Journal of Convex Analysis
Volume 20 (2013), No. 2, 545–571

On one Extension of Decomposition Lemma Dealing with Weakly Converging Sequences of Gradients with Application to Nonconvex Variational Problems*

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Received: October 19, 2010
Revised manuscript received: May 18, 2012

We deal with the variant of Decomposition Lemma due to Kinderlehrer and Pedregal asserting that an arbitrary bounded sequence of gradients of Sobolev mappings $\{\nabla u_k\} \subseteq L^p(\Omega, \mathbf{R}^{m\times n})$, where $p > 1$, can be decomposed into a sum of two sequences of gradients of Sobolev mappings: $\{\nabla z_k\}$ and $\{\nabla w_k\}$, where $\{\nabla z_k\}$ is equintegrable and carries the same oscillations, while $\{\nabla w_k\}$ carries the same concentrations as $\{\nabla u_k\}$. We additionally impose the general trace condition "$u_k = u$" on F, where F is given closed subset of $\bar{\Omega}$. We show that under this assumption the sequence $\{z_k\}$ in decomposition can be chosen to satisfy also the trace condition $z_k = u$ a.e. on F. The result is applied to nonconvex variational problems to regularity results for sequences minimizing functionals. As the main tool we use DiPerna Majda measures.

Keywords: Sequences of gradients, DiPerna Majda measures, concentrations, oscillations

2000 Mathematics Subject Classification: 46E35, 49J45, 35B05

1. Introduction

In this paper we deal with the variant of the following result, known as Decomposition Lemma.

Lemma 1.1 (Decomposition Lemma). *Let $1 < p < \infty$, $\Omega \subseteq \mathbf{R}^n$ be an open, bounded set and let $\{u_k\}$ be a bounded sequence in $W^{1,p}(\Omega, \mathbf{R}^m)$. Then there exists a subsequence, $\{u_j\}$ and a sequence $\{z_j\} \subseteq W^{1,p}(\Omega, \mathbf{R}^m)$ such that*

$$|\{x : z_j(x) \neq u_j(x) \text{ or } \nabla z_j(x) \neq \nabla u_j(x)\}| \to 0 \quad \text{as } j \to \infty,$$

and the sequence $\{|\nabla z_j|^p\}$ is equi-integrable. If Ω is Lipschitz (or, more generally, an extension domain), then each z_j may be chosen to be a Lipschitz function.

It asserts that every sequence of gradients which is bounded in L^p, where $p > 1$, can be decomposed into a sum of two gradients: $\nabla z_j + \nabla w_j$, where ∇z_j carries the

*The work is supported by the Polish Ministry of Science grant no. N N201 397837 (years 2009–2012).

ISSN 0944-6532 / \$ 2.50

same oscillations, while ∇w_j converges to zero in the measure. The lemma was first obtained by Kinderlehrer and Pedregal in [20], then refined and extended by Kristensen in [21] and by Fonseca, Müller and Pedreal in [14], whose version we state. For recent related results we refer to [22, 23].

Decomposition Lemma has its interpretation in the language of Young measures (see e.g. [2], [15], [39]). It expresses the fact that the Young measures generated by the sequence $\{\nabla z_j\}$ are the same as those produced by the original sequence of gradients. Young measure theory is an important tool in the study of nonlinear PDEs, particularly those which arise from physical models, like for example solid-solid phase transitions or shape optimization problems (see e.g. [6], [31], [32], [34], [35], [36], [37] and their references). Decomposition Lemma applied to the Young measure theory is a key tool to describe those Young measures which are generated by sequences of gradients that are bounded in L^p [19, 20].

On the other hand, the Young measure theory has its weak points. It allows to describe limits of sequences of compositions like $\{\phi(y_j)\}_{j\in\mathbf{N}}$ only in the case when this sequence is weakly convergent in L^1. Therefore it controls oscillations of sequences but it does not control their concentrations. In particular we are unable to describe the limit of $\{\phi(y_j)\}$ when whe assume only that it is bounded in L^1. As the need to understand such limits is dictated by physical models, see e.g. [26], [27], [33], many authors are using another concept of measures due to DiPerna and Majda [1], [9], [13], [35]. Roughly speaking, one recognizes the weak-$*$ limit of sequences of measures $\{\phi(y_j(x))dx\}$ (with $y_j(x)\in\mathbf{R}^k$ a.e.) as an integral

$$\int_{\mathcal{B}_{\mathcal{R}}(\mathbf{R}^k)} \hat{\phi}(\lambda)\hat{\nu}_x(d\lambda)\sigma(dx), \tag{1}$$

where σ is a Radon measure on $\bar{\Omega}$, $\mathcal{B}_{\mathcal{R}}(\mathbf{R}^k)$ is the given compactification of $\mathbf{R}^k$, $\hat{\phi}$ is certain transform of ϕ, while $\hat{\nu}_x$ is the probability measure on $\mathcal{B}_{\mathcal{R}}(\mathbf{R}^k)$ (see Section 2 for details).

In this sense DiPerna Majda measures give more precise description of the resulting limits of sequences of compositions.

The purpose of this paper is to obtain a variant of Decomposition Lemma which deals with DiPerna Majda measures. Our main result formulated in Proposition 3.1, describes DiPerna Majda measures generated by "good" and "bad" sequences of gradients in the decomposition. Moreover, if we assume additionally that $\{u_j\}$'s satisfy trace type condition: $u_j(x) = u(x)$ for $x\in F$ (precise definition is given in the next section), where $F\subseteq\bar{\Omega}$ is a given closed set, then good sequence in decomposition can be chosen to satisfy the same trace condition. If we consider for example $F=\partial\Omega$, we are dealing with the classical Dirichlet boundary data.

The variant of Decomposition Lemma concerning DiPerna Majda measures, but without trace condition can be deduced from the proof of Lemma 2.6 in [17], and (for compactification of $\mathbf{R}^{m\times n}$ by the sphere) from the proof of Theorem 1.1 in [14]. Our arguments are inspired by techniques from both mentioned papers [14] and [17], but they are more delicate as we deal with the additional trace condition.

To control traces we apply the new result obtained in [18] about modifications of sequences of gradients (Theorem 2.7 of Section 2). Roughly speaking, the result asserts that having bounded sequence $\{u_k\} \subseteq W^{1,p}(\Omega, \mathbf{R}^m)$ and closed set $F \subseteq \bar{\Omega}$, under certain assumptions, we can modify the previously given sequence to get another one: $\{w_k\} \subseteq W^{1,p}(\Omega, \mathbf{R}^m)$ such that $\{w_k\}$ has a prescribed trace $w_k(x) = u(x)$ for $x \in F$ and moreover, the weak $*$ limits of compositions $\{v(\nabla u_k)dx\}$ and $\{v(\nabla w_k)dx\}$ in the space of measures are the same for large class of the admitted continuous functions v. It is worth to point out that such an approach dealing with F essentially different than $\partial\Omega$ was crucial to provide analysis of supports of DiPerna Majda measures generated by sequences minimizing functionals [18].

Let us mention that the compactification of the space of matrices $\mathbf{R}^{m\times n}$ considered in the expression (1) can be arbitrary and we have no special assumptions on regularity of the domain Ω nor set F.

In Section 4 we propose certain application of our results to the minimization problems in the Calculus of Variations. Namely, in Theorem 4.1 we show that the infimum of certain class of functionals in the form $I_f(v) = \int_\Omega f(x, \nabla v(x))dx$, where $v \in W^{1,p}(\Omega, \mathbf{R}^m)$, $v = u$ on F, is attained for such minimizing sequences $\{u_j\}$ that the sequence $\{|\nabla u_j\|^p\}$ is weakly compact in $L^1(\Omega)$. Such minimization problems with $F = \partial\Omega$ appear for example in pure displacement boundary-value problems for a compressible material in nonlinear elasticity, see e.g. [3], Section 3. The minimization problem with $F \subseteq \partial\Omega, F \neq \partial\Omega$ appears in the analysis of mixed displacement dead load traction boundary conditions in nonlinear elasticity, see e.g. Section 3, page 260 in [5] or Section 3 in [4]. Similar problem has been addressed by Pedregal in [34] (see e.g. Proposition 4.2 and Theorem 5.1) in connection with the theory of magnetostriction and optimal design.

Although some preparations are needed to efficiently use the tool of DiPerna Majda measures, it is worth doing it. This tool explains very well many phenomena in the Calculus of Variations and Convergence Theory and efficiently controls not only oscillations, but also concentration effects.

2. Preliminaries

2.1. Basic notation

By $C(X)$ we denote the space of continuous functions on the topological space X. In the sequel we assume that Ω is a bounded domain in $\mathbf{R}^n$. Symbol $\mathbf{R}^{m\times n}$ denotes the space of $m \times n$ matrices. If S is a metric space, the symbol "rca(S)" denotes the set of regular countably additive set functions on the Borel σ-algebra on S (cf. [10]). By $\mathcal{L}^n$ we denote the Lebegue measure defined on the Euclidean space. If $\sigma \in \mathrm{rca}(\bar{\Omega})$ we write $\sigma = \sigma_a + \sigma_s$ where σ_s is the singular part of σ and $\sigma_a = d_\sigma(x)dx$ its absolutely continuous part with respect to the Lebegue measure in the above Lebesgue decomposition (see e.g. Theorem 3, Section 1.6.2 in [12]). If $A \subseteq \bar{\Omega}$ is a measurable subset and $\sigma \in \mathrm{rca}(\bar{\Omega})$, we denote by $\sigma\angle A$ the restriction of σ to A, i.e. $(\sigma\angle A)(B) = \sigma(A \cap B)$. If $M \subseteq \mathbf{R}^n$ we write $|M|$ for n-dimensional Lebesgue measure of M.

Symbol $W^{1,p}(\Omega;\mathbf{R}^m)$, $1 \le p < +\infty$ denotes the usual Sobolev space, $C_0^\infty(\Omega,\mathbf{R}^m)$ is the space of smooth compactly supported functions, $W_0^{1,p}(\Omega;\mathbf{R}^m)$ is the completion of $C_0^\infty(\Omega,\mathbf{R}^m)$ in $W^{1,p}(\Omega,\mathbf{R}^m)$, while $W_{0,F}^{1,p}(\Omega;\mathbf{R}^m)$ is the completion of the set

$$\{\phi \in C^\infty(\mathbf{R}^n,\mathbf{R}^m) : \phi \equiv 0 \text{ in some neighbourhood of } F\}$$

in $W^{1,p}(\Omega,\mathbf{R}^m)$, provided that $F \subseteq \overline{\Omega}$ is the closed subset. More generally, for $u \in W^{1,p}(\Omega,\mathbf{R}^m)$ we denote

$$W_{u,F}^{1,p}(\Omega;\mathbf{R}^m) := \{w \in W^{1,p}(\Omega,\mathbf{R}^m) : w - u \in W_{0,F}^{1,p}(\Omega;\mathbf{R}^m)\}.$$

If $F = \partial\Omega$, then it is dropped from the notation.

We say that a subset $V \subseteq \Omega$ has a compactness property in $W^{1,p}(\Omega)$ if from every bounded sequence in $W^{1,p}(\Omega)$ one can extract the subsequence which is strongly convergent in $L^p(V)$. Note that for example in case $V = \Omega$ irregular domains may not have this property, as Nikodym's example shows, see e.g. [28], Example 1, Section 1.1.4.

Arrows $\to$, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ denote the strong, weak and weak-$*$ convergence respectively. In particular the notation $u_k \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbf{R}^m)$ means that $u_k \rightharpoonup u$ and $\nabla u_k \rightharpoonup \nabla u$ in $L^p(\Omega)$. As Ω can be irregular, it does not necessarily imply that $u_k \to u$ in $L^p(\Omega)$. If $\omega \subseteq \Omega$ we denote $\partial_\Omega \omega = \partial\omega \cap \Omega = \partial\omega \setminus \partial\Omega$.

2.2. Young measures

The symbol $L_{w*}^\infty(\Omega, \mathrm{rca}(\mathbf{R}^n))$ stands for all mappings $\nu : \Omega \mapsto \mathrm{rca}(\mathbf{R}^m)$ which are weakly $*$ measurable in the sense of Pettis, i.e. for every $f \in C_0(\mathbf{R}^m)$ the mapping $\Omega \ni x \mapsto \int_{\mathbf{R}^n} f(s)\nu_x(ds)$ is $\mathcal{L}^n$-measurable and $\|\nu_x\|_{\mathrm{rca}(\mathbf{R}^n)} \in L^\infty(\Omega)$, see e.g. [38].

We recall one version of the classical theorem of Young (see e.g. [2, 15, 39]).

Theorem 2.1 (Young's Theorem). *Let Ω be an open bounded subset of $\mathbf{R}^n$ and $\{u_k\}_{k\in\mathbf{N}}$ be a sequence of $\mathcal{L}^n$–measurable functions, $u_k : \Omega \to \mathbf{R}^m$. Then we have.*

i) *There exists a subsequence of $\{u_k\}_{k\in\mathbf{N}}$, still denoted by the same expression, and a family of nonnegative measures $\{\nu_x\}_{x\in\Omega} \in L_{w*}^\infty(\Omega, \mathrm{rca}(\mathbf{R}^m))$ such that $\|\nu_x\|_{\mathrm{rca}(\mathbf{R}^m)} \le 1$ for $\mathcal{L}^n$-almost all x and for every function $f \in C_0(\mathbf{R}^m)$ we have*

$$f(u_k) \overset{*}{\rightharpoonup} \overline{f}(x) = \int_{\mathbf{R}^m} f(\lambda)\nu_x(d\lambda) \ \text{ in } L^\infty(\Omega, dx), \text{ as } k \to \infty.$$

ii) *If additionally the sequence $\{u_k\}_{k\in\mathbf{N}}$ satisfies the tightness condition:*

$$\limsup_{k\in\mathbf{N}} |(\{x \in \Omega : |u_k(x)| \ge r\})| \overset{r\to\infty}{\to} 0,$$

then $\{\nu_x\}_{x\in\Omega}$ are probability measures for $\mathcal{L}^n$-almost all $x \in \Omega$.

iii) *If $f = f(x,\lambda) : \Omega \times \mathbf{R}^m \to \mathbf{R}$ is a Carathéodory function (i.e. it is measurable with respect to x and continuous with respect to λ) and sequence $\{f(x,u_k(x))\}$ is weakly compact in $L^1(\Omega)$, then it is weakly convergent in $L^1(\Omega)$ to the function $\int_{\mathbf{R}^m} f(x,\lambda)\nu_x(d\lambda)$.*

Definition 2.2. The sequence $\{u_k\}_{k\in\mathbf{N}}$, where $u_k : \Omega \to \mathbf{R}^m$, is said to generate the classical Young measure $\{\nu_x\}_{x\in\Omega} \in L^\infty_{w^*}(\Omega, \mathrm{rca}(\mathbf{R}^m))$ if for every $f \in C_0(\mathbf{R}^m)$ the sequence $\{f(u_k)\}_{k\in\mathbf{N}}$ converges weakly-$*$ in $L^\infty(\Omega)$ to the function $\overline{f}(x) := \int_{\mathbf{R}^m} f(\lambda)\nu_x(d\lambda)$.

2.3. DiPerna Majda measures

Rings and compactifications of $\mathbf{R}^{m\times n}$. We are now to describe basic properties of DiPerna Majda measures which appeared first in [9]. For details we refer to [24], [25], [35].

Assume that $\mathcal{R} \subseteq C(\mathbf{R}^{m\times n})$ is a separable complete subring (equipped with the Tchebycheff norm $\|\cdot\|_\infty$). There exists a metrizable compactification $\beta_{\mathcal{R}}\mathbf{R}^{m\times n}$ of $\mathbf{R}^{m\times n}$ and dense homeomorphic embedding $i : \mathbf{R}^{m\times n} \to \beta_{\mathcal{R}}\mathbf{R}^{m\times n}$, which establishes the correspondence between $\mathcal{R}$ and $C(\beta_{\mathcal{R}}\mathbf{R}^{m\times n})$ via the isomorphism: $v \in C(\beta_{\mathcal{R}}\mathbf{R}^{m\times n}) \mapsto v\circ i \in \mathcal{R}$ (see e.g. Section 3.12.21 in [11]). We will use the same notation for v and $v\circ i$, as well as for $\mathbf{R}^{m\times n}$ and its image in $\beta_{\mathcal{R}}\mathbf{R}^{m\times n}$.

If K is a compact metric space, $\sigma \in \mathrm{rca}(\bar\Omega)$, by $L^\infty_{w*}(\bar\Omega, \sigma; \mathrm{rca}(K))$ we denote all mappings $\hat\nu : \overline{\Omega} \ni x \mapsto \hat\nu_x \in \mathrm{rca}(K)$ which are weakly $*$ measurable with respect to σ, i.e. for every $f \in C(K)$ the mapping $\bar\Omega \ni x \mapsto \int_K f(s)\hat\nu_x(\mathrm{d}s)$ is σ-measurable in the usual sense.

DiPerna Majda Theorem. DiPerna and Majda obtained the following well known result [9], see also [17], Preliminaries, Section 0.4.

Theorem 2.3. *Let $\{y_k\}_{k\in\mathbf{N}}$ be a bounded sequence in $L^p(\Omega; \mathbf{R}^{m\times n})$, $1 \le p < +\infty$. Then there exists its subsequence (denoted by the same indices), a positive Radon measure $\sigma \in \mathrm{rca}(\bar\Omega)$ and a mapping $\hat\nu \in L^\infty_{\mathrm{w}*}(\bar\Omega, \sigma; \mathrm{rca}(\beta_{\mathcal{R}}\mathbf{R}^{m\times n}))$ such that $\hat\nu_x$ is a probability measure for σ-a.a. $x \in \bar\Omega$ and we have*

i)

$$\forall g\in C(\bar\Omega)\ \forall v_0\in\mathcal{R} : \quad \lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\mathrm{d}x = \int_{\bar\Omega} g(x)\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x), \tag{2}$$

where v is such that $\mathcal{R} \ni v(s)/(1+|s|^p) \sim v_0(s) \in C(\beta_{\mathcal{R}}\mathbf{R}^{m\times n})$.

ii) Putting $v(s) = 1+|s|^p$ in (2) ($v_0 \equiv 1$), we obtain

$$\lim_{k\to\infty}(1+|y_k(x)|^p)dx = \sigma \quad \textit{weakly} * \textit{in } \mathrm{rca}(\bar\Omega).$$

iii) For every $h_0 \in C(\bar\Omega\times\beta_{\mathcal{R}}\mathbf{R}^{m\times n})$ we have

$$\lim_{k\to\infty}\int_\Omega h_0(x, y_k(x))(1+|y_k(x)|^p)dx \to \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x).$$

Definition 2.4. If (2) holds, we say that $\{y_k\}_{\in\mathbf{N}}$ generates DiPerna Majda measure $(\sigma, \hat\nu)$. The set of all DiPerna Majda measures

$$(\sigma,\hat\nu) \in \mathrm{rca}(\bar\Omega)\times L^\infty_{\mathrm{w}*}(\bar\Omega, \sigma; \mathrm{rca}(\beta_{\mathcal{R}}\mathbf{R}^{m\times n}))$$

satisfying (2) is denoted by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n})$. See [25] for its description.

Two facts concerning supports of DiPerna Majda measures. We will use the following facts which will serve as tools. The first lemma can be found in [24] and [35], Lemma. 3.2.14, second lemma is proven in [25], Theorem 1,2 and [35], Proposition 3.2.17 for general compactification of $\mathbf{R}^{m\times n}$ and in [1], Proposition 4.1, part (iii) for compactification by sphere.

Lemma 2.5. *Let the measure $(\alpha,\hat{\mu}_x)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n})$ be generated by the sequence $\{y_k\}_{k\in\mathbf{N}}$. Then sequence $\{|y_k|^p\}_{k\in\mathbf{N}}$ is relatively weekly compact in $L^1(\Omega)$ if and only if*

$$\hat{\mu}_x(\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n})=0 \quad \text{for } \alpha \text{ almost every } x.$$

Lemma 2.6. *Assume that $(\sigma,\hat{\nu})\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n})$. Then for σ_s- almost all $x\in\bar{\Omega}$ we have*

$$\hat{\nu}_x(\mathbf{R}^{m\times n})=0.$$

Special subsets in $C(\mathbf{R}^{m\times n})$. We will deal with the following special subsets in $C(\mathbf{R}^{m\times n})$:

$$\Upsilon^p(\mathbf{R}^{m\times n}) := \{v\in C(\mathbf{R}^{m\times n}) : v(s)/(1+|s|^p)\in\mathcal{R}\}, \tag{3}$$

$$\begin{aligned} S_p := \{v\in\Upsilon^p(\mathbf{R}^{m\times n}) :\ & \text{there exists } C>0,\ 0<\alpha\le 1 : \\ & |v(A)-v(B)|\le C|A-B|^\alpha(1+|A|^{p-\alpha}+|B|^{p-\alpha})\}. \end{aligned} \tag{4}$$

First class of functions is the one to test DiPerna Majda measures on, the same as in Theorem 2.3. Second class of functions was considered for example in the old celebrated Meyer's paper [29]. As typical examples of functions from class S_p we may consider all quasiconvex functions described below.

Quasiconvexity and quasiconvex envelopes. We say that a function $v: R^{m\times n}\to\mathbf{R}$ is quasiconvex (see e.g. [8, 30]) if for any bounded domain $\Omega\subseteq\mathbf{R}^n$, any $s_0\in\mathbf{R}^{m\times n}$ and any $\varphi\in W^{1,\infty}_0(\Omega;\mathbf{R}^m)$

$$v(s_0)|\Omega|\le\int_\Omega v(s_0+\nabla\varphi(x))\,dx.$$

The quasiconvex envelope Qv of $v: R^{m\times n}\to\mathbf{R}$ is defined by

$$Qv=\sup\left\{h\le v : h\in C(\mathbf{R}^{m\times n}) \text{ is quasiconvex}\right\}.$$

It is known (see e.g. [8], [14]) that if $v\in C(\mathbf{R}^{m\times n})$ and $|v(s)|\le C(1+|s|^p)$ then

$$Qv(s_0)=\inf_{\varphi\in W^{1,p}_0(\Omega;\mathbf{R}^m)}\frac{1}{|\Omega|}\int_\Omega v(s_0+\nabla\varphi(x))\,\mathrm{d}x.$$

It is also known that an arbitrary quasiconvex function v of growth p satisfies the inequality

$$|v(A)-v(B)|\le C|A-B|(1+|A|^{p-1}+|B|^{p-1}) \tag{5}$$

with the constant C independent of A, B, in particular it belongs to S_p (see e.g. [8], [14]).

For illustrative discussion of the role of quasiconvexity condition and examples we refer e.g. to [3].

Let us mention that the class S_p is substantially larger than the class of quasiconvex functions. For example identity (5) is satisfied for every function which is convex with respect to each separate variable (remark that every quasiconvex function has this property, see e.g. [8]).

Modifications of DiPerna-Majda measures generated by gradients. We will also apply the following result obtained in [18]. It can be expressed in the language of DiPerna-Majda measures generated by gradients of Sobolev functions, but we leave its original presentation.

Theorem 2.7. *Suppose that the following conditions hold*

i) $\mathcal{R} \subseteq C(\mathbf{R}^{m\times n})$ *is the given separable complete ring and for every* $v_0 \in \mathcal{R}$ *we identify* v_0 *with certain* $v \in \Upsilon^p(\mathbf{R}^{m\times n})$ *by the formulae* $v(s) := v_0(s)(1+|s|^p)$.

ii) $F_1, F \subseteq \bar{\Omega}$ *are closed subsets,* $F \neq \emptyset$ *and either* $\partial\Omega \subseteq F_1 \cup F$ *or* $\Omega \setminus F$ *has a compactness property in* $W^{1,p}(\Omega)$.

iii) $\{u_k\} \subseteq W^{1,p}_{u,F_1}(\Omega, \mathbf{R}^m)$ *is the given sequence,* $u_k \rightharpoonup u$ *in* $W^{1,p}$ *and for every* $v_0 \in \mathcal{R}$ *the sequence of measures* $\{v(\nabla u_k)dx\}_{k\in\mathbf{N}}$ *converges weekly* $*$ *to* $\mu_v \in rca(\bar{\Omega})$*, moreover,*

$$\mu_v \angle F = v(\nabla u(x))dx \angle F \quad \textit{for every } v_0 \in \mathcal{R}. \tag{6}$$

Then there exists a sequence $\{\nabla w_k\}_{k\in\mathbf{N}}$ *such that* $\{w_k - u\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{0,F_1\cup F}(\Omega, \mathbf{R}^{m\times n})$ *and*

$$* \lim_{k\to\infty} v(\nabla w_k)dx = \mu_v \quad \textit{for every } v_0 \in \mathcal{R}.$$

Remark 2.8. The assumption "$\Omega \setminus F$ *has a compactness property in* $W^{1,p}(\Omega)$" in *ii)* can be substantially weakened, see Remark 4.7 in [18].

Remark 2.9. Suppose that the assumptions of Theorem 2.7 are satisfied with F such that $|F| = 0$. In particular (6) is equivalent to $\mu_w(F) = (\mu_w)_s(F) = 0$ where $w(s) = 1+|s|^p$ (because for every $v_0 \in \mathcal{R}$ we have $|v(s)| \leq C(1+|s|^p)$, so that every μ_v is absolutely continuous with respect to μ_w). Taking into account the notation in Theorem 2.3 with $\{y_k\} = \{\nabla u_k\}$ we observe that the condition $\mu_w(F) = 0$ is the same as the condition $\sigma(F) = 0$, where $(\sigma, \hat{\nu})$ is the DiPerna-Majda measure generated by the sequence $\{\nabla u_k\}$. If it is satisfied then the sequence $\{w_k\}$ can be chosen to satisfy additionally the condition (up to the choice of the subsequence of $\{u_k\}$)

$$|\{x \in \Omega : \nabla w_k(x) \neq \nabla u_k(x)\}| \to 0, \quad \text{as } k \to \infty,$$

see Remark 4.1 in [18].

Remark 2.10. Suppose that $F = \partial\Omega$, $F_1 = \emptyset$ and that the sequence of measures $\{(1+|\nabla u_k|^p)dx\}$ converges weakly $*$ to the measure $\sigma \in \mathrm{rca}(\bar{\Omega})$ such that $\sigma(\partial\Omega) =$

0. Theorem 2.7 asserts that the sequence $\{u_k\}$ can be modified to satisfy Dirichlet boundary data: $\{w_k - u\} \subseteq W_0^{1,p}(\Omega, \mathbf{R}^m)$. This follows directly from Theorem 2.3 and formulae (6).

The nonconcentrating modifications. We start by recalling some facts about nonconcentrating modifications of DiPerna Majda measures (see [24, 35] for details). We call $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ nonconcentrating if

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus \mathbf{R}^{m\times n}} \hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) = 0 \ . \tag{7}$$

According to Lemma 2.5 this means that every sequence $\{y_k\}$ generating $(\sigma, \hat{\nu})$ is such that $\{|y_k|^p\}$ is relatively weakly compact in $L^1(\Omega)$.

We will apply the following property, whose proof is given in the Appendix.

Proposition 2.11. *Let $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$. Then we have*

i) *There is precisely one nonconcentrationg measure $(\sigma^\circ, \hat{\nu}^\circ) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ such that*

$$\int_{\Omega} \int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x) = \int_{\bar{\Omega}} \int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x), \tag{8}$$

for any $v_0 \in \mathcal{R}$ such that $\lim_{s\to\infty} v_0(s) = 0$ and any $g \in C(\bar{\Omega})$.

ii) *The nonconcentrationg measure $(\sigma^\circ, \hat{\nu}^\circ) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ satisfies (8) with any $v_0 \in \mathcal{R}$. Moreover, we have for any $v_0 \in \mathcal{R}$*

$$\int_{\Omega} \int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x) = \int_{\bar{\Omega}} \int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)d_\sigma(x)(dx),$$
$$\int_{\bar{\Omega}} \int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x).$$

Definition 2.12. The measure $(\sigma^\circ, \hat{\nu}^\circ)$ in Proposition 2.11 is called the p-nonconcentrating modification of $(\sigma, \hat{\nu})$.

There is a one-to-one correspondence between nonconcentrating DiPerna-Majda measures and Young measures. Lemma 2.13 stated below shows the relations between DiPerna Majda measures, their nonconcentrating modifications, and Young measures. Sketch of its proof is given in the Appendix for readers' convenience.

Lemma 2.13. *Let $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ and $(\sigma^\circ, \hat{\nu}^\circ) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ be the p-nonconcentrating modification of $(\sigma, \hat{\nu})$ and let d_{σ° and d_σ be densities of σ and σ° respectively, with respect to the Lebesgue measure. Then we have*

i) *σ° is absolutely continuous with respect to the Lebesgue measure and for almost every x*

$$d_{\sigma^\circ}(x) = \left(\int_{\mathbf{R}^{m\times n}} \hat{\nu}_x(ds)\right) d_\sigma(x) \quad \text{and} \quad \hat{\nu}^\circ_0 = \frac{\hat{\nu}_x \angle \mathbf{R}^{m\times n}}{\int_{\mathbf{R}^{m\times n}} \hat{\nu}_x(ds)};$$

ii) *For almost every $x \in \Omega$ we have*

$$d_\sigma(x)\chi_{\mathbf{R}^{m\times n}} \cdot \hat{\nu}_x = d_{\sigma^0}(x) \cdot \hat{\nu}_x^0$$

as measures on $\beta_{\mathcal{R}}\mathbf{R}^{m\times n}$.

iii) *if $\{\nu_x\}_{x\in\Omega}$ is the classical Young measure generated by the same sequence as that generating $(\sigma, \hat{\nu})$ then for almost all $x \in \Omega$ we have*

$$\nu_x = d_{\sigma^0}(x) \left(\frac{\hat{\nu}_x^0}{1+|s|^p} \angle \mathbf{R}^{m\times n} \right)$$

as measures on $\mathbf{R}^{m\times n}$. This means that for every $v \in C(\mathbf{R}^{m\times n})$ and almost every $x \in \Omega$ we have $\int_{\mathbf{R}^{m\times n}} v(s)\nu_x(ds) = d_{\sigma^0}(x) \int_{\mathbf{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x^0(ds)$.

Remark 2.14. For almost every $x \in \Omega$ we have $\int_{\mathbf{R}^{m\times n}} \hat{\nu}_x(ds) > 0$. This is shown in [35], Proposition 3.2.15.

3. Main result

3.1. Formulation of the problem and discussion

Our main goal is to obtain the following result.

Theorem 3.1. *Let $1 < p < \infty$, Ω be an arbitrary bounded domain such that $|\partial\Omega| = 0$, $F \subseteq \bar{\Omega}$ be an arbitrary closed subset such that $|\partial_\Omega F| = 0$ and either $\partial\Omega \subseteq F$ or $\Omega \setminus F$ has a compactness property in $W^{1,p}(\Omega)$. Assume that the sequence $\{u_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$ is bounded, $\{\nabla u_k\}_{k\in\mathbf{N}}$ generates $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ and $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbf{R}^m)$.*

Then there exist the subsequence of $\{u_k\}$ (denoted by the same expression) and two bounded sequences $\{z_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$, $\{w_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{0,F}(\Omega, \mathbf{R}^m)$ such that

$$\nabla u_k = \nabla z_k + \nabla w_k.$$

Moreover

i) $\lim_{k\to\infty} |\{x \in \Omega : \nabla z_k(x) \neq \nabla u_k(x)\}| = 0$*;*

ii) *$\{\nabla z_k\}_{k\in\mathbf{N}}$ generates $(\sigma^0, \hat{\nu}^0) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$, which is the nonconcentrating modification of $(\sigma, \hat{\nu})$ (see Section 2.3). In particular the sequence $\{|\nabla z_k|^p\}_{k\in\mathbf{N}}$ is weakly compact in $L^1(\Omega)$;*

iii) *$\{\nabla w_k\}_{k\in\mathbf{N}}$ generates $(\sigma^1, \hat{\nu}^1) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ such that for any $v \in S_p$ (see (4)) and any $g \in C(\bar{\Omega})$*

$$\begin{aligned} &\int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}^1_x(\mathrm{d}s)g(x)\sigma^1(\mathrm{d}x) \\ &= v(0)\int_\Omega g(x)\,dx + \int_{\bar{\Omega}} \int_{\beta_{\mathbf{R}^{m\times n}}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)g(x)\,\sigma(dx). \end{aligned} \tag{9}$$

As an immediate corollary we obtain the following result.

Corollary 3.2. *Under the assumptions of Theorem* 3.1 *the mapping*

$$\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n}) \ni (\sigma,\hat{\nu}) \mapsto (\sigma^0,\hat{\nu}^0) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n})$$

preserves the class of DiPerna Majda measures generated by gradients of Sobolev mappings from the class $W^{1,p}_{u,F}(\Omega,\mathbf{R}^m)$ *where* $\nabla u(x) = \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}_x(ds)$.

We also have an easy observation.

Remark 3.3. According to Lemma 2.13 condition *ii*) in the statement of Theorem 3.1 implies that $\{\nabla u_k\}_{k\in\mathbf{N}}$ and $\{\nabla z_k\}_{k\in\mathbf{N}}$ generate the same classical Young measure.

3.2. Proof of Theorem 3.1

Before we prove the theorem we will need the following lemma. Its proof is given in the Appendix for readers convenience.

Lemma 3.4. *Let* Ω *be a bounded domain in* $\mathbf{R}^n$, $|\partial\Omega| = 0$, $F \subseteq \bar{\Omega}$ *be an arbitrary closed subset such that* $|\partial_\Omega F| = 0$ *and let us denote* $\Omega_F = \Omega \setminus F$, $\tilde{F} = F \cap \bar{\Omega}_F$. *Suppose that* $\{u_k\} \subseteq W^{1,p}_{u,F}(\Omega,\mathbf{R}^m)$ *is a given bounded sequence and* $\{w_k\} \subseteq W^{1,p}_{u,\tilde{F}}(\Omega_F,\mathbf{R}^m)$ *is the sequence of restrictions of* u_k*'s to* Ω_F. *Assume further that* $\{\nabla u_k\}$ *generates DiPerna Majda measure* $(\sigma,\hat{\nu}_x)_{x\in\bar{\Omega}}$, *while* $\{\nabla w_k\}$ *generates DiPerna Majda measure* $(\alpha,\hat{\mu}_x)_{x\in\,\bar{\Omega}_F}$. *Then we have*

$$\begin{aligned} \alpha(A) = \sigma(A) \quad &\text{for every measurable } A \subseteq \bar{\Omega}_F \\ \text{and} \quad \hat{\mu}_x = \hat{\nu}_x \quad &\text{for } \sigma \text{ a.e. } x \in \bar{\Omega}_F. \end{aligned} \tag{10}$$

Now we prove the theorem.

Proof of Theorem 3.1. We consider two cases: $|F| = 0$ and the remaining case: $|F| > 0$.

Case (A): $|F| = 0$. We use Lemma 1.1 and construct the sequence $\{\bar{z}_k\}_{k\in\mathbf{N}}$ such that

$$|\{x\in\Omega : \nabla\bar{z}_k \neq \nabla u_k\}| \to 0 \quad \text{as } k\to\infty$$

and $\{|\nabla\bar{z}_k|^p\}_{k\in\mathbf{N}}$ is weakly compact in $L^1(\Omega)$. It does not follow from the original construction in [14] that $\{\bar{z}_k\} \subseteq W^{1,p}_{u,F}(\Omega,\mathbf{R}^m)$ as we claim and perhaps this condition is not satisfied. From now proof of this part follows by steps.

Step 1A. Passing eventually to a subsequence we can assume that $\{\bar{z}_k\}_{k\in\mathbf{N}}$ generates DiPerna Majda measure $(\beta,\hat{\gamma}_x)$. We show that $(\beta,\hat{\gamma}_x) = (\sigma^0,\hat{\nu}^0)$ – the p-nonconcentrating modification of $(\sigma,\hat{\nu})$. To verify this at first we note that $(\beta,\hat{\gamma}_x)$ is nonconcentrating. Therefore it suffices to show that (8) holds with $(\beta,\hat{\gamma}_x)$ instead of $(\sigma^0,\hat{\nu}^0)$ and apply Proposition 2.11, part *i*).

To show (8) with $(\beta, \hat{\gamma}_x)$ we take an arbitrary $v \in \Upsilon^p(\mathbf{R}^{m\times n})$ such that $v/(1+|\cdot|^p)$ vanishes at ∞. For every such v and every $g \in C(\bar{\Omega})$ we have

$$\left|\int_\Omega (v(\nabla u_k(x)) - v(\nabla \bar{z}_k(x)))\, g(x)dx\right|$$
$$\leq C\int_{R_k} |v(\nabla u_k(x))|dx + C\int_{R_k} |v(\nabla \bar{z}_k(x))|dx,$$

where $R_k = \{x \in \Omega : \nabla \bar{z}_k \neq \nabla u_k\}$, with $C = \sup_\Omega |g|$. Then we will observe that the right hand side above converges to 0 as $k \to \infty$. This will be due to the fact that sequences $\{|v(\nabla u_k(x))|\}_{k\in\mathbf{N}}$ and $\{|v(\nabla \bar{z}_k(x))|\}_{k\in\mathbf{N}}$ are weakly compact in $L^1(\Omega)$ and $|R_k| \to 0$ as $k \to \infty$.

Indeed, in case of second sequence compactness is obvious from construction (as sequence $\{|\nabla \bar{z}_k(x)|^p\}$ is weakly compact in $L^1(\Omega)$). As for first one, this is because $v(s)/(1+|s|^p)$ vanishes at ∞. To see this we take an arbitrary measurable subset $V \subseteq \Omega$ and observe that

$$\int_V |v(\nabla u_k)|dx = \int_V \frac{|v(\nabla u_k)|}{1+|\nabla u_k|^p}(1+|\nabla u_k|^p)dx$$
$$= \int_{V\cap|\nabla u_k|>K} \frac{|v(\nabla u_k)|}{1+|\nabla u_k|^p}(1+|\nabla u_k|^p)dx + \int_{V\cap|\nabla u_k|\leq K} \frac{|v(\nabla u_k)|}{1+|\nabla u_k|^p}(1+|\nabla u_k|^p)dx,$$

where $K > 0$ can be arbitrary. Second term above is upperbounded by $C_v(1+K^p)|V|$, where $C_v = \sup\{|v(s)|/(1+|s|^p), s \in \mathbf{R}^{m\times n}\}$. First term is no bigger than $B_K C$, where $B_K = \sup\{\frac{|v(s)|}{1+|s|^p} : |s| \geq K\}$, $C = \sup_k \int_\Omega (1+|\nabla u_k|^p)dx$. Therefore for any $\epsilon > 0$ we find K_0 with property $B_{K_0}C < \frac{\epsilon}{2}$, and then if $|V| < \frac{\epsilon}{2C_v(1+K_0^p)}$, we get

$$\int_V |v(\nabla u_k)|dx < \epsilon.$$

This shows equiintegrability of sequence $\{|v(\nabla u_k(x))|\}_{k\in\mathbf{N}}$, so also its weak compactness in $L^1(\Omega)$.

Altogether implies that sequence $\int_\Omega v(\nabla u_k)g(x)dx$ has the same limit as $\int_\Omega v(\nabla \bar{z}_k)g(x)dx$. Consequently, the equality

$$\int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x^\circ(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(dx)$$

holds with such a v. By assumption v_0 vanishes on the reminder $\beta_{\mathcal{R}}\mathbf{R}^{m\times n} \setminus \mathbf{R}^{m\times n}$ as $v/(1+|s|^p) \to 0$ when $s \to \infty$. Therefore we can substitute integrals over $\beta_{\mathcal{R}}\mathbf{R}^{m\times n}$ by integrals over $\mathbf{R}^{m\times n}$ (on both sides), getting

$$\int_{\bar{\Omega}}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x^\circ(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) = \int_{\bar{\Omega}}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(dx).$$

Step 2A. We modify sequence $\{\bar{z}_k\}_{k\in\mathbf{N}}$ to another sequence $\{z_k\}$ in such a way that $\{z_k\}$ satisfies *i*), $\{\nabla z_k\}$ generate the same DiPerna Majda measure as $\{\nabla \bar{z}_k\}$ (in particular *ii*) reminds true for $\{z_k\}$) and additionally $\{z_k\} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$. The last condition means that the new sequence has the required trace.

We note that σ^0 is the weak $*$ limit of $\{(1 + |\nabla \bar{z}_k|^p)dx\}_{k\in\mathbf{N}}$ and so it is absolutely continuous with respect to the Lebegue measure. We apply Theorem 2.7 to $\{\nabla \bar{z}_k\}_{k\in\mathbf{N}}$ with $F_1 = \emptyset$ and our F. Note that in such a case (6) is trivially satisfied as $|F| = 0$. This implies existence of the sequence $\{\nabla z_k\}_{k\in\mathbf{N}}$ also generating $(\sigma^0, \hat{\nu}^0)$ such that $\{z_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$. Moreover, according to Remark 2.9 this sequence can be chosen to satisfy $|\{x \in \Omega : \nabla \bar{z}_k(x) \neq \nabla z_k(x)\}| \to 0$ as $k \to \infty$ (up to the choice of the subsequence of $\{\bar{z}_k\}$). Therefore also

$$|\{x \in \Omega : \nabla u_k(x) \neq \nabla z_k(x)\}| \to 0 \quad \text{as } k \to \infty$$

up to the choice of the subsequence of $\{u_k\}$. Moreover, $\{|\nabla z_k|^p\}_{k\in\mathbf{N}}$ is also weakly compact in $L^1(\Omega)$ according to Lemma 2.5.

Step 3A. We verify property *iii*) for the subsequence of:

$$w_k := u_k - z_k = (u_k - u) - (z_k - u) \in W^{1,p}_{0,F}(\Omega, \mathbf{R}^m).$$

After extracting the subsequence (denoted by the same indices) we can assume that $\{w_k\}_{k\in\mathbf{N}}$ generates the measure $(\sigma^1, \hat{\nu}^1)$ and $u_k = z_k + w_k$. Let us prove that $(\sigma^1, \hat{\nu}^1)$ satisfies (9).

To see that at first we compute that for any $v \in S_p$ (see (4)) such that $v(0) = 0$:

$$\begin{aligned} I_k &:= \int_\Omega g(x)v(\nabla w_k(x))dx + \int_\Omega g(x)v(\nabla z_k(x))dx - \int_\Omega g(x)v(\nabla u_k(x))dx \\ &= \int_{R_k} g(x)\left\{v(\nabla u_k(x) - \nabla z_k(x)) - v(\nabla u_k(x))\right\}dx + \int_{R_k} g(x)v(\nabla z_k(x))dx \\ &=: A_k + B_k, \end{aligned}$$

where $R_k = \{x : \nabla u_k(x) \neq \nabla z_k(x)\}$. By the very definition of S_p

$$\begin{aligned} &|(v(\nabla u_k(x) - \nabla z_k(x)) - v(\nabla u_k(x))| \\ &\leq C|\nabla z_k(x)|^\alpha(1 + |\nabla u_k(x)|^{p-\alpha} + |\nabla z_k(x)|^{p-\alpha}). \end{aligned}$$

As $|\nabla z_k|^\alpha \in L^{p/\alpha}(\Omega)$ and $(1 + |\nabla u_k(x)|^{p-\alpha} + |\nabla z_k(x)|^{p-\alpha}) \in L^{(p/\alpha)'}(\Omega)$, where $(p/\alpha)' = \frac{p}{p-\alpha}$ is Hölder conjugate to $\frac{p}{\alpha}$, we easily derive the following estimation

$$|A_k| \leq C\left(\int_{R_k} |\nabla z_k|^p\right)^{\frac{\alpha}{p}},$$

where $C = C(|\Omega|, \sup_\Omega|g|, \sup_k\|\nabla z_k\|_p, \sup_k\|\nabla u_k\|_p)$, independent on k.
Moreover, $|v(\nabla z_k(x))| \leq C_v(1 + |\nabla z_k(x)|^p)$. From there is easily follows that

$$|B_k| \leq \tilde{C}\int_{R_k} (1 + |\nabla z_k(x)|^p)dx,$$

with constant $\tilde{C}$ independent on k. Due to equintegrability property of the sequence $\{|\nabla z_k|^p\}_{k\in\mathbf{N}}$ we observe that

$$I_k \to 0 \ \text{ as } k \to \infty.$$

The above implies

$$\begin{aligned}
&\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x^1(\mathrm{d}s)g(x)\sigma^1(\mathrm{d}x)\\
&= \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x) - \int_{\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x^0(\mathrm{d}s)g(x)\sigma^0(\mathrm{d}x)\\
&=: \mathcal{C} - \mathcal{D},
\end{aligned}$$

whenever $v(0) = 0$ and $v \in S_p$. Decomposing $\sigma(dx) = d_\sigma(x)dx + \sigma_s(dx)$ in first integral on the right hand side above and applying Lemma 2.6 we deduce that

$$\begin{aligned}
\mathcal{C} = &\int_{\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)\mathrm{d}x\\
&+ \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma_s(\mathrm{d}x).
\end{aligned} \tag{11}$$

Moreover, as σ^0 satisfies (7) we get from Proposition 2.11, part *ii*) that

$$\begin{aligned}
\mathcal{D} &= \int_{\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x^0(\mathrm{d}s)g(x)\sigma^0(\mathrm{d}x) = \int_{\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)\\
&= \int_{\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)dx.
\end{aligned}$$

The last equality holds because $\sigma(dx) = d_\sigma(x)dx + \sigma_s(dx)$ and $\hat\nu_x(\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n})$ $= 1$ for σ_s-almost every $x \in \Omega$. Consequently (integrate $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} = \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}}$ $+\int_{\mathbf{R}^{m\times n}}$ in the first term in (11))

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x^1(\mathrm{d}s)g(x)\sigma^1(\mathrm{d}x) = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x), \tag{12}$$

whenever $v(0) = 0$ and $v \in S_p$. We apply this with $v - v(0)$ where $v \in S_p$ is taken arbitrary. Note that

$$\begin{aligned}
&\int_{\Omega} g(x)(v(\nabla w_k(x)) - v(0))dx\\
\overset{k\to\infty}{\to}\ &\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x^1(\mathrm{d}s)g(x)\sigma^1(\mathrm{d}x) - v(0)\int_{\Omega} g(x)dx.
\end{aligned}$$

Moreover, for every constant function C we have $\lim_{s\to\infty}\frac{C}{1+|s|^p} \to 0$ as $s \to \infty$, therefore

$$(v - v(0))_0(s) = v_0(s) \ \text{ for every } s \in \beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}.$$

Hence the right hand side in (12) applied with $v - v(0)$ remains the same as that with v. We arrive at (9). This ends the proof of Case (A).

Case (B): $|F| > 0$. We consider $\Omega_F = \Omega \setminus F$ and $\widetilde{F} = \bar{\Omega}_F \cap F$ instead of Ω and F. We will apply Step 1 to the sequence od restrictions of the original sequence to Ω_F and then extend this sequence by u to the remaining part of Ω. In Steps 1B, 2B and 3B below we explain correctness of such a construction.

Step 1B. We modify the sequence of restrictions to Ω_F.

Before we do this we note that $|\widetilde{F}| = 0$ as $\widetilde{F} \subseteq (\partial\Omega \cap F) \cup \partial_\Omega F$. Moreover, $|\partial\Omega_F| = 0$. Note that $\widetilde{F} \subseteq \partial\Omega_F$ and we have either $\partial\Omega_F = \widetilde{F}$ (in case $\partial\Omega \subseteq F$) or Ω_F has compactness property in $W^{1,p}(\Omega)$. Denote by $\{h_k\} \subseteq W^{1,p}(\Omega_F, \mathbf{R}^m)$ the sequence of restrictions of $\{u_k\}$'s to Ω_F and let $\{(\alpha, \hat{\mu}_x)\}_{x\in\Omega_F}$ be DiPerna Majda measure generated by certain subsequence of $\{\nabla h_k\}_{k\in\mathbf{N}}$. For simplicity this subsequence will also be denoted by $\{\nabla h_k\}_{k\in\mathbf{N}}$. Using Lemma 3.4 we observe that $(\alpha, \hat{\mu}_x) = (\sigma, \hat{\nu}_x)\angle\bar{\Omega}_F$. Let us apply Case (A) to the sequence $\{h_k\} \subseteq W^{1,p}_{u,\widetilde{F}}(\Omega_F, \mathbf{R}^m)$. We find its subsequence (denoted by the same expression) and two sequences $\{\tilde{z}_k\} \subseteq W^{1,p}_{u,\widetilde{F}}(\Omega_F, \mathbf{R}^m)$ and $\{\tilde{w}_k\} \subseteq W^{1,p}_{0,\widetilde{F}}(\Omega_F, \mathbf{R}^m)$ such that (for this subsequence) we have $\nabla h_k = \nabla \tilde{z}_k + \nabla \tilde{w}_k$ and *i)*, *ii)* and *iii)* hold true with Ω substituted by Ω_F and with the same DiPerna Majda measure $(\sigma, \hat{\nu})$ (restricted to $\bar{\Omega}_F$). This ends the construction in Step 1B.

Step 2B. We extend two sequences in decomposition obtained in Step 1B to the whole Ω.

For this, let us denote

$$z_k(x) = \begin{cases} \tilde{z}_k(x) & \text{for } x \in \Omega_F \\ u(x) & \text{for } x \in F \cap \Omega, \end{cases} \qquad w_k(x) = \begin{cases} \tilde{w}_k(x) & \text{for } x \in \Omega_F \\ 0 & \text{for } x \in F \cap \Omega. \end{cases}$$

It remains to show that $\{z_k\}$ and $\{w_k\}$ have the required properties. To verify this at first we observe that sequences defined in this way belong to the required Sobolev class. We show it for sequence $\{z_k\}$ only, as the arguments for the remaining sequence are the same. Using the fact that $\tilde{z}_k - u \in W^{1,p}_{0,\widetilde{F}}(\Omega_F, \mathbf{R}^m)$ we find the sequence $\{\tilde{v}_k^l\}_{l\in\mathbf{N}} \subseteq C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ such that $\tilde{v}_k^l \to \tilde{z}_k - u$ in $W^{1,p}(\Omega_F, \mathbf{R}^m)$ as $l \to \infty$ and $\tilde{v}_k^l \equiv 0$ in some neighborhood of $\widetilde{F}$ in $\mathbf{R}^n$. We will construct a (better) sequence, which vanishes in some neighborhood of F (instead of $\widetilde{F}$) and converges to $z_k - u$ in $W^{1,p}(\Omega, \mathbf{R}^m)$.

For this purpose we take $\epsilon = \epsilon_{l,k}$ such that $\tilde{v}_k^l \equiv 0$ when $\mathrm{dist}(x, \widetilde{F}) < \epsilon_{l,k}$. Let $\psi_{l,k} \in C^\infty(\mathbf{R}^n)$ be such that $\psi_{l,k} \equiv 0$ in a $\epsilon_{l,k}/3$ neighborhood of F and $\psi_{l,k} \equiv 1$ outside $\epsilon_{l,k}$ neighborhood of F. We define $v_k^l(x) := \psi_{l,k}(x)\tilde{v}_k^l(x)$. It is clear that $v_k^l \in C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ and v_k^l vanishes in some neighborhood of F. Moreover,

$$v_k^l(x) = \begin{cases} \tilde{v}_k^l(x) & \text{for } x \in \Omega_F \\ 0 & \text{for } x \in F \cap \Omega. \end{cases}$$

Indeed, on $\Omega_F \cap \{x : \mathrm{dist}(x,F) \le \epsilon_{l,k}\}$ we have $0 = v_k^l(x) = \tilde{v}_k^l(x)$, while on $\Omega_F \cap \{x : \mathrm{dist}(x,F) > \epsilon_{l,k}\}$ we have $v_k^l(x) = \phi_{l,k}(x)\tilde{v}_k^l(x) = \tilde{v}_k^l(x)$. It is also obvious that $\{v_k^l\}$ is a Cauchy sequence in $W^{1,p}(\Omega, \mathbf{R}^m)$. As its limit is $z_k - u$, we have proven that $z_k \in W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$ for every k.

We have

$$\nabla u_k = \nabla z_k + \nabla w_k \ \text{ on } \Omega_F \quad \text{and} \quad \nabla u = \nabla z_k + \nabla w_k \ \text{ on } F \cap \Omega.$$

As $u_k = u$ on $\Omega \cap F$, we get $\nabla u_k = \nabla u$ a.e. on $\Omega \cap F$ (the detailed explanation is given in the proof of Lemma 3.4 in the Appendix) and consequently $\nabla u_k = \nabla z_k + \nabla w_k$ on whole of Ω.

Step 3B. We verify that $\{z_k\}$ and $\{w_k\}$ satisfy the required properties: *i*), *ii*) and *iii*).

At first we note that property *i*) is obvious from construction. Let us verify property *ii*). By the very definition of the noncompact modification we have to verify that

$$\lim_{k\to\infty} \int_\Omega v(\nabla z_k) g(x) dx = \int_\Omega \int_{\mathbf{R}^{m\times n}} v_0(s) \hat{\nu}_x^0(ds) g(x) \sigma^0(dx),$$

whenever $v \in \Upsilon^p(\mathbf{R}^{m\times n})$ and $g \in C(\bar{\Omega})$. We have shown in the proof of Case (A), Step 1A that $(\sigma^0, \hat{\nu}_x^0)$ is DiPerna Majda measure generated by $\{\nabla \overline{z}_k\}$. Therefore it suffices to prove that $\{\nabla z_k\}$ and $\{\nabla \overline{z}_k\}$ generate the same DiPerna Majda measure. To see it we have to check that

$$J = \lim_{k\to\infty} \int_\Omega \left(v(\nabla z_k) - v(\nabla \overline{z}_k) \right) g(x) dx = 0,$$

for an arbitrary $v \in \Upsilon^p(\mathbf{R}^{m\times n}), g \in C(\bar{\Omega})$. The above integral can be as well computed on the set

$$\begin{aligned} \tilde{\mathcal{R}}_k &= \{x \in \Omega : \nabla z_k(x) \neq \nabla \overline{z}_k(x)\} \\ &\subseteq \{x \in \Omega : \nabla z_k(x) \neq \nabla u_k(x)\} \bigcup \{x \in \Omega : \nabla \overline{z}_k(x) \neq \nabla u_k(x)\} =: \mathcal{A}_k \bigcup \mathcal{B}_k. \end{aligned}$$

By construction we have

$$\lim_{k\to\infty} |\mathcal{A}_k| = \lim_{k\to\infty} |\mathcal{B}_k| = 0$$

and both sequences $\{|\nabla z_k|^p\}$ and $\{\nabla \overline{z}_k|^p\}$ are equintegrable. Therefore property *ii*) holds.

We are left with property *iii*). Let $g \in C(\bar{\Omega})$, $v \in S_p$. We can additionally assume that $v(0) = 0$. We have

$$\begin{aligned} \mathcal{L} &= \lim_{k\to\infty} \int_\Omega v(\nabla w_k(x)) g(x) dx \\ &= \lim_{k\to\infty} \left(\int_{\Omega\setminus F} v(\nabla w_k(x)) g(x) dx + \int_F v(\nabla w_k(x)) g(x) dx \right) \\ &= \lim_{k\to\infty} \int_{\Omega_F} v(\nabla \tilde{w}_k(x)) g(x) dx + 0. \end{aligned}$$

The last property follows from construction of the w_k's (Step 2B). By construction in Step 1B the sequence $\{\tilde{w}_k\} \subseteq W^{1,p}_{0,\widetilde{F}}(\Omega_F, \mathbf{R}^m)$ generates such DiPerna Majda measure $(\beta, \hat{\gamma}_x)$ that satisfies (9) with Ω substituted by Ω_F, i.e.

$$\begin{aligned}\mathcal{L} &= \int_{\bar{\Omega}_F} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\gamma}_x(ds)\beta(dx) \\ &= \int_{\bar{\Omega}_F} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx).\end{aligned}$$

As $\int_{\bar{\Omega}} = \int_{\bar{\Omega}_F} + \int_{\bar{\Omega}\setminus\bar{\Omega}_F}$ to complete the arguments we have to check that

$$\int_{\bar{\Omega}\setminus\bar{\Omega}_F} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx) = 0. \tag{13}$$

Note that

$$\bar{\Omega}\setminus\bar{\Omega}_F = \overline{\Omega\cap F}\setminus\overline{\partial_\Omega F}.$$

It is easy to observe that (13) is equivalent to

$$\mathcal{A} := \int_{\overline{\Omega\cap F}} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx) = 0$$

for every $g \in C(\overline{\Omega\cap F})$ such that $g \equiv 0$ on $\overline{\partial_\Omega F}$. Extending every such a function g by putting zero on $\overline{\Omega\setminus F}$ we get a continuous function on $\bar{\Omega}$. Therefore we have to show that

$$\mathcal{A} = \int_{\bar{\Omega}} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx) = 0,$$

where $g \in C(\bar{\Omega})$ and $g \equiv 0$ on $\overline{\Omega\setminus F}$. Let $\psi_\epsilon : [0,\infty) \to [0,1]$ be defined by

$$\psi_\epsilon(t) = \begin{cases} -\frac{1}{\epsilon}t + 1 & \text{for } t \in [0,\epsilon] \\ 0 & \text{for } t > 0 \end{cases}$$

and $v_\epsilon \in \Upsilon^p(\mathbf{R}^{m\times n})$ be defined via its definition on the compactification $\beta_{\mathcal{R}}\mathbf{R}^{m\times n}$ by the expression

$$v^0_\epsilon(s) := v^0(s)\psi_\epsilon(\mathrm{dist}(x, \beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n})).$$

From the Lebegues Dominated Convergence Theorem applied twice we get

$$\begin{aligned}\mathcal{A} &= \int_{\bar{\Omega}} g(x) \lim_{\epsilon\to 0} \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v^\epsilon_0(s)\hat{\nu}_x(ds)\sigma(dx) \\ &= \lim_{\epsilon\to 0}\left(\int_{\bar{\Omega}} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v^\epsilon_0(s)\hat{\nu}_x(ds)\sigma(dx)\right) =: \lim_{\epsilon\to 0}\mathcal{A}_\epsilon.\end{aligned}$$

On the other hand, by the trace condition we have $\nabla u_k(x) = \nabla u(x)$ almost everywhere on $\mathrm{int}(\Omega \cap F)$. Hence

$$\begin{aligned}\mathcal{A}_\epsilon &= \lim_{k\to 0} \int_\Omega g(x) v^\epsilon(\nabla u_k(x))dx \\ &= \lim_{k\to 0} \int_{int(\Omega\cap F)} g(x)v^\epsilon(\nabla u(x))dx = \int_{int(\Omega\cap F)} g(x)v^\epsilon(\nabla u(x))dx.\end{aligned}$$

Let $a := \nabla u(x) \in \mathbf{R}^{m\times n}$. It is well defined for almost every $x \in \mathrm{int}(\Omega \cap F)$. Let $b := i(a) \in \beta_{\mathcal{R}}\mathbf{R}^{m\times n}$ where $i : \mathbf{R}^{m\times n} \to \beta_{\mathcal{R}}\mathbf{R}^{m\times n}$ is the homeomorphic dense embedding we deal with. Then we have

$$v^\epsilon(a) = \frac{v^\epsilon(a)}{1+|a|^p}(1+|a|^p) \quad \text{and} \quad \frac{v^\epsilon(a)}{1+|a|^p} = v_0^\epsilon(b) \overset{\epsilon\to 0}{\to} 0.$$

Therefore $v^\epsilon(\nabla u(x)) \to 0$ for almost every x. Moreover,

$$|v^\epsilon(\nabla u(x))| \leq C(1+|\nabla u(x)|^p).$$

Lebegue's Dominated Convergence Theorem implies $\mathcal{A}_\epsilon \to 0$ as $\epsilon \to 0$. This completes the proof of (13) and ends the proof of the theorem. □

Remark 3.5. Let us note that the assumption "$\Omega \setminus F$ *has a compactness property in* $W^{1,p}$" in the statement of Theorem 3.1 can be weakened. Indeed, we required this assumption to apply Theorem 2.7. As we already mentioned in Remark 2.8, Theorem 2.7 holds also under weaker assumptions.

4. Application. Regularity of minimizing sequences in nonconvex variational problems

4.1. The variational problem and its motivation

Our goal now is to present some applications of our results to the nonconvex variational problems.

Presentation of the problem. We consider variational problem

$$I = \min\{I_f(v) : v \in W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)\}, \quad \text{where } I_f(v) = \int_\Omega f(x, \nabla v(x))dx, \tag{14}$$

under certain conditions to be described later.

Motivation. We can meat this kind of problems in elasticity theory when there is some rigor on deformation of some parts of an elastic body.
Let $\Omega \subseteq \mathbf{R}^3$ be the given body and let $v : \Omega \to v(\Omega) = \Omega' \subseteq \mathbf{R}^3$ be the given deformation. According to the Cauchy-Born role functional of the type

$$\int_\Omega f(x, \nabla v(x))dx$$

describes energy of the given deformation v (with f being the property of the given deformed material, see e.g. [31]). Assume additionally that the fixed part of the material denoted by F is deformed according to the given role. This is interpreted as:

$$v(x) = u(x) \text{ for every } x \in F.$$

In the typical situation $F = \partial\Omega$. This is for example in pure displacement boundary-value problems for a compressible material in nonlinear elasticity, see e.g. [3], Section 3. One can have also situation that $F \subseteq \partial\Omega$ is an essential subset, see e.g. Section 3, page 260 in [5]. Such situation appears in the study of mixed displacement dead load traction boundary conditions in nonlinear elasticity. See also Section 3 in [4]. Weak formulation of energy minimization problem in the class of such deformations is exactly (14).

4.2. Attainment result for minimizing Young measures

The conditions. We will deal with the following conditions.

A) Ω is a bounded domain in $\mathbf{R}^n$, $|\partial\Omega| = 0$;

B) $F \subseteq \bar{\Omega}$ is a closed, $\partial\Omega \subseteq F$ or $\Omega \setminus F$ has compactness property in $W^{1,p}(\Omega)$, $|\partial_\Omega F| = 0$;

C) $\mathcal{R} \subseteq C(\mathbf{R}^{m\times n})$ is a complete subring and $f \in C(\bar{\Omega} \times \mathbf{R}^{m\times n})$ are such that:
   1. for every $x \in \bar{\Omega}$ we have $f(x,s)/(1+|s|^p) \in \mathcal{R}$ and $\frac{f(x,s)}{1+|s|^p} \in C(\bar{\Omega} \times \beta_{\mathcal{R}}\mathbf{R}^{m\times n})$,
   2. $Qf(x,\cdot) > -\infty$,
   3. in case $\partial\Omega \not\subseteq F$ we require:

$$\liminf_{s\to\infty} \frac{f(x,s)}{1+|s|^p} \geq 0 \text{ for every } x \in \partial\Omega \setminus \mathrm{int}_{\partial\Omega}(F \cap \partial\Omega);$$

D) $u \in W^{1,p}(\Omega, \mathbf{R}^m)$.

Regularity result for minimizing sequences of gradients. We are now to present certain application of Theorem 3.1. It can be interpreted as the regularity result for minimizing sequences of gradients. Typically one assumes that function f is nonnegative. Now instead of this assumption we have an substantially weaker one: Assumption 3 in Condition C). If $\partial\Omega \subseteq F$ this condition is empty. Proofs of Theorem 4.1 and Corollary 4.2 given below are postponed to the next subsection.

Theorem 4.1. *Let the assumptions* A)–D) *be satisfied,* $I > -\infty$ *and assume that there exists the minimizing sequence* $\{u_k\} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$ *for* I_f *which is bounded in* $W^{1,p}(\Omega, \mathbf{R}^m)$. *Then there is a (possibly another) sequence* $\{z_k\} \subseteq W^{1,p}_{u,F\cup\partial\Omega}(\Omega, \mathbf{R}^m)$ *minimizing* I_f *such that the sequence* $\{|\nabla z_k|^p\}$ *is weakly compact in* $L^1(\Omega)$.

As a corollary we obtain the following result.

Corollary 4.2. *Suppose that the assumptions of Theorem* 4.1 *are satisfied and additionally I_f satisfies the following coercivity condition:*

$$\text{If the sequence } \{I_f(u_k)\} \text{ is bounded then } \{u_k\} \text{ is bounded in } W^{1,p}(\Omega, \mathbf{R}^m). \tag{15}$$

Let $\mathcal{Y}_{u,F}$ be the set of all classical Young measures generated by sequences of gradients $\{\nabla u_k\}$ where the sequence $\{u_k\}$ is bounded in $W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$. Then we have

$$\begin{aligned} &\inf\{I_f(v) : v \in W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)\} \\ = &\min\left\{\int_\Omega \int_{\mathbf{R}^{m\times n}} f(x,\lambda)\nu_x(d\lambda)dx : \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}_{u,F}\right\}. \end{aligned} \tag{16}$$

Remark 4.3. The problem to represent the minimum of functional in terms of Young measures like in (16) had been addressed in Pedregal's paper [34] were applications to the theory of magnetostriction and optimal design problems were given (see e.g. Proposition 4.2 and Theorem 5.1). For example on page 48 the author deals with functional $\int_\Omega F(x, u(x), y(x), \nabla y(x))dx$ serving for optimal design problems, under the assumption $\lim_{s\to\infty} F(x, u, y, s)|s|^{-2} = 0$ (where F is of growth 2 with respect to the last variable, i.e. $p = 2$). As a corollary the author obtains attainment result (16) for the considered functional. Note that this is the special case of our assumption **C)**, point 3.

Remark 4.4. Contrary to the approach in [7] we do not use directly quasiconvex envelops of functions.

4.3. Proofs of Theorem 4.1 and Corollary 4.2

Before we prove Theorem 4.1 we will need the following result.

Theorem 4.5. *Let Ω is a bounded domain in $\mathbf{R}^n$, $|\partial\Omega| = 0$ and $\mathcal{R} \subseteq C(\mathbf{R}^{m\times n})$ be a complete subring. Assume further that $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$ is generated by the bounded sequence $\{u_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}(\Omega, \mathbf{R}^m)$. Then for σ-almost all $x \in \Omega$ and all $f \in \Upsilon^p(\mathbf{R}^{m\times n})$ with $Qf > -\infty$ it holds that*

$$0 \le \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(s)}{1+|s|^p}\hat{\nu}_x(ds)\ . \tag{17}$$

Moreover, if $F \subseteq \bar{\Omega}$ is the given closed set such that $F \cap \partial\Omega \neq \emptyset$ and additionally $\{u_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$ where $u(x) = d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}_x d\sigma$ a.e., then formulae (17) *holds also for σ-almost every $x \in \mathrm{int}_{\partial\Omega}(F \cap \partial\Omega)$.*

Before we prove the general statement we recall the following result obtained in [18], Theorem 4.6.

Theorem 4.6. *Suppose that the following conditions hold*

i) $\mathcal{R} \subseteq C(\mathbf{R}^{m\times n})$ *is the given separable complete ring and for every $v_0 \in \mathcal{R}$ we write $v(s) := v_0(s)(1 + |s|^p)$.*

ii) $G_1, G \subseteq \bar{\Omega}$ *are closed subsets,* $G \neq \emptyset$ *and either* $\partial\Omega \subseteq G_1 \cup G$ *or* $\Omega \setminus G$ *has a compactness property in* $W^{1,p}$.

iii) $\{u_k\} \subseteq W^{1,p}_{u,G_1}(\Omega, \mathbf{R}^m)$ *is the given sequence,* $u_k \rightharpoonup u$ *in* $W^{1,p}$ *and for every* $v_0 \in \mathcal{R}$ *the sequence of measures* $\{v(\nabla u_k)dx\}_{k\in\mathbf{N}}$ *converges weekly* $*$ *to* $\mu_v \in rca(\bar{\Omega})$.

Then there exists the sequence $\{\nabla w_k\}_{k\in\mathbf{N}}$ *such that* $\{w_k - u\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{0,G_1\cup G}(\Omega, \mathbf{R}^{m\times n})$ *and* $*\lim_{k\to\infty} v(\nabla w_k)dx = \tilde{\mu}_v$ *for every* $v_0 \in \mathcal{R}$, *where* $\tilde{\mu}_v$ *satisfies*

$$\tilde{\mu}_v \angle G = v(\nabla u(x))dx \angle G \quad \text{and} \quad \tilde{\mu}_v \angle (\overline{\Omega} \setminus G) = \mu_v \angle (\overline{\Omega} \setminus G).$$

Proof of Theorem 4.5. Formulae (17) for $x \in \Omega$ and second statement in the case when $F = \partial\Omega$ were obtained in [17], Theorems 2.7 and 2.8. Now we generalize it for arbitrary closed set $F \subseteq \bar{\Omega}$ such that $F \cap \partial\Omega \neq \emptyset$.

We apply Theorem 4.6 to our sequence $\{u_k\}$ with sets $G_1 := F$ and $G := \overline{(\partial\Omega \setminus F)}$. Conditions *i)*–*iii)* are satisfied. Hence there exists the sequence $\{w_k\} \subseteq W^{1,p}_{u,F\cup\partial\Omega}(\Omega, \mathbf{R}^m)$ generating DiPerna Majda measure $(\alpha, \hat{\mu}_x)$ such that for every $g \in C(\bar{\Omega})$ supported in $\bar{\Omega} \setminus G$ and every $v \in \Upsilon^p(\mathbf{R}^{m\times n})$ we have

$$\begin{aligned}
\int_{\bar{\Omega}} g(x)\mu_v(dx) &= \lim_{k\to\infty} \int_{\bar{\Omega}} g(x)v(\nabla u_k(x))dx \\
&= \lim_{k\to\infty} \int_{\bar{\Omega}} g(x)v(\nabla w_k(x))dx = \int_{\bar{\Omega}} g(x)\tilde{\mu}_v(dx).
\end{aligned}$$

As left hand side above equals $\int_{\bar{\Omega}} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx)$, while right hand side equals $\int_{\bar{\Omega}} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\mu}_x(ds)\alpha(dx)$, we recognize (plug at first $v_0 \equiv 1$) that

$$\sigma\angle\left(\bar{\Omega} \setminus G\right) = \alpha\angle\left(\bar{\Omega} \setminus G\right) \quad \text{and} \quad \hat{\nu}_x = \hat{\mu}_x \ \text{ for } \sigma \text{ almost every } x \in \bar{\Omega} \setminus G.$$

Note that

$$\bar{\Omega} \setminus G = \bar{\Omega} \setminus \overline{(\bar{\Omega} \setminus F)} \supseteq \mathrm{int}_{\partial\Omega}\left(F \cap \partial\Omega\right). \tag{18}$$

In particular

$$\hat{\mu}_x = \hat{\nu}_x \ \text{ for } \sigma \text{ almost every } x \in \mathrm{int}_{\partial\Omega}\left(F \cap \partial\Omega\right). \tag{19}$$

On the other hand $\{w_k\} \subseteq W^{1,p}_{u,\partial\Omega}(\Omega, \mathbf{R}^m)$, therefore we deduce from already known case $F = \partial\Omega$ that

$$0 \leq \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(s)}{1+|s|^p}\hat{\mu}_x(\mathrm{d}s) \ .$$

for α- almost every $x \in \partial\Omega$. According to (18) we have $\alpha = \sigma$ when restricted to $\mathrm{int}_{\partial\Omega}\left(F \cap \partial\Omega\right)$ and final conclusion follows from (19). The theorem is proven. $\square$

For our purposes we will need the following generalization of condition (17) to f being x-dependent. For readers convenience its proof is presented in the Appendix.

Lemma 4.7. *Let Ω be a bounded domain in $\mathbf{R}^n$, $|\partial\Omega| = 0$, $\mathcal{R} \subseteq C(\mathbf{R}^{m\times n})$ be a complete subring, $f \in C(\bar{\Omega} \times \mathbf{R}^{m\times n})$ be such that for every $x \in \bar{\Omega}$ we have $f(x,s)/(1+|s|^p) \in \mathcal{R}$ and $\frac{f(x,s)}{1+|s|^p} \in C(\bar{\Omega} \times \beta_{\mathcal{R}}\mathbf{R}^{m\times n})$, moreover, $Qf(x,\cdot) > -\infty$. Assume further that $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n})$ is generated by the bounded sequence $\{\nabla u_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}(\Omega,\mathbf{R}^m)$. Then for σ-almost all $x \in \Omega$ it holds that*

$$0 \leq \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\ . \tag{20}$$

Moreover, if $F \subseteq \bar{\Omega}$ is the given closed set such that $F \cap \partial\Omega \neq \emptyset$ and additionally $\{u_k\}_{k\in\mathbf{N}} \subseteq W^{1,p}_{u,F}(\Omega,\mathbf{R}^m)$ where $u(x) = d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}_x d\sigma$ a.e., then formulae (20) holds also for σ-almost every $x \in \mathrm{int}_{\partial\Omega}(F \cap \partial\Omega)$.

Proof of Theorem 4.1. We can assume that the sequence $\{u_k\} \subseteq W^{1,p}_{u,F}(\Omega,\mathbf{R}^m)$ generates DiPerna Majda measure $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbf{R}^{m\times n})$. We have by Theorem 2.3

$$I = \lim_{k\to\infty} I_f(u_k) = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\sigma(dx).$$

We decompose $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} = \int_{\mathbf{R}^{m\times n}} + \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}}$ and note that

$$\begin{aligned} II := & \int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\sigma(dx) \\ = & \int_{\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\sigma(dx) \\ & + \int_{\partial\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\sigma(dx). \end{aligned}$$

Using Lemma 4.7 we conclude that $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds) \geq 0$ for σ almost every $x \in \Omega$. Moreover, by assumption **C)**, point 3 and by Lemma 4.7 we have $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds) \geq 0$ for σ almost every $x \in \partial\Omega$. Therefore $II \geq 0$. Decomposing $\sigma(dx) = d_\sigma(x)dx + \sigma_s(dx)$ we get

$$\begin{aligned} I \geq & \int_{\bar{\Omega}}\int_{\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\sigma(dx) \\ = & \int_{\Omega}\int_{\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)d_\sigma(x)dx + \int_{\bar{\Omega}}\int_{\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}_x(ds)\sigma_s(dx). \end{aligned}$$

According to Lemma 2.6 second term above is zero. Taking this into account, and also Lemma 2.13, part *ii)* and (7), we get

$$I \geq \int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p}\hat{\nu}^0_x(ds)\sigma^0(dx),$$

where (σ^0, ν_x^0) is the nonconcentrating modification of $(\sigma, \hat{\nu})$. By Corollary 3.2 there exists the sequence $\{\nabla z_k\}$ generating (σ^0, ν^0) such that $\{z_k\} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$. Altogether gives

$$I = \lim_{k\to\infty} I_f(z_k)$$

and by the very definition of nonconcentrating modification the sequence $\{|\nabla z_k|^p\}$ is weakly compact in $L^1(\Omega)$. The fact that $\{z_k\}$ can be chosen to satisfy the condition $\{z_k\} \subseteq W^{1,p}_{u,F\cup\partial\Omega}(\Omega, \mathbf{R}^m)$ follows from Theorem 2.7 applied with $F_1 := F, F := \partial\Omega$. This ends the proof of the theorem. □

Proof of Corollary 4.2. Let us denote the right hand side in (16) by J and the left hand side by I. Let $\{u_k\}$ be the sequence achieving I, i.e. $I_f(u_k)$ converges to I. Using Theorem 4.1 we can additionally assume that the sequence $\{|\nabla u_k|^p\}$ is weakly compact in $L^1(\Omega)$. By Theorem 2.1 we get

$$I = \lim_{k\to\infty} I_f(u_k) = \int_\Omega \int_{\mathbf{R}^{m\times n}} f(x,\lambda)\nu_x(d\lambda)dx \geq J.$$

It reminds to prove that $J \geq I$. To verify this we assume that

$$J = \lim_{k\to\infty} \int_\Omega \int_{\mathbf{R}^{m\times n}} f(x,\lambda)\nu_x^k(d\lambda)dx =: \lim_{k\to\infty} J_k,$$

where $\{\nu_x^k\}_{x\in\Omega}$ are classical Young measures generated by some bounded sequence $\{\nabla w_l^k\}_{l\in\mathbf{N}} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$. After (eventually) extracting the subsequence we can assume that every such sequence generates DiPerna Majda measure $(\sigma^k, \hat{\nu}_x^k) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$, with nonconcentrating modification $(\sigma^{0,k}, \hat{\nu}_x^{0,k}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$. Let this modification be generated by the sequence $\{\nabla \tilde{z}_l^k\}_{l\in\mathbf{N}}$ such that $\{\tilde{z}_l^k\} \subseteq W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$ (we use Corollary 3.2). Using Lemma 2.13 and (7) we get for every k

$$J_k = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p} \hat{\nu}_x^{0,k}(ds)\sigma^{0,k}(dx) = \lim_{l\to\infty} I_f(\tilde{z}_l^k).$$

Let us fix $\epsilon > 0$, chose for every k the number $l(k)$ such that

$$|J_k - I_f(\tilde{z}^k_{l(k)})| \leq \frac{\epsilon}{k}$$

and denote $v_k = \tilde{z}^k_{l(k)} \in W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$. We observe that the sequence $\{I_f(v_k)\}_{k\in\mathbf{N}}$ converges to J as $k \to \infty$. Coercivity condition (15) implies that the sequence $\{v_k\}$ is bounded in $W^{1,p}(\Omega, \mathbf{R}^m)$. Passing eventually to the subsequence again we can assume that it generates DiPerna Majda measure $(\alpha, \hat{\mu}_x) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbf{R}^{m\times n})$, so that

$$J = \lim_{k\to\infty} I_f(v^k) = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} \frac{f(x,s)}{1+|s|^p} \hat{\mu}_x(ds)\alpha(dx) \geq I.$$

This completes our arguments. □

A. Proofs of Proposition 2.11 and of Lemmas 2.13, 3.4 and 4.7

Proof of Proposition 2.11. *i)*: See e.g. [35], Propositions 3.2.17, 3.4.17 and 3.4.18.

ii): Let $v \in \Upsilon^p(\mathbf{R}^{m\times n})$ be arbitrary now and $\phi \in C_0^\infty(\mathbf{R}^{m\times n})$ be such that $\phi \equiv 1$ in a neighborhood of 0. Define $\phi_R(s) = \phi(s/R)$, $v_R(s) = \phi_R(s)v(s)$. Then (8) holds with $v = v_R$. After letting $R \to \infty$ and applying Lebegue's Dominated Convergence Theorem, we get (8) with an arbitrary v.

Let us decompose $\sigma = d_\sigma(x)dx + \sigma_s$ into an absolutely continuous and singular part in the Lebegue decomposition. We have

$$\begin{aligned}&\int_{\bar\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(dx)\\ =&\int_{\bar\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)(dx) + \int_{\bar\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma_s(dx).\end{aligned}$$

Second term above is zero, because according to Lemma 2.6 we have $\hat\nu_x(\mathbf{R}^{m\times n}) = 0$ for σ_s-almost all $x \in \bar\Omega$. Therefore right hand side in (8) equals $\int_{\bar\Omega}\int_{\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)(dx)$.

Note that sequence $\{|\nabla \bar z_k|^p\}_{k\in\mathbf{N}}$ is weakly compact in $L^1(\Omega)$. It implies that σ^0 is absolutely continuous with respect to the Lebegue measure and according to Lemma 2.5 we get $\hat\nu_x^0(\beta_{\mathcal{R}}\mathbf{R}^{m\times n} \setminus \mathbf{R}^{m\times n}) = 0$ for σ^0 almost all x.

Therefore the right hand side in (8) equals $\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x^\circ(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x)$.

Sketch of the proof of Lemma 2.13. *i)*: The proof can be found in [25, Lemma 1, Th. 1, 2] and [35, Prop. 3.2.17].

ii): This follows directly from Proposition 2.11, part *ii)*.

iii): It easily follows from Theorems 2.1 and 2.3. □

Proof of Lemma 3.4. Let $g \in C(\bar\Omega_F)$ and $v \in \Upsilon^p(\mathbf{R}^{m\times n})$ be taken arbitrary. Let $\tilde g$ be an arbitrary extension of g to a continuous function on $\bar\Omega$. We define the function $y_\epsilon : [0,\infty) \to [0,1]$ by expression $y_\epsilon(t) = -\frac{1}{\epsilon}t + 1$ when $t \in [0,\epsilon]$ and $y_\epsilon \equiv 0$ on $[\epsilon,\infty)$. Then we set: $\phi_\epsilon(x) = y_\epsilon(\mathrm{dist}(x,\Omega_F))$, $\phi_\epsilon \in C(\bar\Omega)$. According to Theorem 2.3 we have

$$I_{g,v} := \lim_{k\to 0}\int_{\Omega_F} g(x)v(\nabla u_k(x))dx = \int_{\bar\Omega_F} g(x)\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\mu_x(ds)\alpha(dx),$$

and at the same time

$$\begin{aligned}J_{g,v} :=& \lim_{\epsilon\to 0}\lim_{k\to\infty}\int_\Omega \tilde g(x)\phi_\epsilon(x)v(\nabla u_k(x))dx\\ =& \lim_{\epsilon\to 0}\int_{\bar\Omega}\tilde g(x)\phi_\epsilon(x)\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(ds)\sigma(dx)\\ =& \int_{\bar\Omega_F} g(x)\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat\nu_x(ds)\sigma(dx).\end{aligned}$$

It suffices to prove that

$$J_{g,v} = I_{g,v} \quad \text{for every } g \text{ and } v. \tag{21}$$

Indeed, having (21) we substitute at first $v(s) = (1 + |s|^p)$, so that $v_0 \equiv 1$, to get:

$$\int_{\bar{\Omega}_F} g(x)\alpha(dx) = \int_{\bar{\Omega}_F} g(x)\sigma(dx)$$

for every $g \in C(\bar{\Omega}_F)$, so that $\alpha = \sigma$ as measures on $\bar{\Omega}_F$. Then inequality:

$$\int_{\bar{\Omega}_F} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\mu}_x(ds)\sigma(dx) = \int_{\bar{\Omega}_F} g(x) \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx)$$

implies that $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\mu}_x(ds)\sigma(dx) = \int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)\sigma(dx)$ as measures on $\bar{\Omega}_F$, so that their densities with respect to σ: $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\mu}_x(ds)$ and $\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}} v_0(s)\hat{\nu}_x(ds)$ are equal σ-almost everywhere. This will finish the proof. We are left with the proof of (21).

We have

$$\begin{aligned} J_{g,v} - I_{g,v} &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\left(\int_{\Omega} \tilde{g}(x)\phi_\epsilon(x)v(\nabla u_k(x))dx - \int_{\Omega_F} g(x)v(\nabla u_k(x))dx\right) \\ &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{\Omega\cap F} \tilde{g}(x)\phi_\epsilon(x)v(\nabla u_k(x))dx = \mathcal{L}. \end{aligned}$$

By an assumption the set $\Omega \cap F$ is of positive measure. It is crucial now that $\nabla u_k(x) = \nabla u(x)$ almost everywhere on $\Omega \cap F$. Indeed, by an assumption $u_k \in W^{1,p}_{u,F}(\Omega, \mathbf{R}^m)$, equivalently $u_k - u \in W^{1,p}_{0,F}(\Omega, \mathbf{R}^m)$. This implies that $u_k - u \equiv 0$ a.e. on $\Omega \cap F$. To see that we consider a sequence $\{\tilde{v}^l_k\}_{l\in\mathbf{N}} \subseteq C^\infty_0(\Omega, \mathbf{R}^m)$ such that $v^l_k \to u_k - u$ in $W^{1,p}(\Omega, \mathbf{R}^m)$ as $l \to \infty$ and $v^l_k \equiv 0$ in a neighborhood of F (by the very definition of the space $W^{1,p}_{0,F}(\Omega, \mathbf{R}^m)$ such sequence exists). Then we have $0 \equiv v^l_k \to u_k - u$ in $L^p(\Omega \cap F)$, therefore $u_k = u$ a.e. on $\Omega \cap F$. Consequently $\nabla u_k = \nabla u$ a.e. on $\Omega \cap F$, which together with the fact that $\phi_\epsilon \to 0$ a.e. on $\Omega \cap F$ as $\epsilon \to 0$ and Lebegue's Dominated Convergence Theorem gives

$$\mathcal{L} = \lim_{\epsilon\to 0}\int_{\Omega\cap F} \tilde{g}(x)\phi_\epsilon(x)v(\nabla u(x))dx = 0$$

and finishes the proof of the lemma. □

Proof of Lemma 4.7. We prove the lemma under the condition $u_k = u$ on $\partial\Omega$ as the general statement follows by almost the same arguments. We use the technique from [16], where we dealt with classical Young measures generated by gradients. Let $\{A_j\}_{j\in\mathbf{N}}$ be the countable dense subset in $\bar{\Omega}$. For every fixed j we consider set $\Omega(j)$ of full measure σ in $\bar{\Omega}$ such that for every $x \in \Omega(j)$ $\hat{\nu}_x$ is well defined and satisfies inequality

$$\int_{\beta_{\mathcal{R}}\mathbf{R}^{m\times n}\setminus\mathbf{R}^{m\times n}} \frac{f(A_j,s)}{1+|s|^p}\hat{\nu}_x(ds) \geq 0. \tag{22}$$

Then set $\tilde{\Omega} = \cap_j \Omega(j)$ is also of full measure σ in $\bar{\Omega}$ and for every $x \in \tilde{\Omega}$ inequality (22) holds with every j. Let us fix $x \in \tilde{\Omega}$ and chose the sequence $\{A_{j_k}\}_{k\in\mathbf{N}}$ such that $A_{j_k} \to x$ as $k \to \infty$. Now it suffices to apply inequality (22) with A_{j_k} instead of A_j, let $k \to \infty$ and apply Lebegue's Dominated Convergence Theorem. $\square$

Acknowledgements. I would like to thank Martin Kružík for many helpful discussions. I am also indebted to Piotr Kalita for his helpful advice and interesting discussion about mathematical models.

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