{\pi}{n}}{1-\cos^p\frac{\pi}{n}} = \lim_{n\to\infty}\tan^{p-2}\left(\frac{\pi}{n}\right) = \infty. \qquad \square$$

Lemma 2.7. *Let W_n be a convex polygon in $\mathbb{R}^2$ with vertices p_i, $i = 1,\dots,n$, $n \ge 3$; and let $\overline{p_i}$, $i = 1,\dots,n$ be a sequence of points in $\mathbb{R}^2$ such that*

$$\|p_i - \overline{p_i}\| < \varepsilon \quad \textit{for } i = 1,\dots,n. \tag{4}$$

If $\varepsilon > 0$ is sufficiently small, then the sequence $\overline{p_i}$, $i = 1,\dots,n$ defines convex polygon $\overline{W_n}$ with vertices $\overline{p_i}$, $i = 1,\dots,n$ such that

$$|b(P^k(W_n)) - b(P^k(\overline{W_n}))| < 2\varepsilon \quad \textit{for } k \in \mathbb{N}_0. \tag{5}$$

Proof. It is obvious that we can choose so small $\varepsilon > 0$ that $\overline{W_n}$ is a convex polygon. Assume that we have chosen ε in such a way. Then from (4) we obtain

$$|\|\overline{p_i} - \overline{p_{i+1}}\| - \|p_i - p_{i+1}\|| < 2\varepsilon \quad \text{for } i = 1,\dots,n-1,$$
$$|\|\overline{p_n} - \overline{p_1}\| - \|p_n - p_1\|| < 2\varepsilon.$$

Whence, for sufficiently small ε, we get

$$|b(W_n) - b(\overline{W_n})| < 2\varepsilon. \tag{6}$$

Now we prove that

$$|b(P(W_n) - bP(\overline{W_n})| < 2\varepsilon. \tag{7}$$

Polygons $P(W_n)$ and $P(\overline{W_n})$ have the vertices

$$\frac{p_1+p_2}{2},\dots,\frac{p_{n-1}+p_n}{2},\frac{p_n+p_1}{2},$$
$$\frac{\overline{p_1}+\overline{p_2}}{2},\dots,\frac{\overline{p_{n-1}}+\overline{p_n}}{2},\frac{\overline{p_n}+\overline{p_1}}{2},$$

respectively. Making use of (4) we obtain

$$\left\|\frac{p_i+p_{i+1}}{2} - \frac{\overline{p_i}+\overline{p_{i+1}}}{2}\right\| = \frac{1}{2}\|(p_i-\overline{p_i}) + (p_{i+1}-\overline{p_{i+1}})\| < \varepsilon$$

for $i = 1,\dots,n-1$ and

$$\left\|\frac{p_n+p_1}{2} - \frac{\overline{p_n}+\overline{p_1}}{2}\right\| < \varepsilon.$$

Applying now the previous part of the proof we get (7). Repeating the above procedure and making use of (6) we finally obtain (5). $\square$

Lemma 2.8. *For $p \in (0,2)$ there are no p-strongly midquasiconvex functions defined on $[-1,1] \times [-1,1]$.*

Proof. Let $p \in (0,2)$ be arbitrary. Suppose, for an indirect proof, that there exist an $\varepsilon > 0$ and $f : [-1,1] \times [-1,1] \to \mathbb{R}$ which is (ε, p)-strongly midquasiconvex. We denote

$$M := \max\{f(-1,-1), f(-1,1), f(1,-1), f(1,1)\},$$
$$B_1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Consider arbitrary $K > 0$ and $\delta > 0$. In virtue of Lemma 2.6 we can find $k \in \mathbb{N}$, $n \in \mathbb{N}$ and regular $2n$-polygon $\overline{W} \subset B_1$ such that

$$S_p^k(\overline{W}) > K + \delta.$$

Take $\varepsilon > 0$ so small that Lemma 2.7 is valid and

$$(b(\overline{W}) - 2\varepsilon)^p + (b(P(\overline{W})) - 2\varepsilon)^p + \ldots + (b(P^{k-1}(\overline{W})) - 2\varepsilon)^p \geq K.$$

Since the set D is dense in $\mathbb{R}$ we can find a convex, symmetric with respect to the origin $2n$-polygon $W \subset B_1$ with all vertices in $D \times D \cap ([-1,1] \times [-1,1])$ such that all distances between respective vertices of $\overline{W}$ and W are smaller than ε. By Lemma 2.7 we get

$$\begin{aligned} S_p^k(W) &= b^p(W) + b^p(P(W)) + \ldots + b^p(P^{k-1}(W)) \\ &\geq (b(\overline{W}) - 2\varepsilon)^p + (bP(\overline{W}) - 2\varepsilon)^p + \ldots + (bP^{k-1}(\overline{W}) - 2\varepsilon)^p \geq K, \end{aligned}$$

i.e.

$$S_p^k(W) \geq K. \tag{8}$$

Let q_i^j for $i = 1, \ldots, 2n$; $j = 0, \ldots, k-1$ be the vertices of $P^j(W)$. Then by Lemma 2.4 we have

$$f(q_i^0) \leq M \quad \text{for } i = 1, \ldots, 2n.$$

Applying sequentially Lemma 2.5 we obtain

$$f(q_i^1) \leq M - \frac{\varepsilon}{2^p} b^p(W),$$
$$f(q_i^2) \leq M - \frac{\varepsilon}{2^p}(b^p(W) + b^p(P(W)),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f(q_i^{k-1}) \leq M - \frac{\varepsilon}{2^p}(b^p(W) + b^p(P(W)) + \ldots + b^p(P^{k-1}(W)) = M - \frac{\varepsilon}{2^p}(S_p^k(W))$$

for $k = 1, \ldots, 2n$.

Making use of the last inequality and (8) we get

$$f(q_i^{k-1}) < M - \frac{\varepsilon}{2^p} K \quad \text{for } k = 1, \ldots, 2n. \tag{9}$$

Since W is a polygon symmetric with respect to the origin, hence also $P^{k-1}(W)$ has this property. Consider an arbitrary vertex q of $P^{k-1}(W)$. Then $-q$ is also its vertex. By midquasiconvexity of f and (9) we obtain

$$f(0,0) = f\left(\frac{q+(-q)}{2}\right) \leq \max(f(q), f(-q)) < M - \frac{\varepsilon}{2^p}K.$$

Since K was arbitrary, we have

$$f(0,0) = -\infty,$$

a contradiction. □

If V is a convex subset of a vector space X, then by $\dim V$ we understand the dimension of the affine subspace of X spanned over V.

Theorem 2.9. *Let X be a normed space, and let $V \subset X$ be a convex set with* $\dim V \geq 2$. *Let $p \in (0,2)$ be arbitrary. Then there is no p-strongly midquasiconvex function defined on V.*

Proof. We divide the proof into two parts. First we consider the situation when V is convex subset of $\mathbb{R}^2$ with nonempty interior (in the second part we reduce the general case to this one).

Suppose, for the proof by contradiction, that there exist a convex subset $V \subset \mathbb{R}^2$, $\operatorname{int} V \neq \emptyset$, $\varepsilon > 0$ and an ε-strongly midquasiconvex function $f : V \to \mathbb{R}$. Let $q \in \operatorname{int} V$ and let $\delta > 0$ be such that

$$Q := q + [-\delta, \delta] \times [-\delta, \delta] \subset V.$$

We denote

$$Q_1 := [-1,1] \times [-1,1].$$

Then we have

$$Q = \delta Q_1 + q.$$

We define a function $\tilde{f} : Q_1 \to \mathbb{R}$ by the formula

$$\tilde{f}(x) = f(\delta x + q) \quad \text{for } x \in Q_1.$$

Consider arbitrary $x, y \in Q_1$. Then we have

$$\begin{aligned}
\tilde{f}\left(\frac{x+y}{2}\right) &= f\left(\frac{\delta x + q + \delta y + q}{2}\right) \\
&\leq \max[f(\delta x + q), f(\delta y + q)] - \frac{\varepsilon}{2}\delta\|x-y\| \\
&= \max[\tilde{f}(x), \tilde{f}(y)] - \frac{\varepsilon\delta}{2}\|x-y\|.
\end{aligned}$$

It means that $\tilde{f}$ is $\varepsilon\delta$-strongly midquasiconvex, which contradicts Lemma 2.8.

Now we proceed to the general case when V is a convex set with $\dim V \geq 2$. Suppose that there exists a strongly midquasiconvex function $f : V \to \mathbb{R}$. Let H be an arbitrary two-dimensional affine subspace of the affine space spanned over V and let $\tilde{V} := V \cap H$. Then $\tilde{V}$ is convex, has nonempty interior in H and obviously $f_{|\tilde{V}}$ is strongly midquasiconvex. We may identify H and $\mathbb{R}^2$ with a certain norm $\|\cdot\|_1$. Obviously $\|\cdot\|_1$ is equivalent to Euclidean norm, which yields by the first part of the proof a contradiction. □

3. Case $p = 1$

As we see from the previous section, two cases are of special interest: $p = 1$ and $p = 2$. Since the 2-strongly quasiconvex (or simply strongly quasiconvex) functions were already considered, see for example [8], we restrict our attention to the case when $p = 1$ (and consequently the domain of the function is one-dimensional). We show that 1-strongly midquasiconvex functions behave similarily to quasiconvex ones.

To shorten formulations of our statements we introduce some additional denotations. Let $f : I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$ be any function. If

$$\frac{f(y) - f(x)}{y - x} \leq -\varepsilon \quad \text{for } x, y \in I,\ x \neq y$$

then we will write that $f \in \nearrow (\varepsilon)$ and if

$$\frac{f(y) - f(x)}{y - x} \geq \varepsilon \quad \text{for } x, y \in I,\ x \neq y$$

then we write that $f \in \searrow (\varepsilon)$.

One can directly verify the following characterization of the above defined classes.

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be an arbitrary function. Then*

- *$f \in \searrow (\varepsilon)$ if and only if the function $I \ni x \mapsto f(x) + \varepsilon x$ is decreasing,*
- *$f \in \nearrow (\varepsilon)$ if and only if the function $I \ni x \mapsto f(x) - \varepsilon x$ is increasing.*

Applying this lemma we obtain

Lemma 3.2. *Let $f : I \to \mathbb{R}$ be a given function. We assume that for every $a \in I$ there exists a neighbourhood U of a such that $f|_U \in \searrow (\varepsilon)$. Then $f \in \searrow (\varepsilon)$.*

Dually, if for every $a \in I$ there exists a neighbourhod U of a such that $f|_U \in \nearrow (\varepsilon)$, then $f \in \nearrow (\varepsilon)$.

Proof. We consider the first case (the second one is analogous). Let $g(x) := f(x) + \varepsilon x$. Then by the previous lemma we obtain, that g is locally increasing. Since g is defined on an interval, we obtain that g is globally increasing, and therefore by once more applying Lemma 3.1 we obtain that $f \in \searrow (\varepsilon)$. □

Now we present a convenient sufficient condition for ε-strong midquasiconvexity.

Theorem 3.3. *Let $a \in \operatorname{cl} I$ in $\overline{\mathbb{R}}$. If a function $f : I \to \mathbb{R}$ satisfies one of the following conditions:*

(i) $\quad f|_{I_{<a}} \in \searrow(\varepsilon)$ *and* $f|_{I_{\geq a}} \in \nearrow(\varepsilon)$,

or

(ii) $\quad f|_{I_{\leq a}} \in \searrow(\varepsilon)$ *and* $f|_{I_{>a}} \in \nearrow(\varepsilon)$,

then f is ε-strongly midquasiconvex.

Proof. Suppose that condition (i) is valid and consider arbitrary $x, y \in I$, $x < y$. Then

$$x < \frac{x+y}{2} < y.$$

If $\frac{x+y}{2} < a$ then we obtain

$$\frac{f(\frac{x+y}{2}) - f(x)}{\frac{x+y}{2} - x} \leq -\varepsilon,$$

and consequently

$$f\left(\frac{x+y}{2}\right) \leq f(x) - \frac{\varepsilon}{2}(y-x) \leq \max[f(x), f(y)] - \frac{\varepsilon}{2}|y-x|.$$

Suppose now that $\frac{x+y}{2} \geq a$. Then we get

$$\frac{f(y) - f(\frac{x+y}{2})}{y - \frac{x+y}{2}} \geq \varepsilon.$$

Hence

$$f\left(\frac{x+y}{2}\right) \leq f(y) - \frac{\varepsilon}{2}(y-x) \leq \max[f(x), f(y)] - \frac{\varepsilon}{2}|y-x|.$$

In the case of condition (ii) we only change in the above proof the inequality $\frac{x+y}{2} < a$ on $\frac{x+y}{2} \leq a$ and $\frac{x+y}{2} \geq a$ on $\frac{x+y}{2} > a$. $\square$

Applying the above result we can easily construct ε-strongly midquasiconvex functions.

Example 3.4. Let $a_1, a_2 \geq \varepsilon > 0$, $b_1, b_2, x_0 \in \mathbb{R}$. Then the function

$$f(x) = \begin{cases} -a_1 x + b_1 & \text{for } x \leq x_0 \\ a_2 x + b_2 & \text{for } x > x_0, \end{cases}$$

is ε-strongly midquasiconvex. In particular the function

$$\mathbb{R} \ni x \to \varepsilon|x - x_0| + b_1 \quad \text{for } x \in \mathbb{R}$$

is ε-strongly midquasiconvex.

We prove implication partially converse to that from Theorem 3.3.

Lemma 3.5.

(i) *If $f : I \to \mathbb{R}$ is decreasing and ε-strongly midquasiconvex then $f \in \searrow (\varepsilon)$.*
(ii) *If $f : I \to \mathbb{R}$ is increasing and ε-strongly midquasiconvex then $f \in \nearrow (\varepsilon)$.*

Proof. We prove (i).

Fix an $a \in \mathrm{int}I$ and let U be a neighbourhood of a such that $2U - U \subset I$. Consider arbitrary $x, y \in U$, $x < y$. Then $2y - x \in I$ and $x < 2y - x$. Whence we obtain

$$f(y) = f\left(\frac{x + (2y - x)}{2}\right) \leq \max[f(y), f(2y - x)] - \varepsilon(y - x) \leq f(x) - \varepsilon(y - x),$$

and consequently

$$\frac{f(y) - f(x)}{y - x} \leq -\varepsilon.$$

By Lemma 3.2 we obtain that $f|_{\mathrm{int}I} \in \searrow (\varepsilon)$.

To show that $f \in \searrow (\varepsilon)$ choose arbitrary $x, y \in I$, $x < y$. Then we can find sequences $x_n, y_n \in \mathrm{int}I$ such that $x_n \to x$, $y_n \to y$ and

$$x \leq x_n < y_n \leq y.$$

Since f is decreasing we get

$$f(y) - f(x) \leq \lim_{n\to\infty} f(y_n) - f(x_n) \leq \lim_{n\to\infty} -\varepsilon(y_n - x_n) \leq -\varepsilon(y - x).$$

To get (ii) we apply (i) for the function $-I \ni x \mapsto f(-x)$. □

From Theorem 3.3 and Lemma 3.5 we obtain directly the following result.

Corollary 3.6. *A function $f : I \to \mathbb{R}$ is decreasing and ε-strongly midquasiconvex if and only if $f \in \searrow (\varepsilon)$.*

A function $f : I \to \mathbb{R}$ is increasing and ε-strongly midquasiconvex if and only if $f \in \nearrow (\varepsilon)$.

We would like to pose the following problem.

Problem 3.7. *Does the result converse to Theorem 3.3 hold?*

We know an answer to the above problem under the additional assumption that f is lower semicontinuous.

Theorem 3.8. *A lower semicontinuous function $f : I \to \mathbb{R}$ is ε-strongly midquasiconvex if and only if there exists an $a \in \mathrm{cl}\, I$ in $\overline{R}$ such that $f_{\leq a} \in \searrow (\varepsilon)$ and $f_{\geq a} \in \nearrow (\varepsilon)$.*

Proof. By Theorem 3.3 we know that if there exists an a such that $f_{\leq a} \in \searrow (\varepsilon)$ and $f_{\geq a} \in \nearrow (\varepsilon)$, then f is ε-strongly midquasiconvex.

To prove the converse implication by Proposition 1.1 we can apply Theorem CM and obtain that there exists an a such that $f_{\leq a}$ is decreasing and $f_{\geq a}$ is increasing. Now by Lemma 3.5 we obtain that $f_{\leq a} \in \searrow (\varepsilon)$ and $f_{\geq a} \in \nearrow (\varepsilon)$. □

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