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Smallness of Singular Sets of Semiconvex Functions in Separable Banach Spaces

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Let X be a separable superreflexive Banach space and f be a semiconvex function (with a general modulus) on X. For $k \in \mathbb{N}$, let $\Sigma_k(f)$ be the set of points $x \in X$, at which the Clarke subdifferential $\partial f(x)$ is at least k-dimensional. Note that $\Sigma_1(f)$ is the set of all points at which f is not Gâteaux differentiable. Then $\Sigma_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k which are described by functions, which are differences of two semiconvex functions. If X is separable and superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type 2 (e.g., if X is a Hilbert space or $X = L^p(\mu)$ with $2 \le p$), we give, for a fixed modulus ω and $k \in \mathbb{N}$, a complete characterization of those $A \subset X$, for which there exists a function f on X which is semiconvex on X with modulus ω and $A \subset \Sigma_k(f)$. Namely, $A \subset X$ has this property if and only if A can be covered by countably many Lipschitz surfaces S_n of codimension k which are described by functions, which are differences of two Lipschitz semiconvex functions with modulus $C_n\omega$.

Keywords: Semiconvex function with general modulus, Clarke subdifferential, singular set, singular point of order k, Lipschitz surface, DSC surface, superreflexive space

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1. Introduction

In this article, we study smallness of singular sets $\Sigma_k(f)$ ($k \in \mathbb{N}$) of semiconvex functions in separable Banach spaces. By a *semiconvex function* we mean a semiconvex function with a general modulus in the present article. (Note that the definition of a "semiconvex function in $\mathbb{R}^n$" usually refers to a semiconvex function with a linear modulus, i.e. to a function, which can be represented in the form $f(x) = c(x) - K\|x\|^2$, where $K \ge 0$ and $c(x)$ is a convex function.)

Semiconvex functions (and dual semiconcave functions) with general modulus form an important class of functions, cf. the monograph [5]. Note that (see [7, p. 239]) semiconvex functions on a Banach space X essentially coincide with strongly para-

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convex functions of Rolewicz and also with uniformly Fréchet subdifferentiable functions and, if $X = \mathbb{R}^n$, then locally semiconvex functions coincide with Spingarn's lower C^1 functions, with approximately convex functions (in the sense of [9]), and also with weakly convex functions in Nurminskii's sense.

If f is a semiconvex function on a Banach space and $k \in \mathbb{N}$, then we consider the set $\Sigma_k(f)$ of *singular points of order* k (or of magnitude k by [5]) at which the Clarke subdifferential $\partial f(x)$ is at least k-dimensional. Note that $\Sigma_1(f)$ is the set of all points at which f is not Gâteaux differentiable.

The smallness of singular sets of semiconvex functions was investigated in $\mathbb{R}^n$ in [2] and in [1] in separable Banach spaces (cf. also [5]). It was essentially proved in [1], that, for a semiconvex function f on a separable Banach space, the singular set $\Sigma_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k (see Definition 2.15 below). Even though the definition of an $\infty - k$ dimensional rectifiable set used in [1] is too weak, since the whole space can be such a set, the proof gives the result stated above.

It is proved in [15] that the mentioned result of [1] holds even for all Clarke regular functions (and so also for each approximately convex function in the sense of [9]).

If X is a separable Banach space, then a complete characterization of the smallness of sets $\Sigma_k(f)$, where f is a continuous convex function on X, was given in [13, Proposition 2 and 3]. In the case of convex f, the corresponding Lipschitz surfaces of codimension k are DC-surfaces: they are described by Lipschitz functions, which are differences of two convex continuous functions. It is thus natural to conjecture that in the case of a semiconvex function, the corresponding Lipschitz surfaces are described by functions which are both Lipschitz and representable as a difference of two semiconvex functions. Actually, we show that, in some spaces, the method of [13] works and so the conjecture is true. The only essential problem of this generalization is that it is not easy to transform the key lemma [13, Lemma 1] to the "semiconvex case". However, we were able to prove in [7] a corresponding result (see Proposition 2.12 below) on semiconvex functions in some important special spaces.

The structure of the article is the following. In Section 2 we recall a number of known notions and results. We prove also some needed simple results on semiconvex functions. Further, we introduce necessary terminology concerning surfaces of finite codimension and prove some facts we need. Section 3 contains our main results concerning smallness of singular sets $\Sigma_k(f)$, whose informal description (without precise terminology concerning surfaces) follows.

If X is separable and superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type 2 (e.g., if X is a Hilbert space or $X = L^p(\mu)$ with $2 \le p$), we give, for a fixed modulus ω and $k \in \mathbb{N}$, a complete characterization (Theorem 3.8(i)) of those $A \subset X$, for which there exists a function f on X which is semiconvex on X with modulus ω and $A \subset \Sigma_k(f)$. Namely, $A \subset X$ has this property if and only if A can be covered by countably many Lipschitz surfaces S_n of codimension k which are described by functions, which are differences of two Lipschitz semiconvex functions with modulus $C_n\omega$ (for some $C_n > 0$). Moreover, Proposition 3.9 states that this result holds in some more general spaces X in the case of moduli of a power type.

Theorem 3.5 implies that if f is a semiconvex function on a separable superreflexive Banach space, then $\Sigma_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k which are described by functions, that are differences of two Lipschitz semiconvex functions. Moreover, if X is a separable and superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type 2, then this result on smallness of $\Sigma_k(f)$ is the best possible (see Theorem 3.8(ii)).

In Section 4, we prove, in some important cases, the plausible hypothesis that the assertions of our result (Theorem 3.8(i)) concerning smallness of sets of Gâteaux nondifferentiable points $\Sigma_1(f)$ for functions which are semiconvex with a given modulus are different for essentially different moduli ω and $\tilde{\omega}$. More precisely (see Corollary 4.8), if ω and $\tilde{\omega}$ are concave moduli such that $\omega(t) = o(\tilde{\omega}(t))$, $t \to 0+$, and X admits an equivalent norm with modulus of smoothness of power type 2, then there exists $A \subset X$ such that $A \subset \Sigma_1(\tilde{f})$ for an $\tilde{f}$ semiconvex with modulus $\tilde{\omega}$, but $A \subset \Sigma_1(f)$ for no function f that is semiconvex with modulus ω. Since already this proof is rather technical, we do not consider the more complicated case of $\Sigma_k(f)$ for $k > 1$.

2. Preliminaries

2.1. Basic notation

In the following, X will always be a real Banach space. By $B(x, r)$ we denote the open ball with center x and radius r. If X is a Banach space and $X = E \oplus F$, then we denote by $\pi_{E,F}$ the projection of X on E along the space F. The symbol $\langle \cdot, \cdot \rangle$ is used for the usual duality on $X \times X^*$. For $A \subset X$, we set $A^\perp := \{x^* \in X^* : \langle x, x^* \rangle = 0 \text{ for each } x \in A\}$. For $t \in \mathbb{R}$, we set $t^+ := \max(t, 0)$ and $t^- := \max(-t, 0)$. If $a_n \in \mathbb{R}$, the equality $\sum_{i\in\mathbb{N}} a_i = s \in [-\infty, \infty]$ means (as usual) that $\sum_{i=1}^\infty (a_i)^+ - \sum_{i=1}^\infty (a_i)^- = s$. Recall that, if $\sum_{i\in\mathbb{N}} a_i = s$, then $\sum_{i=1}^\infty a_i = s$. The symbol $s = \pm\infty$ means $s \in \{\infty, -\infty\}$.

Definition 2.1. Let X be a Banach space, $x \in X$, $v \in X$, and f a real function defined on a subset of X. Then we define the *upper (Dini) one-sided directional derivative of f at x in the direction v* by

$$d^+ f(x; v) := \limsup_{h\to 0+} (f(x + hv) - f(x))h^{-1}.$$

The *directional* and *one-sided directional derivatives of f at $x \in G$ in the direction $v \in X$* are defined respectively by

$$f'(x, v) := \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad f'_+(x, v) := \lim_{t\to 0+} \frac{f(x + tv) - f(x)}{t}.$$

If f is locally Lipschitz on an open set $G \subset X$, then

$$f^0(a,v) := \limsup_{z \to a, t \to 0+} \frac{f(z+tv) - f(z)}{t}$$

is the *Clarke derivative of f at a in the direction v* and

$$\partial f(a) := \{x^* \in X^* : \ x^*(v) \leq f^0(a,v) \text{ for all } v \in X\}$$

is the *Clarke subdifferential of f at a* (which is always non-empty and convex). We say that *f is Clarke regular at $x \in G$* if $f^0(x,v) = f'_+(x,v)$ for each $v \in X$. We say that *f is Clarke regular on G*, if f is Clarke regular at each point of G.

2.2. Semiconvex functions and their differences.

Definition 2.2. We denote by $\mathcal{F}$ the set of all functions $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which are non-decreasing and right continuous at 0.

The following definition is taken from [5]; the definitions in [2] and [1] are slightly different, but essentially equivalent.

Definition 2.3. A real valued continuous function f on an open convex set $\Omega \subset X$ is called *semiconvex with modulus $\omega \in \mathcal{F}$* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|, \qquad (1)$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

A function is called *semiconvex on* Ω if it is semiconvex on Ω with some modulus $\omega \in \mathcal{F}$.

It follows directly from the definition that if f and g are semiconvex on Ω with modulus ω and $c > 0$, then

$$\begin{gathered} \max(f,g) \text{ is semiconvex with modulus } \omega \\ \text{and } cf \text{ is semiconvex with modulus } c\omega. \end{gathered} \qquad (2)$$

Semiconvex functions have many properties of continuous convex functions. In particular, the following facts are well-known.

Lemma 2.4. *Let f be a semiconvex function on an open convex subset $G \subset X$. Then the following hold.*

(*i*) *f is locally Lipschitz and the one-sided directional derivative $f'_+(x,v)$ exists for all $x \in G$ and $v \in X$.*
(*ii*) *f is Clarke regular on G.*
(*iii*) *$f'_+(x,v) + f'_+(x,-v) \geq 0$ for all $x \in G$ and $v \in X$.*

For (i) and (ii) see, e.g., [11, Lemma 2.5], which deals with more general approximately convex functions and (iii) follows from (ii), since it is well-known that the function $v \mapsto f^0(x,v)$ is convex.

We will also need the following lemma.

Lemma 2.5. *Let f be a semiconvex function on an open interval $I \subset \mathbb{R}$. Then for, each $x \in I$,*

(i) $f'_+(x) \geq f'_-(x)$,
(ii) $f'_-(x) = \lim_{t\to x-} f'_+(t)$, *and*
(iii) $f'_+(x) = \lim_{t\to x+} f'_-(t)$.

Proof. The inequality (i) immediately follows from Lemma 2.4(iii). Since f is lower C^1 function in the sense of [10] (this follows immediately, e.g., from [7, Theorem 5.5]), we obtain by [10, Proposition 2.3] that f is semismooth, which easily implies (ii) and (iii). □

Lemma 2.6. *Let f_n, $n \in \mathbb{N}$, be functions on an open convex set $\Omega \subset X$ which are semiconvex with modulus $\omega \in \mathcal{F}$. Let (c_n) be a sequence of nonnegative numbers such that $c := \sum_{n=1}^\infty c_n < \infty$ and $f(x) := \sum_{n=1}^\infty c_n f_n(x)$ is finite for each $x \in \Omega$. Then f is semiconvex with modulus $c\omega$ on Ω.*

Proof. Let $x, y \in A$ and $\lambda \in [0,1]$. Then

$$
\begin{aligned}
&f(\lambda x + (1-\lambda)y) = \sum_{n=1}^\infty c_n f_n(\lambda x + (1-\lambda)y)\\
\leq\ &\sum_{n=1}^\infty c_n \lambda f_n(x) + \sum_{n=1}^\infty c_n (1-\lambda) f_n(y) + \sum_{n=1}^\infty c_n \lambda(1-\lambda)\omega(\|x-y\|)\,\|x-y\| \qquad (3)\\
=\ &\lambda f(x) + (1-\lambda)f(y) + \lambda(1-\lambda)c\omega(\|x-y\|)\|x-y\|.
\end{aligned}
$$
□

Lemma 2.7. *Let X, Y be Banach spaces, $A \subset X$ an open convex set, and $B \subset Y$ an open convex set. Let $\varphi : X \to Y$ be a continuous affine mapping of the form $\varphi(x) = c + L(x)$, where $c \in Y$ and $L : X \to Y$ is a continuous linear mapping, with $\varphi(A) \subset B$. Let f be a function on B semiconvex with modulus $\omega \in \mathcal{F}$. Then there exists an $E > 0$ such that $g := f \circ \varphi$ is semiconvex on A with modulus $E\omega$.*

Proof. By [7, Corollary 3.7] f is semiconvex with a concave modulus $\omega^* \in \mathcal{F}$ with $\omega^* \leq 4\omega$. Let $x, y \in A$ and $\lambda \in [0,1]$. Then

$$
\begin{aligned}
&g(\lambda x + (1-\lambda)y) = f(\varphi(\lambda x + (1-\lambda)y)) = f(\lambda\varphi(x) + (1-\lambda)\varphi(y))\\
\leq\ &\lambda f(\varphi(x)) + (1-\lambda)f(\varphi(y)) + \lambda(1-\lambda)\omega^*(\|\varphi(x)-\varphi(y)\|)\cdot\|\varphi(x)-\varphi(y)\|\\
=\ &\lambda f(\varphi(x)) + (1-\lambda)f(\varphi(y)) + \lambda(1-\lambda)\omega^*(\|L(x)-L(y)\|)\cdot\|L(x)-L(y)\|\\
\leq\ &\lambda g(x) + (1-\lambda)g(y) + \lambda(1-\lambda)\omega^*(\|L\|\cdot\|x-y\|)\cdot\|L\|\cdot\|x-y\|. \qquad (4)
\end{aligned}
$$

So g is semiconvex on A with modulus $\|L\|\,\omega^*(\|L\|t)$. Setting $D := \max(\|L\|, 1)$ and $E := 4D^2$, we obtain, using concavity of ω^*, $\|L\|\,\omega^*(\|L\|t) \leq D\omega^*(Dt) \leq D^2\omega^*(t) \leq E\omega(t)$, which completes the proof. □

Proposition 2.8. *Let f be a real function on an open interval $I \subset \mathbb{R}$ and $\omega \in \mathcal{F}$. Then the following assertions hold.*

(i) *If f is semiconvex with modulus ω on I, then*

$$f'_+(x) - f'_+(x+\tau) \le 2\omega(\tau) \quad \textit{whenever } x, x+\tau \in I \textit{ and } \tau > 0. \tag{5}$$

(ii) *If f is continuous, $f'_+(x)$ exists for each $x \in I$ and*

$$f'_+(x) - f'_+(x+\tau) \le \omega(\tau) \quad \textit{whenever } x, x+\tau \in I \textit{ and } \tau > 0, \tag{6}$$

then f is semiconvex with modulus ω on I.

Proof. The proof of (i) easily follows from from [7, Lemma 3.4]. Indeed, if f is semiconvex with modulus ω on I, then $\omega_f \le \omega$, where ω_f is the minimal modulus of semiconvexity of f (see [7, Definition 3.1]). Thus, Lemma 2.4(i) and [7, Lemma 3.4] give that $\omega_f^* \le 2\omega_f \le 2\omega$, where

$$\begin{aligned}\omega_f^*(t) := \ \sup\Big\{ \big(d^+ f(x;v) - d^+ f(x+\tau v; v)\big)^+ :& \\ \|v\| = 1,\ x, x+\tau v \in \Omega,\ 0 \le \tau \le t&\Big\}.\end{aligned}$$

It is easy to see (using Lemma 2.4(i)) that the inequality $\omega_f^* \le 2\omega$ implies (5).

Part (ii) can be proved by slightly modifying the proof of [7, Lemma 3.4]. However, we prefer to present an alternative proof, which is independent of the results from [7]. To that end, suppose that the assumptions of the implication in (ii) hold. Then f'_+ is clearly locally bounded on I. Indeed, if $x < y$ are points of I with $y - x < \tau$, than $f'_+(u) \le f'_+(y) + \omega(\tau)$ for each u from some neigbourhood of x and $f'_+(v) \ge f'_+(x) - \omega(\tau)$ for each v from some neigbourhood of y. Thus, the classical Dini's theorem (see [4, Theorem 1.2, p. 39]) clearly shows that f is locally Lipschitz. Consequently,

$$\begin{aligned} f(v) - f(u) &= \int_u^v f'_+(x)\ dx \\ &= (v-u)\int_0^1 f'_+(u + t(v-u))\ dt \ \text{ if } u < v \text{ belong to } I. \end{aligned} \tag{7}$$

To prove that f is semiconvex with modulus ω, it is clearly sufficient to prove (1) for all points $x < y$ from I and $\lambda \in [0,1]$. So, let such x, y, λ be fixed. Denote $z := \lambda x + (1-\lambda)y$. Observing that $z - x = (1-\lambda)(y-x)$, $y - z = \lambda(y-x)$ and using (7), (6), we obtain

$$\begin{aligned}
&f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)\\
&=\lambda(f(z)-f(x))-(1-\lambda)(f(y)-f(z))\\
&=\lambda(z-x)\int_0^1 f'_+(x+t(z-x))\,dt-(1-\lambda)(y-z)\int_0^1 f'_+(z+t(y-z))\,dt\\
&=\lambda(1-\lambda)(y-x)\int_0^1 (f'_+(x+t(z-x))-f'_+(z+t(y-z))\,dt\\
&\le \lambda(1-\lambda)(y-x)\omega(\|y-x\|)
\end{aligned}$$

and (1) follows. □

The introduction (in [7]) of the following notion (which generalizes an older notion to the case of a non-convex A) was motivated by an application in the present article (via Proposition 2.12).

Definition 2.9. Let f be a real valued function defined on an arbitrary nonempty subset A of a Banach space X, and let $\alpha\in\mathcal{F}$ with $\lim_{t\to 0+}\alpha(t)/t=0$. We say that $x^*\in X^*$ is an $[\alpha]$-*subgradient* of f at $x\in A$ provided

$$\langle h,x^*\rangle-(f(x+h)-f(x))\le\alpha(\|h\|)\quad\text{whenever } x+h\in A.$$

Then we will write $x^*\in\partial_A^\alpha f(x)$.

We will say that f *is* $[\alpha]$-*subdifferentiable at* $x\in A$ (resp. $[\alpha]$-*subdifferentiable on* A) if $\partial_A^\alpha f(x)\neq\emptyset$ (resp. $\partial_A^\alpha f(y)\neq\emptyset$ for each $y\in A$).

The following fact is an immediate consequence of [7, Theorem R, p. 248] and [7, Remark 2.11(a), (i)].

Lemma 2.10. *Let $\varphi\in\mathcal{F}$, let f be a semiconvex function with modulus φ on a nonempty open convex subset Ω of a Banach space X, and $x\in\Omega$. Then $\partial_\Omega^\alpha f(x)=\partial f(x)$, where $\alpha(t):=t\varphi(t)$.*

Let us now recall some facts about superreflexive Banach spaces that we will need later on. For the original definition of superreflexive spaces, and a number of their characterizations, see, e.g., [3] or [6]. For example, X is superreflexive if and only if it admits an equivalent norm, which is uniformly rotund (resp. uniformly smooth).

If X is a Banach space, then the *modulus of smoothness* of the norm of X (or of the space X) is defined as the function

$$\rho_X(\tau)=\rho(\tau)=\sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1:\|x\|=\|y\|=1\right\},\quad \tau>0.$$

We say ([6, p. 157]) that X has the *modulus of smoothness of power type* p, if there exists a $K>0$ such that $\rho(\tau)\le K\tau^p$, $\tau>0$.

Remark 2.11. (i) It is well-known (see [3, p. 412] or [6]) that a Banach space X is superreflexive if and only if X admits an equivalent norm which has modulus of smoothness of power type p for some $1 < p \leq 2$.

(ii) Suppose that X admits an equivalent norm which has modulus of smoothness of power type p (or X is superreflexive). Then any closed subspace Y of X has the same property. This follows immediately from the definition of modulus of smoothness and from (i).

Further note that (see [6, Corollary V.1.2]), if μ is an arbitrary measure, then $L_p(\mu)$ has a modulus of smoothness of power type p (resp. 2), if $1 < p \leq 2$ (resp. $p > 2$). In particular, each Hilbert space has a modulus of smoothness of power type 2.

We can now formulate the following result ([7, Proposition 5.12]) which is one of basic tools in the present article.

Proposition 2.12. *Let X be a superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type $1+\beta$, where $0 < \beta \leq 1$. Let $\emptyset \neq A \subset X$ be a bounded set, let $\psi \in \mathcal{F}$ be concave, and $K \geq 0$. Let f be a Lipschitz function on A which has at each point $a \in A$ a $[t\psi(t)]$-subgradient $h_a \in X^*$ with $\|h_a\| \leq K$. Then there exists a Lipschitz semiconvex function F on X such that $F|_A = f$. Moreover, if*

(i) $\beta = 1$ or
(ii) $\psi(t) = Ct^\alpha$ for some $C > 0$ and $0 < \alpha \leq \beta$,

then F is semiconvex with modulus $D\psi$ for some $D > 0$.

Using semiconvex functions instead of convex functions, we obtain the following generalization of DC functions and DC mappings with finite-dimensional range. For our purposes, it is sufficient to consider functions and mappings defined on the whole space.

Definition 2.13. Let E be a Banach space and $\omega \in \mathcal{F}$.

(i) A real function f on E is called a *DSC* function, if it is the difference of two Lipschitz semiconvex functions.
(ii) A real function f on E is called a DSC_ω function, if there exists $C > 0$ and two Lipschitz functions g, h on E which are semiconvex with modulus $C\omega$ such that $f = g - h$.
(iii) If F is a finite-dimensional Banach space, then a mapping $\varphi : E \to F$ is called a DSC_ω (or *DSC*) mapping, if $f^* \circ \varphi$ is a DSC_ω (or *DSC*) function for each functional $f^* \in F^*$.

Note that an "arbitrary multiplicative constant" appears even in the definition of a semiconvex function with modulus ω in [1]. It is obvious that f is a *DSC* function if and only if it is a DSC_ω function for some $\omega \in \mathcal{F}$.

Lemma 2.14. *Let E be a Banach space, $\omega \in \mathcal{F}$, and F a finite-dimensional*

Banach space.

(*i*) *If g, h are DSC_ω (or DSC) functions on E and $a, b \in \mathbb{R}$, then $ag + bh$ is a DSC_ω (or DSC) function.*

(*ii*) *If $g : E \to F$, $h : E \to F$ are DSC_ω (or DSC) mappings and $a, b \in \mathbb{R}$, then $ag + bh$ is a DSC_ω (or DSC) mapping.*

(*iii*) *If B is a basis of F, then $\varphi : E \to F$ is a DSC_ω (or DSC) mapping, if and only if $f^* \circ \varphi$ is a DSC_ω (or DSC) function for each functional $f^* \in B$.*

(*iv*) *A mapping $\varphi : E \to F$ is a DSC mapping if and only if it is a DSC_ω mapping for some $\omega \in \mathcal{F}$.*

(*v*) *Let $B \subset E$ be an open convex set and φ a DSC_ω mapping $\varphi : B \to F$ (or a DSC_ω function). Let X be a Banach space, $A \subset X$ an open convex set and $\psi : A \to E$ a continuous affine mapping with $\psi(A) \subset B$. Then $\varphi \circ \psi$ is a DSC_ω mapping (or function).*

Proof. The assertion (*i*) is obvious (cf. (2) and Lemma 2.6); (*ii*) and (*iii*) clearly follow from (*i*). The assertion (*iv*) clearly follows from (*iii*). Using (*iii*), we see that it is sufficient to prove (*v*) for the case of a function φ. Write $\varphi = \varphi_1 - \varphi_2$, where φ_1 and φ_2 are Lipschitz and semiconvex with modulus $C\omega$ for some $C > 0$. Lemma 2.7 implies that $\varphi_1 \circ \psi$ and $\varphi_2 \circ \psi$ are Lipschitz and semiconvex with modulus $E\omega$ for some $E > 0$. So $\varphi \circ \psi = \varphi_1 \circ \psi - \varphi_2 \circ \psi$ is a DSC_ω function. □

2.3. Surfaces of finite codimension

Recall that if X is a Banach space and $X = E \oplus F$, then by $\pi_{E,F}$ we denote the projection of X on E along the space F.

Definition 2.15. Let X be a Banach space and $A \subset X$.

(i) Let F be a finite-dimensional subspace of X. We say that A is an *F-Lipschitz surface* if there exists a topological complement E of F and a Lipschitz mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : \ x \in E\}$.

(ii) Let $1 \le n < \dim X$ be a natural number. We say that A is a *Lipschitz surface of codimension n* if A is an F-Lipschitz surface for some n-dimensional space $F \subset X$.

(iii) If $\omega \in \mathcal{F}$, and we consider in (i) DSC mappings (resp. DSC_ω mappings) $\varphi : E \to F$, we obtain the notions of an *F-DSC (resp. F-DSC_ω) surface* and of a *DSC (resp. DSC_ω) surface of codimension n*. A Lipschitz surface (resp. DSC surface, resp. DSC_ω surface) of codimension 1 is said to be a *Lipschitz hypersurface* (resp. *DSC hypersurface*, resp. *DSC_ω hypersurface*).

(iv) The σ-ideals of sets which can be covered by countably many Lipschitz surfaces (DSC surfaces, DSC_ω surfaces) of codimension n will be denoted by $\mathcal{L}^n(X)$ ($\mathcal{DSC}^n(X)$, $\mathcal{DSC}^n_\omega(X)$), respectively.

We will need the following easy facts.

Lemma 2.16. *Let X be a Banach space, $F \subset X$ a space of dimension k ($1 \le k < \dim X$), $A \subset X$, and $\omega \in \mathcal{F}$ a modulus. Then the following assertions are* *equivalent.*

(i) *A is an F-Lipschitz surface (resp. an F-DSC surface, resp. an F-DSC_ω surface).*

(ii) *If $X = F \oplus E$, then there exists a Lipschitz (resp. DSC, resp. DSC_ω) mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : \ x \in E\}$.*

(iii) *Let $X = F \oplus E$ and $x_1^* \in X^*, \dots, x_k^* \in X^*$ be functionals from $E^\perp \subset X^*$ such that $x_1^* \restriction_F, \dots, x_k^* \restriction_F$ form a basis of F^*. Then there exist Lipschitz (resp. DSC, resp DSC_ω) functions $\varphi_1, \dots, \varphi_k$ on E such that*

$$A = \{x \in X : \ x_i^*(x) = \varphi_i(\pi(x)), \ i = 1, \dots, k\}, \tag{8}$$

where $\pi := \pi_{E,F}$.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial. The opposite implication for the case of Lipschitz surfaces is contained in [14, Lemma 3.2]. (Note that the proof given below for DSC_ω surfaces also works for F-Lipschitz surfaces.) Using Lemma 2.14(iv), it is easy to see that it is sufficient to prove the case of DSC_ω surfaces.

So suppose that A is an F-DSC_ω surface. Then there exists a closed space $\tilde{E} \subset X$ and a Lipschitz DSC_ω mapping $\tilde{\varphi} : \tilde{E} \to F$ such that $X = F \oplus \tilde{E}$ and $A = \{y + \tilde{\varphi}(y) : \ y \in \tilde{E}\}$. Let $X = F \oplus E$. It is easy to see that the mapping $L := \pi_{\tilde{E},F} \restriction_E$ is a linear homeomorphism $L : E \to \tilde{E}$ and $L(x) - x \in F$ for each $x \in E$. So $\eta(x) := L(x) - x$ is a continuous linear mapping $\eta : E \to F$. Clearly

$$A = \{L(x) + \tilde{\varphi}(L(x)) : \ x \in E\} = \{x + \eta(x) + \tilde{\varphi}(L(x)) : \ x \in E\}.$$

Note that $\tilde{\varphi} \circ L$ is a DSC_ω mapping by Lemma 2.14(v). Using Lemma 2.14(ii), we see that it is sufficient to put $\varphi = \eta + \tilde{\varphi} \circ L$.

To prove $(ii) \Rightarrow (iii)$, observe that if φ is as in (ii), then (8) holds for $\varphi_i := x_i^* \circ \varphi$. For the opposite implication, suppose that (8) holds for Lipschitz (resp. DSC, resp DSC_ω) functions $\varphi_1, \dots, \varphi_k$ on E. Then Lemma 2.14(iii) shows that the unique mapping $\varphi : E \to F$ for which $\varphi_i := x_i^* \circ \varphi$ is Lipschitz (resp. DSC, resp DSC_ω) and clearly $A = \{x + \varphi(x) : \ x \in E\}$. □

Remark 2.17. Let X be a Banach space, $F \subset X$ a space of finite dimension and $X = E \oplus F$. Denote $\pi := \pi_{E,F}$. Let A be an F-Lipschitz surface. Then the following assertions hold.

(i) The mapping $\pi \restriction_A : A \to E$ is bilipschitz.

(ii) The set A is closed in X.

Indeed, (i) is obvious, since π is Lipschitz and $(\pi \restriction_A)^{-1}(x) = x + \varphi(x), \ x \in E$, for a Lipschitz $\varphi : E \to F$ by Lemma 2.16(ii). The assertion (ii) easily follows from (i).

To prove Proposition 2.19 below, we need the following easy fact.

Lemma 2.18. *Let $\omega_i \in \mathcal{F}$, $i \in \mathbb{N}$. Then there exists a concave $\omega \in \mathcal{F}$ and a sequence $(\delta_i)_{i=1}^\infty$ such that $\delta_i > 0$ and $\omega(x) \ge \omega_i(x)$ for $x \in (0, \delta_i)$.*

Proof. For each $i \in \mathbb{N}$, find $\delta_i > 0$ such that $\max(\omega_1(x), \ldots, \omega_i(x)) < 1/i$ for $x \in (0, \delta_i)$. We can clearly suppose that $\delta_1 > \delta_2 > \ldots$. For $i \in \mathbb{N}$, let α_i be an affine function with $\alpha_i(0) = 1/i$, $\alpha_i(\delta_i) = 1$ and $\omega(x) := \inf_{i \in \mathbb{N}} \alpha_i(x)$. It is easy to see that ω and $(\delta_i)_{i=1}^{\infty}$ have the desired properties. □

Proposition 2.19. *Let X be a separable Banach space which admits an equivalent norm with modulus of smoothness of power type 2. Then, for each $n \in \mathbb{N}$,*

$$\mathcal{DSC}^n(X) = \bigcup_{\omega \in \mathcal{F}} \mathcal{DSC}^n_{\omega}(X).$$

Proof. The inclusion "$\supset$" is obvious. To prove the opposite inclusion, consider an arbitrary $B \in \mathcal{DSC}^n(X)$. By Definition 2.15(iv) there exist DSC surfaces B_i, $i \in \mathbb{N}$, of codimension n such that $B \subset \bigcup_{i=1}^{\infty} B_i$. By Lemma 2.14(*iv*) and Definition 2.15, each B_i is a DSC_{ω_i} surface of codimension n for some $\omega_i \in \mathcal{F}$. By Lemma 2.18, there exists a concave $\omega \in \mathcal{F}$ and a sequence $(\delta_i)_{i=1}^{\infty}$ such that $\delta_i > 0$ and

$$\omega(x) \geq \omega_i(x) \quad \text{for } x \in (0, \delta_i). \tag{9}$$

Now fix an arbitrary $i \in \mathbb{N}$ and denote $A := B_i$. It is clearly sufficient to prove $A \in \mathcal{DSC}^n_{\omega}(X)$.

By Definition 2.15, there exists a closed n-dimensional space $F \subset X$, a topological complement E to F and a DSC_{ω_i} mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : x \in E\}$. Let $f_1, \ldots, f_n$ be a basis of F and $f_1^*, \ldots, f_n^*$ be a basis of F^* such that $\langle f_j, f_j^* \rangle = 1$ and $\langle f_j, f_k^* \rangle = 0$ for $j \neq k$. For each $j \in \{1, \ldots, n\}$, we can choose $C_j > 0$ and Lipschitz functions g_j, h_j on E, which are semiconvex with modulus $C_j \omega_i$ and $\langle \varphi, f_j^* \rangle = g_j - h_j$.

Using separability of E, we can write $E = \bigcup_{p=1}^{\infty} A_p$ where $\operatorname{diam} A_p < \delta_i$ for each $p \in \mathbb{N}$. For each $p \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$, we will apply Proposition 2.12 (with $X := E$, $A := A_p$, $f := g_j \restriction_{A_p}$, $\psi := C_j \omega$) and obtain a function G_j^p on E which is semiconvex with modulus $D_j^p \omega$ for some $D_j^p > 0$, and $G_j^p \restriction_{A_p} = g_j \restriction_{A_p}$. To show that the assumptions of Proposition 2.12 are satisfied, first observe that E admits an equivalent norm with modulus of smoothness of power type 2 by Remark 2.11. Further, consider an arbitrary $a \in A_p$. By Lemma 2.10 there exists a $[tC_j \omega_i(t)]$-subgradient $h_a \in E^*$ of g_j at a (with respect to E). It is easy to see (cf. [7, Remark 4.2]) that $\|h_a\| \leq K$, where K is the Lipschitz constant of g_j. Further, since $\operatorname{diam} A_p < \delta_i$, (9) implies that h_a is a $[tC_j \omega(t)]$-subgradient of $g_j \restriction_{A_p}$ at a.

In the same way we obtain a function H_j^p on E which is semiconvex with modulus $\tilde{D}_j^p \omega$ for some $\tilde{D}_j^p > 0$, and $H_j^p \restriction_{A_p} = h_j \restriction_{A_p}$. Setting

$$\Phi_p := \sum_{j=1}^{n} (G_j^p - H_j^p) f_j \quad \text{and} \quad M_p := \{x + \Phi_p(x) : x \in E\},$$

we easily see (using Lemma 2.14(*iii*)) that M_p is a DSC_{ω} surface of codimension n and $A \subset \bigcup_{p=1}^{\infty} M_p$. Thus $A \in \mathcal{DSC}^n_{\omega}(X)$. □

3. Main results on smallness of singular sets

In this section, we will use the notation of the following definition.

Definition 3.1. Let $n \in \mathbb{N}$ and let X be a separable Banach space with dimension at least $n+1$. Then:

(i) Choose a countable dense set $E \subset X$ and denote by $\mathcal{K} = \mathcal{K}(E, n)$ the family of all linear subspaces $K \subset X$ that are generated by n linearly independent elements of E.

(ii) For each $K \in \mathcal{K}$, choose a countable dense set of functionals $D_K \subset K^*$ and denote by $\mathcal{M}_K$ the system of all $M = (m_0, m_1, \dots, m_n) \in (D_K)^{n+1}$ such that $m_i \in D_K,\ i = 0, \dots, n$, and the vectors $m_i,\ i = 0, \dots, n$ are affinely independent.

(iii) Let f be a semiconvex function defined on an open convex set $\Omega \subset X$. Let $K \in \mathcal{K}$ and $M = (m_0, \dots, m_n) \in \mathcal{M}_K$ be given. Denote by $\Sigma_{K,M}(f)$ the set of all $x \in \Omega$, for which there exist functionals $y_0, \dots y_n \in \partial f(x),\ i = 0, \dots, n$, such that $y_i|_K = m_i, i = 0, \dots, n$.

We will need the following lemma.

Lemma 3.2. *Let $n \in \mathbb{N}$ and let X be a separable Banach space with dimension at least $n+1$. Let f be a semiconvex function defined on an open convex set $\Omega \subset X$. Then*

$$\Sigma_n(f) = \bigcup \{\Sigma_{K,M}(f) : \ K \in \mathcal{K}, M \in \mathcal{M}_K\}.$$

Proof. The inclusion "$\supset$" is obvious, since the functionals $y_0, \dots y_n,\ i = 0, \dots, n$ from Definition 3.1(iii) are clearly affinely independent.

To prove the other inclusion, suppose that $x \in \Sigma_n(f)$ is given. Thus there exist $z_0, \dots, z_n \in \partial f(x)$ such that the functionals $p_1 = z_1 - z_0, \dots, p_n = z_n - z_0$ are linearly independent. Therefore, we can choose $K \in \mathcal{K}(E, n)$ such that $p_1|_K, \dots, p_n|_K$ are linearly independent (and so generate K^*). (For an elementary proof of this easy fact see [1, p. 175, the construction of $e_{k_1}, \dots, e_{k_n}$ and $q_1, \dots, q_n$].)

Consequently, $z_0|_K, \dots, z_n|_K$ are affinely independent elements of the set $T := \{z|_K : z \in \partial f(x)\}$. So, since T is clearly convex, it is easy to find $M = (m_0, \dots, m_n) \in \mathcal{M}_K$ such that $m_i \in T,\ i = 0, \dots, n$. So, $x \in \Sigma_{K,M}(f)$. □

We will need the following lemma, which is an easy consequence of results of [15]

Lemma 3.3. *Let $n \in \mathbb{N}$ and let X be a separable Banach space with dimension at least $n+1$. Let f be a semiconvex function defined on an open convex set $\Omega \subset X$. Let $K \in \mathcal{K}$ and $M \in \mathcal{M}_K$.*

Then the set $\Sigma_{K,M}(f)$ can be covered by countably many K-Lipschitz surfaces.

Proof. First, we will show that

$$f'(x, v) \text{ does not exist for each } x \in \Sigma_{K,M}(f) \text{ and } 0 \neq v \in K. \tag{10}$$

To this end, suppose that $x \in \Sigma_{K,M}(f)$ and $0 \neq v \in K$. Choose functionals $y_0, \dots y_n \in \partial f(x)$, $i = 0, \dots, n$, such that $y_i|_K = m_i$, $i = 0, \dots, n$, where $(m_0, \dots, m_n) = M$. Since $m_1 - m_0, \dots, m_n - m_0$ generate K^*, there exists $1 \leq i \leq n$ such that

$$\langle v, y_i - y_0 \rangle = \langle v, m_i - m_0 \rangle \neq 0. \tag{11}$$

Since f is Clarke regular by Lemma 2.4(*ii*), it is easy to see (cf. [15, (2.13)]) that (11) implies (10).

Since f is locally Lipschitz, the directional derivative (resp. the one-sided directional derivative) coincides with the Hadamard directional derivative (resp. the Hadamard one-sided directional derivative); cf. [15]. Thus, the statement of the lemma follows from Lemma 2.4(*i*), (10) and [15, Lemma 3.2]. □

In the proof of the following lemma and theorem, we use ideas from [13] (see also the proof of [3, Theorem 4.20]).

Lemma 3.4. *Let X be a separable superreflexive Banach space, $X = H \oplus K$, where $\dim H \geq 1$ and $1 \leq \dim K < \infty$. Denote $\pi_H := \pi_{H,K}$ and $\pi_K := \pi_{K,H}$. Let $S \subset X$ be a K-Lipschitz surface. Let $\Omega \subset X$ be a bounded open convex set and $f : \Omega \to \mathbb{R}$ a Lipschitz semiconvex function. Let $m \in K^*$ and*

$$A_m \subset \{x \in S \cap \Omega : \ m = p_x|_K \ \text{ for some } p_x \in \partial f(x)\}.$$

Then there exists a Lipschitz semiconvex function g_m on H such that

$$f(x) - \langle \pi_K(x), m \rangle = g_m(\pi_H(x)) \ \text{ for each } x \in A_m.$$

Moreover, if X admits an equivalent norm with modulus of smoothness of power type $1 + \beta$ ($0 < \beta \leq 1$), f is semiconvex with modulus ω, and

(*i*) $\beta = 1$ *or*

(*ii*) $\omega(t) = Ct^\alpha$ *for some $C > 0$ and $0 < \alpha \leq \beta$,*

then we can find g_m to be semiconvex with modulus $E\omega$ for some $E > 0$.

Proof. Suppose that f is semiconvex with modulus ω. By [7, Corollary 3.6], f is semiconvex with a concave modulus φ for which $\varphi \leq 4\omega$. Set $\alpha(t) := t\varphi(t)$, $t > 0$. Since S is a K-Lipschitz surface, it is easy to see that there exists a (unique) function γ_m on $\pi_H(A_m)$ such that

$$f(x) - \langle \pi_K(x), m \rangle = \gamma_m(\pi_H(x)) \ \text{ for each } x \in A_m. \tag{12}$$

So, our task is to find a Lipschitz semiconvex extension g_m of γ_m (with modulus $E\omega$, if (*i*) or (*ii*) holds).

Since S is a K-Lipschitz surface there exists (see Remark 2.17(i)) $V > 1$ such that

$$\psi := (\pi_H \restriction_S)^{-1} \text{ is } V\text{-Lipschitz}, \quad \text{i.e.,} \quad \|z - x\| \leq V \cdot \|\pi_H(z - x)\|, \ x, z \in S. \tag{13}$$

Since also π_K and f are Lipschitz and $\gamma_m(y) = f(\psi(y)) - \langle \pi_K(\psi(y)), m \rangle$ for $y \in \pi_H(A_m)$, we obtain that γ_m is Lipschitz.

Let $x, z \in A_m$, and $p_x \in \partial f(x)$ be such that $m = p_x|_K$. Since p_x is an $[\alpha]$-subgradient of f at x by Lemma 2.10, we have

$$\langle z - x, p_x \rangle - (f(z) - f(x)) \leq \alpha(\|z - x\|).$$

Using (13), we have

$$\langle \pi_H(z - x), p_x \rangle + \langle \pi_K(z - x), p_x \rangle - (f(z) - f(x)) \leq \alpha(V \cdot \|\pi_H(z - x)\|).$$

Since $p_x|_K = m$ and concavity of φ implies $\varphi(Vy) \leq V\varphi(y),\ y > 0$, we obtain

$$\begin{aligned} &\langle \pi_H(z) - \pi_H(x), p_x \rangle + \langle \pi_K(z), m \rangle - f(z) - \langle \pi_K(x), m \rangle + f(x) \\ &\leq V^2 \|\pi_H(z - x)\| \varphi(\|\pi_H(z - x)\|). \end{aligned}$$

Denoting $\tilde{z} := \pi_H(z)$, $\tilde{x} := \pi_H(x)$ and using (12), we obtain that

$$\langle \tilde{z} - \tilde{x}, p_x \rangle - (\gamma_m(\tilde{z}) - \gamma_m(\tilde{x})) \leq V^2 \|\tilde{z} - \tilde{x}\| \varphi(\|\tilde{z} - \tilde{x}\|) \text{ for each } \tilde{x}, \tilde{z} \in \pi_H(A_m).$$

Consequently, the function γ_m is a Lipschitz function with the bounded domain $\pi_H(A_m)$ and, for each $\tilde{x} \in \pi_H(A_m)$, the functional $h_{\tilde{x}} := p_x|_H$ is a $[V^2 t\varphi(t)]$-subgradient of γ_m at $\tilde{x}$. Since f is Lipschitz with some constant $M \geq 0$, it is easy to see (cf. [7, Remark 4.2]) that $\|p_x\| \leq M$ and therefore $\|h_{\tilde{x}}\| \leq M$. Using Remark 2.11(i), (ii) and Proposition 2.12 we obtain that there exists a Lipschitz semiconvex function g_m on H such that $g_m|_{\pi_H(A_m)} = \gamma_m$. Moreover, if (i) or (ii) holds, g_m is semiconvex with modulus $DV^2\varphi$ for some $D > 0$, and so (since $\varphi \leq 4\omega$) with modulus $E\omega$ for some $E > 0$. □

Theorem 3.5. *Let X be a separable superreflexive space, $\Omega \subset X$ a nonempty open convex set, $f : \Omega \to \mathbb{R}$ a semiconvex function, and* $\dim X > k \in \mathbb{N}$. *Let* $\Sigma_k(f) := \{x \in \Omega : \ \dim(\partial f(x)) \geq k\}$. *Then*

$$\Sigma_k(f) \in \mathcal{DSC}^k(X).$$

Moreover, if X admits an equivalent norm with modulus of smoothness of power type $1 + \beta$ $(0 < \beta \leq 1)$, *f is semiconvex with modulus $\omega \in \mathcal{F}$, and*

(i) $\quad \beta = 1$ *or*

(ii) $\quad \omega(t) = Ct^\alpha$ *for some* $C > 0$ *and* $0 < \alpha \leq \beta$,

then

$$\Sigma_k(f) \in \mathcal{DSC}^k_\omega(X).$$

Proof. Since X is separable and f is locally Lipschitz, we can (and will) suppose that Ω is bounded and f is Lipschitz. By Lemma 3.2 and Lemma 3.3 it is sufficent to prove the following claim:

(C) If $K \subset X$ is a k-dimensional space and $m_0 \in K^*, \dots, m_k \in K^*$ are affinely independent, then each subset A of

$$\{x \in \Omega : \ m_0 = p_0^x|_K, \dots, m_k = p_k^x|_K \text{ for some } p_0^x \in \partial f(x), \dots, p_k^x \in \partial f(x)\}$$

which is contained in a K-Lipschitz surface is a subset of a K-DSC surface (resp. K-DSC_ω surface if (i) or (ii) is satisfied).

To prove the claim, choose a topological complement H of K. Using Lemma 3.4 for $m := m_i$, we obtain Lipschitz semiconvex functions (with modulus $E\omega$ if (i) or (ii) is satisfied) $g_i := g_{m_i},\ i = 0, \dots, k$, on H such that

$$f(x) - \langle \pi_K(x), m_i \rangle = g_i(\pi_H(x)) \ \text{ for each } x \in A.$$

Consequently, for each $x \in A$, we have (letting $h := \pi_H(x)$, $\kappa := \pi_K(x)$)

$$f(x) - m_i(\kappa) = g_i(h), \quad i = 0, \dots, k, \quad \text{and so}$$

$$(m_i - m_0)(\kappa) = g_0(h) - g_i(h), \quad i = 1, \dots, k.$$

Since $m_i - m_0,\ i = 1, \dots, k$, form a basis of K^*, we easily obtain (using Lemma 2.14(iii)) that A is a subset of a K-DSC surface (resp. K-DSC_ω surface if (i) or (ii) is satisfied). □

Proposition 3.6. *Let X be a separable Banach space with $\dim X > k \in \mathbb{N}$. Let $\omega \in \mathcal{F}$ and $A \in \mathcal{DSC}^k_\omega(X)$. Then there exists a Lipschitz function f on X which is semiconvex with modulus ω and $A \subset \Sigma_k(f)$.*

Proof. Let $A_n (n \in \mathbb{N})$ be K_n-DSC_ω surfaces such that $\dim K_n = k$ and $A \subset \bigcup_{n=1}^\infty A_n$.

Now fix $n \in \mathbb{N}$ and choose a topological complement H of $K := K_n$ and functionals $x_1^* \in X^*, \dots, x_k^* \in X^*$ such that $x_i^* \in H^\perp$, $i = 1, \dots, k$, and $x_1^* \restriction_K, \dots, x_k^* \restriction_K$ form a basis of K^*. By Lemma 2.16(iii) we can choose functions $Z_1, \dots, Z_k$ and $T_1, \dots, T_k$ on H which are Lipschitz and semiconvex with modulus $D_n\omega$ for some $D_n > 0$ and

$$A_n = \{x \in X :\ x_i^*(x) = Z_i(\pi(x)) - T_i(\pi(x)),\ i = 1, \dots, k\}, \tag{14}$$

where $\pi := \pi_{H,K}$. Put

$$g_0(x) := T_1(\pi(x)) + \dots + T_k(\pi(x)) + x_1^*(x) + \dots + x_k^*(x), \quad x \in X,$$
$$\text{and} \quad g_i(x) := Z_i(\pi(x)) + g_0(x) - T_i(\pi(x)) - x_i^*(x), \quad x \in X,$$

for $i = 1, \dots, k$.

The functions $g_0, g_1, \dots, g_k$ are clearly Lipschitz and by Lemma 2.7 and Lemma 2.6 they are semiconvex with modulus $E_n\omega$ for some $E_n > 0$. Set

$$f_n := \max(g_0, g_1, \dots, g_k).$$

Then f_n is clearly Lipschitz and (by (2)) it is semiconvex with modulus $E_n\omega$. We will prove that

$$f_n'(a, v) \text{ does not exists for every } a \in A_n \text{ and } 0 \neq v \in K. \tag{15}$$

To prove (15), choose arbitrary $a \in A_n$ and $0 \neq v \in K$. Choose $j \in \{1, \dots, k\}$ such that $x_j^*(v) \neq 0$ and set $\varphi := \max(g_0, g_j)$. By (14) we obtain $g_0(a) = g_1(a) = \dots = g_k(a) = f_n(a)$. Consequently

$$(f_n)'_+(a, v) \geq \varphi'_+(a, v) = \max((g_0)'_+(a, v), (g_j)'_+(a, v))$$
$$= \sum_{i=1}^k x_i^*(v) - \min(0, x_j^*(v)),$$

$$(f_n)'_+(a,-v) \geq \varphi'_+(a,-v) = \max((g_0)'_+(a,-v),(g_j)'_+(a,-v))$$
$$= \sum_{i=1}^{k} x_i^*(-v) - \min(0, x_j^*(-v)).$$

Summing these inequalities, we obtain

$$(f_n)'_+(a,v)+(f_n)'_+(a,-v) \geq -\min(0,x_j^*(v))-\min(0,-x_j^*(v)) = |x_j^*(v)| > 0, \quad (16)$$

which proves (15).

We can clearly choose $c_n > 0$ so small that $h_n := c_n f_n$ is Lipschitz with constant 1, h_n is semiconvex with modulus ω and $|h_n(0)| < 1$. It is easy to see that the function $f := \sum_{i=1}^{\infty} 2^{-i} h_i$ is Lipschitz on X; by Lemma 2.6 we obtain that f is semiconvex with modulus ω. To prove $A \subset \Sigma_k(f)$, it is sufficient to show that $f'(a,v)$ does not exist for every $a \in A$ and $0 \neq v \in K$. To this end, fix such a and v. Choose $n \in \mathbb{N}$ such that $a \in A_n$. Since $\tilde{f} := \sum_{i\in\mathbb{N}\setminus\{n\}} 2^{-i} h_i$ is semiconvex with modulus ω by Lemma 2.6, we obtain $(\tilde{f})'_+(a,v) + (\tilde{f})'_+(a,-v) \geq 0$ by Lemma 2.4(*iii*). So, using (16), we obtain

$$f'_+(a,v) + f'_+(a,-v)$$
$$= (\tilde{f})'_+(a,v) + (\tilde{f})'_+(a,-v) + (c_n 2^{-n} f_n)'_+(a,v) + (c_n 2^{-n} f_n)'_+(a,-v) > 0,$$

which completes the proof. □

Theorem 3.5, Proposition 3.6 and Proposition 2.19 easily give some best possible results on smallness of sets $\Sigma_k(f)$ for semiconvex functions f. For their concise formulation, we introduce the following notation.

Definition 3.7. Let X be a Banach space, $k \in \mathbb{N}$ and $\omega \in \mathcal{F}$.

(i) Denote by $\mathcal{A}^k(X)$ (resp. $\mathcal{A}^k_\omega(X)$) the system of all sets $A \subset X$ such that $A \subset \Sigma_k(f)$ for some f semiconvex (resp. semiconvex with modulus ω) on X.
(ii) Denote by $\sigma(\mathcal{A}^k(X))$ (resp. $\sigma(\mathcal{A}^k_\omega(X))$) the smallest σ-ideal of subsets of X containing $\mathcal{A}^k(X)$ (resp. $\mathcal{A}^k_\omega(X)$).

Theorem 3.8. *Let X be a separable superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type 2 (e.g., X is a Hilbert space or $X = L^p(\mu)$ with $2 \leq p$) and* $\dim X > k \in \mathbb{N}$*. Then*

(i) $\mathcal{A}^k_\omega(X) = \mathcal{DSC}^k_\omega(X)$ *for each $\omega \in \mathcal{F}$, and*
(ii) $\mathcal{A}^k(X) = \mathcal{DSC}^k(X)$*.*

Proof. The assertion (*i*) immediately follows from Theorem 3.5 and Proposition 3.6.

To prove (*ii*), observe that the inclusion $\mathcal{A}^k(X) \subset \mathcal{DSC}^k(X)$ immediately follows from Theorem 3.5. Further, let $A \in \mathcal{DSC}^k(X)$ be given. By Proposition 2.19, there exists $\omega \in \mathcal{F}$ such that $A \in \mathcal{DSC}^k_\omega(X)$ and thus $A \in \mathcal{A}^k(X)$ by Proposition 3.6. □

Theorem 3.5 and Proposition 3.6 also give immediately the following definitive result in more general superreflexive spaces for the case of moduli of power type.

Proposition 3.9. *Let X be a separable superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type $1+\beta$ $(0<\beta\le 1)$ and $\dim X > k \in \mathbb{N}$. Let $0<\alpha\le\beta$ and $\omega(t)=Ct^\alpha$ for some $C>0$. Then $\mathcal{A}^k_\omega(X)=\mathcal{DSC}^k_\omega(X)$.*

If X is a general separable superreflexive space, we obtain the following weaker result.

Proposition 3.10. *Let X be a separable superreflexive Banach space and $\dim X > k \in \mathbb{N}$. Then $\sigma(\mathcal{A}^k(X))=\mathcal{DSC}^k(X)$.*

Proof. Since $\mathcal{A}^k(X)\subset\mathcal{DSC}^k(X)$ by Theorem 3.5 and $\mathcal{DSC}^k(X)$ is a σ-ideal, the inclusion $\sigma(\mathcal{A}^k(X))\subset\mathcal{DSC}^k(X)$ follows.

To prove the opposite inclusion, suppose that $A\in\mathcal{DSC}^k(X)$. Then there exist moduli $\omega_i\in\mathcal{F}$ $(i\in\mathbb{N})$ and DSC_{ω_i} surfaces P_i $(i\in\mathbb{N})$ of codimension k such that $A\subset\bigcup_{i\in\mathbb{N}}P_i$. Since each P_i belongs to $\mathcal{A}^k(X)$ by Proposition 3.6, we obtain $A\in\sigma(\mathcal{A}^k(X))$. □

Remark 3.11. For simplicity, we formulated Theorem 3.8 and Propositions 3.9, 3.10 for classes of subsets of the whole space X. Of course, our proofs give analogous results if we work on an open convex subset Ω of X.

4. A comparision of σ-ideals $\mathcal{DSC}^1_\omega(X)$

Obviously, if $\tilde\omega\in\mathcal{F}$, $\omega\in\mathcal{F}$ are moduli such that $\omega\le\tilde\omega$, then $\mathcal{A}^1_\omega(X)\subset\mathcal{A}^1_{\tilde\omega}(X)$ and $\mathcal{DSC}^1_\omega(X)\subset\mathcal{DSC}^1_{\tilde\omega}(X)$.

We will prove that if X is separable and $\tilde\omega\in\mathcal{F}$, $\omega\in\mathcal{F}$ are concave moduli such that $\omega(t)=o(\tilde\omega(t))$, $t\to 0+$, then $\mathcal{DSC}^1_{\tilde\omega}(X)\setminus\mathcal{DSC}^1_\omega(X)\neq\emptyset$. Our proof of this plausible result is based on the same basic idea as the proof of [5, Proposition 2.1.4], but is much more technical. Thus, if $\omega,\tilde\omega$ are as above and X moreover admits an equivalent norm with modulus of smoothness of power type 2, then Theorem 3.8(i) implies that $\mathcal{A}^1_{\tilde\omega}(X)\setminus\mathcal{A}^1_\omega(X)\neq\emptyset$, which is a result generalizing [11, Proposition 6.1] (in which ω is linear and $X=\mathbb{R}^2$).

To formulate technical Lemma 4.2 below, which we will use to show that a function is not a DSC_ω function, we will need the following terminology.

Definition 4.1. Let $\omega\in\mathcal{F}$ and let $I\subset\mathbb{R}$ be an open interval. We will say that a function $f: I\to\mathbb{R}$ has property $P(\omega,I)$, if there exists $a\in I$ and a strictly monotone sequence $(t_i)_{i=1}^\infty$ with $t_i\in I$ and $\lim t_i=a$ such that

$$\sum_{i=1}^\infty \omega(|t_{i+1}-t_i|)<\infty \quad\text{and}\quad \sum_{i\in\mathbb{N}}(f'_+(t_i)-f'_-(t_i))=\pm\infty. \tag{17}$$

Lemma 4.2. *Let $\omega\in\mathcal{F}$ and let $I\subset\mathbb{R}$ be an open interval. Let a function $f: I\to\mathbb{R}$ have property $P(\omega,I)$. Then f is not a DSC_ω function.*

Proof. Without any loss of generality, we will suppose that $\sum_{i\in\mathbb{N}}(f'_+(t_i)-f'_-(t_i))=\infty$ (otherwise we can consider $-f$ instead of f). We can also suppose that $(t_i)_{i=1}^\infty$ is strictly increasing (otherwise we consider $f^*(x)=f(-x)$ on $I^*=-I$ and points $a^*=-a$, $t_i^*=-t_i$). Suppose to the contrary that $f=g-h$, where both g and h are semiconvex on I with modulus $C\omega$ for some $C>0$. Since $h'_+(t_i)-h'_-(t_i)\geq 0$ by Lemma 2.4(*iii*) and $g=f+h$, we obtain that

$$\sum_{i=1}^{\infty}(g'_+(t_i)-g'_-(t_i))=\infty. \tag{18}$$

Proposition 2.8(*i*) implies that $g'_+(t_i)-g'_+(t)\leq 2C\omega(t_{i+1}-t_i)$ for each $t\in(t_i,t_{i+1})$. Since $g'_-(t_{i+1})=\lim_{t\to t_{i+1}-}g'_+(t)$ by Lemma 2.5(*ii*), we obtain

$$\begin{aligned} g'_+(t_{i+1})-g'_+(t_i) &= g'_+(t_{i+1})-g'_-(t_{i+1})-(g'_+(t_i)-g'_-(t_{i+1}))\\ &\geq g'_+(t_{i+1})-g'_-(t_{i+1})-2C\omega(t_{i+1}-t_i). \end{aligned}$$

So, using (17) and (18), we obtain

$$\infty=\sum_{i=1}^{\infty}(g'_+(t_{i+1})-g'_+(t_i))=\lim_{n\to\infty}(g'_+(t_{n+1})-g'_+(t_1)).$$

Thus g is not locally Lipschitz which contradicts Lemma 2.4(*i*). □

Lemma 4.3. *Suppose that X is a Banach space, $F:(a_1,b_1)\to X$, $G:(a_2,b_2)\to X$ are continuous mappings, and $\eta:(a_1,b_1)\to(a_2,b_2)$ is a homeomorphism such that $F=G\circ\eta$. Let $s_0\in(a_1,b_1)$, $r_0:=\eta(s_0)$, and let the one-sided derivatives $F'_+(s_0)$, $F'_-(s_0)$,$G'_+(r_0)$, $G'_-(r_0)$ exist. Then the following assertions hold.*

(*i*) *If η is increasing, then the vectors $F'_+(s_0)$, $G'_+(r_0)$ are linearly dependent, and also $F'_-(s_0)$, $G'_-(r_0)$ are linearly dependent.*

(*ii*) *If η is decreasing, then the vectors $F'_+(s_0)$, $G'_-(r_0)$ are linearly dependent, and also $F'_-(s_0)$, $G'_+(r_0)$ are linearly dependent.*

Proof. We will prove the first part of (*ii*). We can suppose that $F'_+(s_0)\neq 0$ and $G'_-(r_0)\neq 0$. It is easy to see that

$$A:=\lim_{s\to s_0+}\frac{F(s)-F(s_0)}{\|F(s)-F(s_0)\|}=\lim_{s\to s_0+}\frac{(F(s)-F(s_0))/(s-s_0)}{\|(F(s)-F(s_0))/(s-s_0)\|}=\frac{F'_+(s_0)}{\|F'_+(s_0)\|},$$

$$B:=\lim_{r\to r_0-}\frac{G(r)-G(r_0)}{\|G(r)-G(r_0)\|}=-\lim_{r\to r_0-}\frac{(G(r)-G(r_0))/(r-r_0)}{\|(G(r)-G(r_0))/(r-r_0)\|}=-\frac{G'_-(r_0)}{\|G'_-(r_0)\|}.$$

Since clearly

$$A:=\lim_{s\to s_0+}\frac{F(s)-F(s_0)}{\|F(s)-F(s_0)\|}=\lim_{s\to s_0+}\frac{G(\eta(s))-G(r_0)}{\|G(\eta(s))-G(r_0)\|}=B,$$

the assertion follows. The other three cases are quite analoguous. □

Lemma 4.4. *Let $\omega \in \mathcal{F}$ be a concave function. Let $\alpha(s) = q + ks,\ s \in \mathbb{R}$, be a non-constant real affine function. Let I, I_1 be open intervals such that $I = \alpha(I_1)$. Suppose that $h : I \to \mathbb{R}$ has property $P(\omega, I)$. Then $\varphi := h \circ \alpha$ has property $P(\omega, I_1)$.*

Proof. By Definition 4.1, there exists $a \in I$ and a strictly monotone sequence $(t_i)_{i=1}^{\infty}$ with $t_i \in I$ and $\lim t_i = a$ such that

$$\sum_{i=1}^{\infty} \omega(|t_{i+1} - t_i|) < \infty \quad \text{and} \quad \sum_{i \in \mathbb{N}} (h'_+(t_i) - h'_-(t_i)) = \pm\infty.$$

Set $a^* := \alpha^{-1}(a)$ and $t_i^* := \alpha^{-1}(t_i)$. Then (t_i^*) is strictly monotone, $\lim t_i^* = a^*$ and $|t_{i+1}^* - t_i^*| = |1/k| \cdot |t_{i+1} - t_i|$. If $|1/k| \le 1$, then $\omega(|t_{i+1}^* - t_i^*|) \le \omega(|t_{i+1} - t_i|)$, and if $|1/k| > 1$, then the concavity of ω implies $\omega(|t_{i+1}^* - t_i^*|) \le |1/k| \cdot \omega(|t_{i+1} - t_i|)$. Consequently, $\sum_{i=1}^{\infty} \omega(|t_{i+1}^* - t_i^*|) < \infty$. Further observe that, for each i,

$$\varphi'_+(t_i^*) - \varphi'_-(t_i^*) = |k|\,(h'_+(t_i) - h'_-(t_i)). \tag{19}$$

Indeed, if $k > 0$, then clearly $\varphi'_+(t_i^*) = kh'_+(t_i)$, $\varphi'_-(t_i^*) = kh'_+(t_i)$, and (19) follows. If $k < 0$, it is easy to see that $\varphi'_+(t_i^*) = kh'_-(t_i)$, $\varphi'_-(t_i^*) = kh'_+(t_i)$, which implies (19) as well.

So, $\sum_{i \in \mathbb{N}} (\varphi'_+(t_i^*) - \varphi'_-(t_i^*)) = \sum_{i \in \mathbb{N}} |k|\,(h'_+(t_i) - h'_-(t_i)) = \pm\infty$. Thus, φ has property $P(\omega, I_1)$. □

Lemma 4.5. *Let $\omega \in \mathcal{F}$ and $\tilde{\omega} \in \mathcal{F}$ be concave functions with $\lim_{t \to 0+} \frac{\omega(t)}{\tilde{\omega}(t)} = 0$. Then, for each $\delta > 0$, there exists a sequence $(\Delta_i)_{i=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \omega(\Delta_i) < \infty$, $\sum_{i=1}^{\infty} \Delta_i < \infty$ and $\sum_{i=1}^{\infty} \tilde{\omega}(\Delta_i) = \infty$.*

Proof. For each $n \in \mathbb{N}$, choose $\delta_n \in (0, 1/n)$ such that

$$\tilde{\omega}(s) \ge n^2 \omega(s) \quad \text{for } 0 < s < \delta_n.$$

Using the continuity of ω, we can clearly choose for each $n \in \mathbb{N}$ a number $s_n \in (0, \delta_n)$ and $m_n \in \mathbb{N}$ such that $n^2 m_n \omega(s_n) = 1$. Then $m_n \tilde{\omega}(s_n) \ge n^2 m_n \omega(s_n) = 1$. Now define $(\Delta_i)_{i=1}^{\infty}$ as the sequence

$$s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots,$$

in which each symbol s_n appears m_n-times. Then

$$\sum_{i=1}^{\infty} \omega(\Delta_i) = \sum_{n=1}^{\infty} m_n \omega(s_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$\text{and} \quad \sum_{i=1}^{\infty} \tilde{\omega}(\Delta_i) = \sum_{n=1}^{\infty} m_n \tilde{\omega}(s_n) \ge \sum_{n=1}^{\infty} 1 = \infty.$$

Since $\omega \in \mathcal{F}$ is concave, we can clearly find $c > 0$ and $\delta > 0$ such that $\omega(x) > cx$ for each $x \in (0, \delta)$. Since $\Delta_i \to 0$, we have $\omega(\Delta_i) > c\Delta_i$ for all sufficienty large i, and so $\sum_{i=1}^{\infty} \Delta_i < \infty$. □

Lemma 4.6. *Let $\omega \in \mathcal{F}$ and $\tilde\omega \in \mathcal{F}$ be concave functions, and let $\lim_{t\to 0+} \frac{\omega(t)}{\tilde\omega(t)} = 0$. Then there exists a function h on $\mathbb{R}$ which is semiconvex with modulus $\tilde\omega$ on $\mathbb{R}$ and has property $P(\omega, I)$ for each open interval $I \subset \mathbb{R}$. In particular, h is not a DSC_ω function on any open interval I.*

Proof. By Lemma 4.5 there exists a sequence $(\Delta_i)_{i=1}^\infty$ of positive numbers such that $\sum_{i=1}^\infty \omega(\Delta_i) < \infty$, $\sum_{i=1}^\infty \Delta_i < \infty$ and $\sum_{i=1}^\infty \tilde\omega(\Delta_i) = \infty$. Obviously, there exists a strictly increasing sequence $t_1 < t_2 < \cdots < 0$ such that $t_n \to 0$ and $t_{i+1} - t_i = \Delta_i$. Set

$$g(t) := 0 \quad \text{for } t \in (-\infty, t_1) \cup [0, \infty)$$
$$\text{and} \quad g(t) := \tilde\omega(t_{i+1} - t_i) - \tilde\omega(t - t_i) \quad \text{for } t \in [t_i, t_{i+1}).$$

Obviously, g is nonnegative and bounded, g is decreasing on each $[t_i, t_{i+1})$, g is right continuous at all points and continuous at each point $x \neq t_i$. Further, g is discontinuous at each t_i, since $\lim_{t\to t_i-} g(t) = 0$ and $g(t_i) = \tilde\omega(t_{i+1} - t_i) = \tilde\omega(\Delta_i) > 0$.

Set

$$f(t) = \int_0^t g(s)\, ds \quad \text{for each } t \in \mathbb{R}.$$

Then f is clearly a bounded Lipschitz function, $f'_+(t) = g(t)$, $t \in \mathbb{R}$, and $f'_-(t_i) = 0$. Thus $f'_+(t_i) - f'_-(t_i) = \tilde\omega(\Delta_i)$.

Now, we will prove that f is semiconvex with modulus $\tilde\omega$ on $\mathbb{R}$. By Proposition 2.8(ii), it is sufficient to show that

$$f'_+(x) - f'_+(x+\tau) = g(x) - g(x+\tau) \leq \tilde\omega(\tau) \quad \text{for each } x \in \mathbb{R},\ \tau > 0.$$

This inequality is obvious if $x < t_1$ or $x \geq 0$. So, let $x \in [t_i, t_{i+1})$. If $x+\tau \in [t_i, t_{i+1})$, then $g(x) - g(x+\tau) = \tilde\omega(x+\tau-t_i) - \tilde\omega(x-t_i) \leq \tilde\omega(\tau)$, since $\tilde\omega$ is concave, and therefore subadditive. If $x+\tau \geq t_{i+1}$, then $g(x) - g(x+\tau) \leq g(x) = \tilde\omega(t_{i+1} - t_i) - \tilde\omega(x - t_i) \leq \tilde\omega(t_{i+1} - x) \leq \tilde\omega(\tau)$.

Now choose a sequence $(r_n)_{n=1}^\infty$ dense in $\mathbb{R}$ and a sequence $(c_n)_{n=1}^\infty$ with $c_n > 0$ and $\sum_{n=1}^\infty c_n = 1$. Set

$$h(x) := \sum_{n=1}^\infty c_n\, f(x - r_n), \quad x \in \mathbb{R}.$$

Since f is bounded, h is a real function. Since each function $f_n(x) := f(x - r_n)$ is clearly semiconvex with modulus $\tilde\omega$, Lemma 2.6 implies that h is semiconvex with modulus $\tilde\omega$.

Now consider an arbitrary open interval $I \subset \mathbb{R}$. Choose $k \in \mathbb{N}$ with $r_k \in I$ and find $p \in \mathbb{N}$ so large that all points $\tau_i := r_k + t_{i+p}$, $i = 1, 2, \ldots$, belong to I. Then

$$\sum_{i=1}^\infty \omega(\tau_{i+1} - \tau_i) = \sum_{i=1}^\infty \omega(t_{i+p+1} - t_{i+p}) = \sum_{i=1}^\infty \omega(\Delta_{i+p}) < \infty.$$

Obviously,

$$\sum_{i=1}^{\infty}((f_k)'_+(\tau_i)-(f_k)'_-(\tau_i))=\sum_{i=1}^{\infty}(f'_+(t_{i+p})-f'_-(t_{i+p}))=\sum_{i=1}^{\infty}\tilde{\omega}(\Delta_{i+p})=\infty.$$

Observe that $h_k(x):=\sum_{n\in\mathbb{N}\setminus\{k\}}c_nf_n(x)$ is semiconvex with modulus $\tilde{\omega}$ by Lemma 2.6. Consequently

$$\begin{aligned}h'_+(\tau_i)-h'_-(\tau_i)&=((h_k)'_+(\tau_i)-(h_k)'_-(\tau_i))+((c_kf_k)'_+(\tau_i)-(c_kf_k)'_-(\tau_i))\\&\geq c_k((f_k)'_+(\tau_i)-(f_k)'_-(\tau_i)).\end{aligned}$$

Thus $\sum_{i=1}^{\infty}(h'_+(\tau_i)-h'_-(\tau_i))=\infty$. Since Lemma 2.4(*iii*) implies $h'_+(\tau_i)-h'_-(\tau_i)\geq 0$, we obtain that h has property $P(\omega,I)$. □

Theorem 4.7. *Let X be a separable Banach space with $\dim X>1$. Let $\tilde{\omega}\in\mathcal{F}$, $\omega\in\mathcal{F}$ be concave moduli such that $\omega(t)=o(\tilde{\omega}(t))$, $t\to 0+$. Then*

$$\mathcal{DSC}^1_{\tilde{\omega}}(X)\setminus\mathcal{DSC}^1_{\omega}(X)\neq\emptyset.$$

Proof. By Lemma 4.6, we can choose a function h on $\mathbb{R}$ which is semiconvex with modulus $\tilde{\omega}$ on $\mathbb{R}$ and has property $P(\omega,I)$ for each open interval $I\subset\mathbb{R}$ (and thus h is a DSC_ω function on no open interval I).

Now write $X=V\oplus\operatorname{span}\{v\}$, where $\|v\|=1$. Further choose $z\in V$ with $\|z\|=1$, and then $z^*\in V^*$ such that $\|z^*\|=1$ and $z^*(z)=1$. Set

$$f(x):=h(z^*(x)),\quad x\in V.$$

By Lemma 2.7, the function f is semiconvex on V with modulus $E\tilde{\omega}$ ($E>0$), and so

$$P:=\{x+f(x)v:x\in V\}$$

is a $DSC_{\tilde{\omega}}$ hypersurface in X.

Thus it is sufficient to prove that $P\notin\mathcal{DSC}^1_{\omega}(X)$. So suppose to the contrary that $P\subset\bigcup_{i=1}^{\infty}Q_i$, where each Q_i is a DSC_ω hypersurface in X. Since P and each Q_i is closed in X by Remark 2.17(ii), the Baire theorem implies that there exist $i\in\mathbb{N}$, $p_0\in P$ and $\rho>0$ such that $(P\cap B(p_0,\rho))\subset Q_i=:Q$.

Let $w\in X$, $\|w\|=1$, be a vector such that Q is a $\operatorname{span}\{w\}-DSC_\omega$ surface. Denote $\pi:=\pi_{V,\operatorname{span}\{v\}}$ and $u:=\pi(w)$.

Now we will distinguish the following three cases:

a) $w\notin V$ and $z^*(u)\neq 0$,
b) $w\in V$ and $z^*(w)\neq 0$,
c) $w\notin V$ and $z^*(u)=0$.

These are all the possible cases, since the case $w\in V$ and $z^*(w)=0$ cannot occur. Indeed, in this case clearly $p_1:=p_0+(\rho/2)w\in(P\cap B(p_0,\rho))\subset Q$, which is impossible, since $p_0\in Q$, $p_1\in Q$ and $p_1-p_0\in\operatorname{span}\{w\}$.

First, consider the case a) $w \notin V$ and $z^*(u) \neq 0$. Then clearly $w = u + cv$, where $c \neq 0$. We can (and will) suppose that $c > 0$ (otherwise we can consider $-w$ instead of w). By Lemma 2.16, there exists a DSC_ω function $g : V \to \mathbb{R}$ such that

$$Q = \{y + g(y)w : \; y \in V\}.$$

Let x_0, y_0 be the points of V for which $p_0 = x_0 + f(x_0)v = y_0 + g(y_0)w$. Denote

$$\tilde{\pi} := \pi_{V,\mathrm{span}\,\{w\}}, \qquad \pi^P := \pi \restriction_P, \qquad \pi^Q := \tilde{\pi} \restriction_Q$$

and observe (see Remark 2.17(i)) that both $\pi^P : P \to V$ and $\pi^Q : Q \to V$ are bilipschitz mappings. So we can find $\varepsilon > 0$ such that

$$(\pi^P)^{-1}(B(x_0, \varepsilon) \cap V) \subset B(p_0, \rho) \cap P \subset Q.$$

Then $T := \pi^Q \circ (\pi^P)^{-1} \restriction_{B(x_0,\varepsilon)\cap V}$ is a bilipschitz mapping $T : (B(x_0, \varepsilon) \cap V) \to T(B(x_0, \varepsilon) \cap V) \subset V$.

Choose $\delta > 0$ so small that, for each $s \in (-\delta, \delta)$, we have $x_0 + su \in (B(x_0, \varepsilon) \cap V)$ and so $T(x_0 + su)$ is well-defined. Since $T(x_0 + su) - (x_0 + su) \in \mathrm{span}\{v, w\} = \mathrm{span}\{v, u\}$ and $T(x_0 + su) - (x_0 + su) \in V$, we have $T(x_0 + su) = x_0 + \eta(s)u$ for some (uniquely defined) $\eta(s) \in \mathbb{R}$. Since T is bilipschitz, we easily obtain that the mapping $\eta : (-\delta, \delta) \to \eta((-\delta, \delta))$ is bilipschitz. Consequently, $J := \eta((-\delta, \delta))$ is a bounded open interval and η is strictly monotone. Set $F(s) := (\pi^P)^{-1}(x_0 + su) = (x_0 + su) + f(x_0 + su)v$ for $s \in (-\delta, \delta)$ and $G(r) := (\pi^Q)^{-1}(x_0 + ru) = (x_0 + ru) + g(x_0 + ru)w$ for $r \in J$. By the definition of T and η, we have

$$F(s) := (\pi^P)^{-1}(x_0 + su) = (\pi^Q)^{-1}(x_0 + \eta(s)u) = G(\eta(s)), \quad s \in (-\delta, \delta).$$

Denote $\varphi(s) := f(x_0 + su)$, $s \in (-\delta, \delta)$ and $\gamma(r) := g(x_0 + ru)$, $r \in J$. Using Lemma 2.7 and Lemma 2.14(v), we obtain that φ is semiconvex on $(-\delta, \delta)$ and

$$\gamma \text{ is a } DSC_\omega \text{ function on } J, \tag{20}$$

and therefore both φ and γ have all one-sided derivatives at all points.

Now, we will distinguish the case when η is strictly increasing and the case when η is strictly decreasing. First, suppose that η is strictly increasing. Then Lemma 4.3(i) gives that, for each $s \in (-\delta, \delta)$, the vectors $F'_+(s)$, $G'_+(\eta(s))$ (and also $F'_-(s)$, $G'_-(\eta(s))$) are linearly independent. Clearly,

$$\begin{aligned} F'_+(s) &= u + \varphi'_+(s)\, v \\ \text{and} \quad G'_+(\eta(s)) &= u + \gamma'_+(\eta(s))\, w = u + \gamma'_+(\eta(s))u + \gamma'_+(\eta(s))cv. \end{aligned}$$

Since u, v (and also u, w) are linearly independent, we see that $F'_+(s) \neq 0$ and $G'_+(\eta(s)) \neq 0$. Thus there exists $\lambda \neq 0$ such that

$$1 + \gamma'_+(\eta(s)) = \lambda, \qquad c\gamma'_+(\eta(s)) = \lambda\varphi'_+(s).$$

So, repeating this calculation for left derivatives, we obtain that

$$\varphi'_+(s) = \frac{c\gamma'_+(\eta(s))}{1+\gamma'_+(\eta(s))} \quad \text{and} \quad \varphi'_-(s) = \frac{c\gamma'_-(\eta(s))}{1+\gamma'_-(\eta(s))}.$$

Denoting $\nu(z) := \frac{cz}{1+z}$, $z \neq -1$, we have for each $s \in (-\delta, \delta)$

$$\varphi'_+(s) = \nu(\gamma'_+(\eta(s))) \quad \text{and} \quad \varphi'_-(s) = \nu(\gamma'_-(\eta(s))). \tag{21}$$

If η is strictly decreasing, then Lemma 4.3(ii) gives that, for each $s \in (-\delta, \delta)$, the vectors $F'_+(s)$, $G'_-(\eta(s))$ (and also $F'_-(s)$, $G'_+(\eta(s))$) are linearly independent. Proceeding as above, we obtain that for each $s \in (-\delta, \delta)$

$$\varphi'_+(s) = \nu(\gamma'_-(\eta(s))) \quad \text{and} \quad \varphi'_-(s) = \nu(\gamma'_+(\eta(s))). \tag{22}$$

Since γ is Lipschitz, we can choose $s_1 \in (-\delta, \delta)$ such that $\gamma'(\eta(s_1))$ exists. By (21) we have $d := \gamma'(\eta(s_1)) \neq -1$. Thus it is easy to see that we can choose $\Delta > 0$ and $K > 0$ such that ν is strictly monotone, ν is Lipschitz with constant K on $(d-\Delta, d+\Delta)$, and (observe that $\nu^{-1}(y) = \frac{y}{c-y}$, $y \neq c$) also $(\nu \restriction_{(d-\Delta,d+\Delta)})^{-1}$ is Lipschitz with constant K. Further, using Lemma 2.5, we can choose an open interval $J_1 \subset J$ containing $r_1 := \eta(s_1)$ such that $\gamma'_+(r) \in (d-\Delta, d+\Delta)$ and $\gamma'_-(r) \in (d-\Delta, d+\Delta)$ for each $r \in J_1$. Set $I_1 := \eta^{-1}(J_1)$. Then $I_1 \subset (-\delta, \delta)$ and, using (21) and (22), we obtain (independently on the type of monotonicity of η) that

$$\frac{1}{K}|\varphi'_+(s) - \varphi'_-(s)| \leq |\gamma'_+(\eta(s) - \gamma'_-(\eta(s)| \leq K|\varphi'_+(s) - \varphi'_-(s)| \text{ for each } s \in I_1. \tag{23}$$

Denote $\alpha(s) := z^*(x_0 + su)$, $s \in (-\delta, \delta)$. Obviously, α is a nonconstant affine function and $\varphi = h \circ \alpha$. So, by the choice of h and Lemma 4.4, we obtain that φ has property $P(\omega, I)$ on each subinterval I of $(-\delta, \delta)$. Thus we can choose $a \in I_1$ and a strictly monotone sequence $(t_i)_{i=1}^\infty$ with $t_i \in I_1$ and $\lim t_i = a$ such that $\sum_{i=1}^\infty \omega(|t_{i+1} - t_i|) < \infty$ and $\sum_{i\in\mathbb{N}}(\varphi'_+(t_i) - \varphi'_-(t_i)) = \pm\infty$.

Set $a^* := \eta(a)$ and $t_i^* := \eta(t_i)$, $i = 1, 2, \dots$. Then $a^* \in J_1$, $t_i^* \in J_1$, (t_i^*) is strictly monotone and $\lim_{i\to\infty} t_i^* = a^*$.

Let η be Lipschitz with constant $L \geq 1$. Then $|t_{i+1}^* - t_i^*| \leq L|t_{i+1} - t_i|$ and so, since ω is concave, we have

$$\sum_{i=1}^\infty \omega(|t_{i+1}^* - t_i^*|) \leq \sum_{i=1}^\infty \omega(L|t_{i+1} - t_i|) \leq \sum_{i=1}^\infty L\omega(|t_{i+1} - t_i|) < \infty. \tag{24}$$

Observe that (23) implies

$$\frac{1}{K}|\varphi'_+(t_i) - \varphi'_-(t_i)| \leq |\gamma'_+(t_i^*) - \gamma'_-(t_i^*)| \leq K|\varphi'_+(t_i) - \varphi'_-(t_i)|.$$

Further, by (21) and (22), it is easy to see that either $\operatorname{sgn}(\varphi'_+(t_i) - \varphi'_-(t_i)) = \operatorname{sgn}(\gamma'_+(t_i^*) - \gamma'_-(t_i^*))$ for each i or $\operatorname{sgn}(\varphi'_+(t_i) - \varphi'_-(t_i)) = -\operatorname{sgn}(\gamma'_+(t_i^*) - \gamma'_-(t_i^*))$ for each i (depending on the type of monotonicity of η and ν). So, since $\sum_{i\in\mathbb{N}}(\varphi'_+(t_i) - \varphi'_-(t_i)) = \pm\infty$, we easily obtain that also $\sum_{i\in\mathbb{N}}(\gamma'_+(t_i^*) - \gamma'_-(t_i^*)) = \pm\infty$. Thus γ has property $P(\omega, J_1)$, and so (24) and Lemma 4.2 imply that γ is not a DSC_ω function, which contradicts (20).

Now we will consider the case b) $w \in V$ and $z^*(w) \neq 0$, which is similar to the case a), but is simpler.

We can clearly choose a closed space $W \subset X$ of codimension 1 such that $v \in W$ and $W \oplus \operatorname{span}\{w\} = X$. By Lemma 2.16, there exists a DSC_ω function $g : W \to \mathbb{R}$ such that

$$Q = \{y + g(y)w : \ y \in W\}.$$

Let x_0 be the point of V for which $p_0 = x_0 + f(x_0)v$. Denote

$$\tilde{\pi} := \pi_{W,\operatorname{span}\{w\}}, \qquad \pi^P := \pi\restriction_P, \qquad \pi^Q := \tilde{\pi}\restriction_Q .$$

Recall that both $\pi^P : P \to V$ and $\pi^Q : Q \to V$ are bilipschitz mappings. So we can find $\varepsilon > 0$ such that

$$(\pi^P)^{-1}(B(x_0, \varepsilon) \cap V) \subset B(p_0, \rho) \cap P \subset Q.$$

Then $T := \pi^Q \circ (\pi^P)^{-1}\restriction_{B(x_0,\varepsilon)\cap V}$ is a bilipschitz mapping $T : (B(x_0, \varepsilon) \cap V) \to T(B(x_0, \varepsilon) \cap V)) \subset W$. Choose $\delta > 0$ so small that, for each $s \in (-\delta, \delta)$, we have $x_0 + su \in (B(x_0, \varepsilon) \cap V)$ and so $T(x_0 + su)$ is well-defined. We have $(\pi^P)^{-1}(x_0 + sw) = x_0 + sw + f(x_0 + sw)v$. Denote $\varphi(s) := f(x_0 + sw)$, $s \in (-\delta, \delta)$. Since $v \in W$ and $\tilde{\pi}$ is linear, we obtain

$$T(x_0 + sw) = \tilde{\pi}(x_0 + sw + f(x_0 + sw)v) = \tilde{\pi}(x_0) + \varphi(s)v.$$

Since T is bilipschitz, we easily obtain that the mapping $\varphi : (-\delta, \delta) \to \varphi((-\delta, \delta))$ is bilipschitz. Consequently, $J := \varphi((-\delta, \delta))$ is a bounded open interval and φ is strictly monotone. Without any loss of generality, we can suppose that φ is strictly increasing (otherwise we can consider $-w$ instead of w). By the definition of T, we have

$$x_0 + sw + \varphi(s)v = \tilde{\pi}(x_0) + \varphi(s)v + g(\tilde{\pi}(x_0) + \varphi(s)v)w, \quad s \in (-\delta, \delta).$$

Now observe that $\tilde{\pi}(x_0) - x_0 = bw$ for some $b \in \mathbb{R}$, and so $s = b + g(\tilde{\pi}(x_0) + \varphi(s)v)$. Thus, denoting $\gamma(t) := b + g(\tilde{\pi}(x_0) + tv)$ for $t \in J$, we have $s = \gamma(\varphi(s))$, $s \in (-\delta, \delta)$. In other words, $\varphi = \gamma^{-1}$. Observe that γ is a DSC_ω function by Lemma 2.14(v), and thus it has all one-sided derivatives. So, since φ is strictly increasing, it is easy to see that $\varphi'_+(s) = \frac{1}{\gamma'_+(\varphi(s))}$ and $\varphi'_-(s) = \frac{1}{\gamma'_-(\varphi(s))}$. Denoting $\nu(z) := 1/z$, we have for each $s \in (-\delta, \delta)$

$$\varphi'_+(s) = \nu(\gamma'_+(\varphi(s))) \quad \text{and} \quad \varphi'_-(s) = \nu(\gamma'_-(\varphi(s))). \tag{25}$$

Further we proceed as in the case a) and so we sketch the arguments only. First, we observe that $\varphi = h \circ \alpha$ for the affine function $\alpha(s) = z^*(x_0 + sw)$ and thus φ

has property $P(\omega, I)$ on each subinterval I of $(-\delta, \delta)$. Then, using the facts that φ is bilipschitz and $\nu(z) = 1/z$ is decreasing and locally bilipschitz on $(0, \infty)$, we obtain as in the case a) (now using (25) instead of (21) and (22)) that γ is not a DSC_ω function, which is a contradiction.

Finally, we will consider the (easy) case c) $w \notin V$ and $z^*(u) = 0$. By Lemma 2.16, there exists a DSC_ω function $g : V \to \mathbb{R}$ such that

$$Q = \{y + g(y)w : \ y \in V\}.$$

As in the case a), we have $w = u + cv$, where $c \neq 0$, and we can (and will) suppose that $c > 0$. Now we will observe that

$$\text{If } y \in V \text{ and } d \in \mathbb{R}, \text{ then } y + dw \in P \text{ if and only if } d = f(y)/c. \tag{26}$$

Indeed, $y+dw = y+du+dcv \in P$ if and only if $dc = f(y+du)$ and $f(y+du) = f(y)$, since $z^*(u) = 0$, and so $f(y + du) = h(z^*(y + du)) = h(z^*(y)) = f(y)$.

Let y_0 be the point of V for which $y_0 + g(y_0)w = p_0$. By (26), we have $g(y_0) = f(y_0)/c$ and so $p_0 = y_0 + (1/c)f(y_0)w$. Obviously, we can find $\delta > 0$ such that

$$y_0 + tz + (1/c)f(y_0 + tz)w \in B(p_0, \rho) \ \text{ for each } t \in (-\delta, \delta).$$

Using (26), we obtain

$$y_0 + tz + (1/c)f(y_0 + tz)w \in B(p_0, \rho) \cap P \subset Q \ \text{ for each } t \in (-\delta, \delta),$$

and so

$$g(y_0 + tz) = (1/c)f(y_0 + tz) = (1/c)h(z^*(y_0) + t) \ \text{ for each } t \in (-\delta, \delta),$$
$$\text{and so} \quad h(s) = cg(y_0 + (s - z^*(y_0))z) \ \text{ for each } s \in (z^*(y_0) - \delta, z^*(y_0) + \delta).$$

Since g is a DSC_ω function, Lemma 2.14(i), (v) gives that h is a DSC_ω function on $(z^*(y_0) - \delta, z^*(y_0) + \delta)$, which contradicts the choice of h. $\square$

Using Theorem 3.8, we immediately obtain the following corollary.

Corollary 4.8. *Let X be a separable superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type* 2 *(e.g., if X is a Hilbert space or $X = L^p(\mu)$ with $2 \leq p$) and* $\dim X > 1$. *Let $\tilde{\omega} \in \mathcal{F}$, $\omega \in \mathcal{F}$ be concave moduli such that $\omega(t) = o(\tilde{\omega}(t))$, $t \to 0+$. Then $\mathcal{A}^1_{\tilde{\omega}}(X) \setminus \mathcal{A}^1_\omega(X) = \mathcal{A}^1_{\tilde{\omega}}(X) \setminus \sigma(\mathcal{A}^1_\omega(X)) \neq \emptyset$.*

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