

Strongly Adequate Functions on Banach Spaces

Michel Volle

*Department of Mathematics, University of Avignon, France
michel.volle@univ-avignon.fr*

Constantin Zălinescu

*University Alexandru Ioan Cuza Iași, Faculty of Mathematics, Iași, Romania;
and: Institute of Mathematics Octav Mayer, Iași, Romania*

Received: February 10, 2012

Revised manuscript received: September 17, 2012

The notion of adequate function has been recently introduced in order to characterize the essentially strictly convex functions on a reflexive Banach space among the weakly lower semicontinuous ones. In this paper we reinforce this concept and show that a lower semicontinuous function is essentially firmly subdifferentiable if and only if it is strongly adequate.

Keywords: Convex duality, well posed minimization problem, essential firm subdifferentiability, essential strong convexity, essential Fréchet differentiability, total convexity, E-space

2000 Mathematics Subject Classification: 46G05, 49J50, 46N10

1. Introduction

The notion of adequate function on a reflexive Banach space has been recently introduced to obtain a characterization for the class of essentially strictly convex functions (in the sense of [4]) among the weakly lower semicontinuous ones ([19, Th. 1]). In the present paper we reinforce this concept (Definition 3.3) in order to treat lower semicontinuous (lsc) functions on general Banach spaces instead of weakly lower semicontinuous ones. The corresponding concept of convexity no longer produces the class of essentially strictly convex functions but the class of essentially firmly subdifferentiable (convex) functions that we introduce for this purpose (Definition 3.10); this notion benefits from nice properties in terms of optimization problems (Proposition 3.13). We prove in Theorem 3.15 that any lsc strongly adequate function on a Banach space X is essentially firmly subdifferentiable and that the converse holds for X reflexive. In fact, Proposition 3.16 says that the concept of essentially firmly subdifferentiable function is intermediate between the concept of essentially strongly convex function recently introduced in [18] and the concept of totally convex function ([7], [8], [9], [16], ...). In the reflexive case, the class of lsc strongly adequate functions coincides with the one of essentially strongly convex functions (Proposition 3.22). In the finite dimensional case, the two classes above coincide with the essentially strictly convex functions in the sense of [17] (Proposition 3.16). We provide an example of an essentially

strictly convex function on $\mathbb{R}^2$ with convex subdifferential domain which is not totally convex (Example 3.12). A case of essentially strictly convex function on $\mathbb{R}^2$ which is not totally convex on the domain of its subdifferential is given in Example 3.17.

An important tool we use is the natural notion of essential Fréchet differentiability introduced in [18], which strengthens the concept of essential smoothness of [4]. In this way, a dual characterization of lsc strongly adequate functions on general Banach spaces is given in Proposition 3.7 in terms of essential Fréchet differentiability of the Legendre–Fenchel conjugate functions.

Section 4 is devoted to relative projections introduced in [9] which are natural generalizations of the Bregman and Alber’s projections ([1], [6], ...). In this context, we define the f -strongly Tchebychev sets with respect to a lsc function f , non necessarily convex (see also [5]), and study the convexity of this kind of sets (Theorem 4.2). As an application, we get a farthest point like result (Corollary 4.3, Remark 4.4). In Section 5, we characterize the so-called E-spaces ([10], [11], [13], ...) by using our notion of strongly adequate function (Proposition 5.1) and the firm subdifferentiability of the square of the norm (Proposition 5.2). Finally, Proposition 5.3 gives a variational characterization of the closed convex sets in an E-space, including the fact that a closed set in a Hilbert space is convex if and only if it is strongly Tchebychev.

2. Notation and preliminaries

In the sequel X is a Banach space whose topological dual and bidual we denote by X^* and X^{**} , respectively; the dual norm on X^* is denoted by $\|\cdot\|_*$, and $x^*(x)$ for $x \in X$ and $x^* \in X^*$ is denoted by $\langle x, x^* \rangle$. We set $F(X)$ for the class of proper extended real-valued functions $J : X \rightarrow \mathbb{R} \cup \{+\infty\}$ (i.e. $\text{dom } J := \{x \in X \mid J(x) < \infty\} \neq \emptyset$). As usual $\Gamma(X)$ denotes the set of the lower semicontinuous (lsc) convex proper functions on X , and J^* denotes the Legendre–Fenchel conjugate of $J \in F(X)$:

$$J^*(x^*) := \sup_{x \in X} (\langle x, x^* \rangle - J(x)), \quad x^* \in X^*.$$

The subdifferential (in the sense of Convex analysis) of $J \in F(X)$ (J not necessarily convex) at a point $x \in X$ is the set

$$\partial J(x) := \{x^* \in X^* \mid J(u) \geq J(x) + \langle u - x, x^* \rangle \quad \forall u \in X\};$$

clearly, $\partial J(x) = \emptyset$ if $J(x) \notin \mathbb{R}$. One has

$$x^* \in \partial J(x) \iff J^*(x^*) + J(x) = \langle x, x^* \rangle. \quad (1)$$

Consider the inverse multimap

$$MJ := (\partial J)^{-1} : X^* \rightrightarrows X;$$

of course, one has

$$MJ(x^*) = \arg \min(J - \langle \cdot, x^* \rangle), \quad x^* \in X^*.$$

Taking into account (1) and the Fenchel–Young inequality one easily observes that

$$MJ(x^*) \subset \partial J^*(x^*) = \{x^{**} \in X^{**} \mid J^*(u^*) \geq J^*(x^*) + \langle u^* - x^*, x^{**} \rangle \ \forall u^* \in X^*\},$$

where $x \in X$ is identified with $\varphi \in X^{**}$ defined by $\varphi(x^*) := \langle x, x^* \rangle$. Since J^* is subdifferentiable on $\text{int}(\text{dom } J^*)$ we have

$$\text{dom } MJ \cup \text{int}(\text{dom } J^*) \subset \text{dom } \partial J^* \subset \text{dom } J^*. \quad (2)$$

A function $J \in F(X)$ is said to be adequate ([19]) if

$$\begin{cases} \text{dom } MJ = \text{dom}(\partial J^*) \text{ is a nonempty open set, and} \\ MJ \text{ is single-valued on its domain.} \end{cases}$$

It has been proved in [19, Th. 1] that for X reflexive and $J \in F(X)$, J weakly lsc, J is adequate iff J is essentially strictly convex in the sense of [4].

3. Strongly adequate functions

Given an adequate function $J \in F(X)$ and $x^* \in \text{dom } MJ$, the map $J - x^*$ attains a single minimum point over X . We are specially interested in the case when this minimum is a strong minimum, that means every minimizing sequence norm-converges to this minimum. To this end we recall below an important result (see [3, Cor. 6] and [14, Prop. 4]):

Lemma 3.1. *Let $J \in F(X)$ be lsc and $x^* \in \text{int}(\text{dom } J^*)$. Then $J - x^*$ attains a strong minimum over X iff J^* is Fréchet differentiable at x^* .*

Such a situation occurs for instance in the following case:

Proposition 3.2. *Assume that X has the Radon–Nikodym property and let $J \in F(X)$ be lsc. Then the set of $x^* \in \text{int}(\text{dom } J^*)$ such that $J - x^*$ attains a strong minimum over X is a dense G_δ in $\text{int}(\text{dom } J^*)$.*

Proof. By [12, Th. 3.5.8], J^* is Fréchet differentiable on a dense G_δ subset S of $\text{int}(\text{dom } J^*)$. By Lemma 3.1 $J - x^*$ attains a strong minimum over X for every $x^* \in S$. $\square$

In the light of the previous considerations we introduce now the notion of a strongly adequate function.

Definition 3.3. A function $J \in F(X)$ is said to be *strongly adequate* if $\text{dom } \partial J^*$ is a nonempty open set and $J - x^*$ attains a strong minimum over X for every $x^* \in \text{dom } \partial J^*$.

According to (2) any strongly adequate $J \in F(X)$ satisfies

$$\text{dom } MJ = \text{int}(\text{dom } J^*) = \text{dom } \partial J^* \neq \emptyset, \quad (3)$$

and, of course, any strongly adequate function is adequate.

An important example of strongly adequate function is furnished by the lsc functions $J \in F(X)$ whose conjugate J^* is Fréchet differentiable on X^* . In order to go further in our investigation, let us quote the following concept (see [18, Def. 2]).

Definition 3.4. Given a Banach space Y , we say that the function $G \in \Gamma(Y)$ is *essentially Fréchet differentiable* if G is Fréchet differentiable at each point of $\text{int}(\text{dom } \partial G) \neq \emptyset$ and $\|\nabla G(x_n)\|_* \rightarrow \infty$ whenever (x_n) is a sequence in $\text{int}(\text{dom } G)$ converging to some boundary point of $\text{dom } G$.

Proposition 3.5. *A function $G \in \Gamma(Y)$ is essentially Fréchet differentiable iff G is Fréchet differentiable at each point of $\text{dom } \partial G$.*

Proof. It is similar to that of the equivalence of (i) and (v) in [4, Th. 5.6]. We give the proof for reader's convenience.

Sufficiency: here $\text{dom } \partial G$ is open, and so $\text{dom } \partial G = \text{int}(\text{dom } G)$. Hence G is Fréchet differentiable on $\text{int}(\text{dom } G)$. Let $(x_n) \subset \text{int}(\text{dom } G)$ be convergent to $x \in \text{bd}(\text{dom } G)$, and assume that $\|\nabla G(x_n)\|_* \not\rightarrow \infty$. Passing to a subsequence if necessary, we may (and do) assume that $(\|\nabla G(x_n)\|_*)$ is bounded. Therefore, $(\|\nabla G(x_n)\|_*)$ has a subnet $(\|\nabla G(x_{\varphi(i)})\|_*)_{i \in I}$ converging weakly-star to $x^* \in Y^*$. By [20, Th. 2.4.2(ix)] we obtain that $(x, x^*) \in \partial G$, and so we get the contradiction $x \in \text{dom } \partial G = \text{int}(\text{dom } G)$. Therefore, G is essentially Fréchet differentiable.

Necessity: since G is essentially Fréchet differentiable on $\text{int}(\text{dom } G) \neq \emptyset$, we have to show that $\text{dom } \partial G = \text{int}(\text{dom } G)$. Assume that there exists $x \in \text{dom } \partial G \setminus \text{int}(\text{dom } G)$. Fix $\bar{x} \in \text{int}(\text{dom } G)$; clearly, $]x, \bar{x}] \subset \text{int}(\text{dom } G)$. Using [4, Lem. 4.4], we have that $\nabla G(]x, \bar{x}])$ is bounded. Taking $x_n := (1 - n^{-1})x + n^{-1}\bar{x} \in]x, \bar{x}]$, we have that $x_n \rightarrow x$ and $(\nabla G(x_n))$ is bounded. This contradiction proves that $\text{dom } \partial G \subset \text{int}(\text{dom } G)$. $\square$

An essentially Fréchet differentiable function $G \in \Gamma(Y)$ satisfies $\text{dom } \partial G = \text{int}(\text{dom } G) \neq \emptyset$ and ∂G is both single-valued and locally bounded on its domain (see e.g. [20, Cor. 2.4.13]). Consequently, any essentially Fréchet differentiable function $G \in \Gamma(Y)$ is essentially smooth in the sense of [4, Def. 5.2].

It is worthwhile noting that if Y is finite dimensional the two notions above coincide with the usual one introduced in [17, Section 2.6].

Remark 3.6. If $G \in \Gamma(Y)$ is finite-valued, then G is essentially Fréchet differentiable iff G is Fréchet differentiable at each point of Y .

We now provide a dual characterization for a strongly adequate function.

Proposition 3.7. *A lsc function $J \in F(X)$ is strongly adequate iff its conjugate J^* is essentially Fréchet differentiable.*

Proof. In both cases $\text{dom } \partial J^*$ is open, nonempty and, according to (2) and (3), coincides with $\text{int}(\text{dom } J^*)$. It then suffices to apply Lemma 3.1. $\square$

As in [2] (see also [20, p. 188] and [18]), let us introduce the set

$$\Gamma_0 := \{\psi : \mathbb{R}_+ \rightarrow [0, \infty] \mid \psi \text{ lsc convex, } \psi(t) = 0 \iff t = 0\}.$$

Any $\psi \in \Gamma_0$ is a forcing function in the sense of [10, p. 6]:

$$\forall (t_n) \subset \mathbb{R}_+ : \psi(t_n) \rightarrow 0 \Rightarrow t_n \rightarrow 0.$$

Also, any $\psi \in \Gamma_0$ satisfies $\lim_{t \rightarrow \infty} \psi(t) = \infty$.

The following concept has been introduced in [18, Def. 2].

Definition 3.8. A function $H \in \Gamma(X)$ is said to be *essentially strongly convex* if it is essentially strictly convex in the sense of [4, Def. 5.2] and if for every $x \in \text{dom } \partial H$ there exist $x^* \in \partial H(x)$ and $\psi \in \Gamma_0$ such that:

$$H(u) \geq H(x) + \langle u - x, x^* \rangle + \psi(\|u - x\|) \quad \forall u \in X. \quad (4)$$

By [18, Th. 3] we know that for any $J \in F(X)$, one has:

$$J^* \text{ essentially Fréchet differentiable} \Rightarrow J \text{ essentially strongly convex.}$$

From our Proposition 3.7 above we thus have:

Corollary 3.9. Any lsc strongly adequate function $J \in F(X)$ is essentially strongly convex.

In fact, more can be said. To this end, let us introduce the following notion (which appears in [18, Prop. 2] in the framework of essentially strictly convex functions on reflexive Banach spaces):

Definition 3.10. A convex function $H \in F(X)$ is said to be *firmly subdifferentiable at* $x \in \text{dom } \partial H$ if for any $x^* \in \partial H(x)$ there exists $\psi \in \Gamma_0$ such that (4) holds. If H is firmly subdifferentiable at each point of $\text{dom } \partial H$ we will say that H is *essentially firmly subdifferentiable*.

In order to illustrate Definition 3.10 let us recall that a convex function $H \in F(X)$ is said to be *totally convex at* a point $x \in \text{dom } H$ if, denoting by $H'(x, d)$ the right hand side derivative of H at x in the direction d , one has ([7])

$$\inf \{H(u) - H(x) - H'(x, u - x) \mid u \in \text{dom } H, \|u - x\| = t\} > 0 \quad \forall t > 0.$$

Given $x \in \text{dom } \partial H$ we know ([8, Lem. 3.3]) that H is totally convex at x iff there exists $\xi \in \Gamma_0$ such that $(\text{int}(\text{dom } \xi) \neq \emptyset)$ and

$$H(u) \geq H(x) + H'(x, u - x) + \xi(\|u - x\|) \quad \forall u \in X.$$

Since for any $x^* \in \partial H(x)$ and any direction d one has $\langle d, x^* \rangle \leq H'(x, d)$, it holds that if H is totally convex at $x \in \text{dom } \partial H$ then H is firmly subdifferentiable at x .

It follows from [8, Prop. 3.5] that for the proper convex function H which is continuous at $x \in \text{dom } H$ we have that H is totally convex at x iff H is uniformly firmly subdifferentiable at x (that is the same ψ is valid for all $x^* \in \partial H(x)$).

Proposition 3.11. *Let X be finite dimensional, $H \in F(X)$, H convex, and $\bar{x} \in \text{rint}(\text{dom } H)$. Then H is totally convex at $\bar{x}$ iff H is firmly subdifferentiable at $\bar{x}$.*

Proof. Clearly $\partial H(\bar{x}) \neq \emptyset$. We may (and do) assume that H is lsc. The implication $\Rightarrow$ was observed above. Replacing H by $H(\bar{x} + \cdot) - H(\bar{x})$ we may assume that $\bar{x} = 0$ and $H(0) = 0$. Moreover, taking $X_0 = \text{aff}(\text{dom } H) = \text{lin}(\text{dom } H)$, we have that $\text{dom } H'(0, \cdot) = X_0$. It follows that $H'(0, \cdot)|_{X_0}$ is continuous on X_0 . Assume that H is firmly subdifferentiable at $\bar{x} = 0$ but H is not totally convex at $\bar{x}$. Then there exists $\bar{t} > 0$ such that

$$\begin{aligned} & \inf \{ H(x) - H'(0, x) \mid x \in \text{dom } H, \|x\| = \bar{t} \} \\ &= \inf \{ H(x) - H'(0, x) \mid x \in X_0, \|x\| = \bar{t} \} = 0. \end{aligned}$$

Because $H|_{X_0} - H'(0, \cdot)|_{X_0}$ is lsc on X_0 and $A := \{x \in X_0 \mid \|x\| = \bar{t}\}$ is compact, there exists $\bar{u} \in A$ such that $H(\bar{u}) - H'(0, \bar{u}) = 0$. But $H'(0, u) = \max\{\langle u, x^* \rangle \mid x^* \in \partial H(0)\}$ for every $u \in X_0$, and so there exists $\bar{x}^* \in \partial H(0)$ with $H'(0, \bar{u}) = \langle \bar{u}, \bar{x}^* \rangle$. Therefore, $\inf \{ H(x) - \langle x, \bar{x}^* \rangle \mid \|x\| = \bar{t} \} = 0$, contradicting the fact that H is firmly subdifferentiable at 0. $\square$

In [16] it was posed the problem if, in finite dimensions, the converse of [16, Prop. 2.1] is true, that is if an essentially strictly convex function $H \in \Gamma(X)$ with $\text{dom } \partial H$ convex is totally convex (in the case $\dim X < \infty$). We provide an example of an essentially strictly convex function $H \in \Gamma(\mathbb{R}^2)$ with $\text{dom } \partial H$ convex which is not totally convex.

Example 3.12. Let $H : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$ be defined by $H(x, y) := -\sqrt[4]{(1-x^2)(1-y^2)}$ for $(x, y) \in [-1, 1] \times [-1, 1]$, $H(x, y) := \infty$ otherwise. Observe that $H|_{\text{dom } H}$ is continuous and

$$\begin{aligned} \frac{\partial^2 H(x, y)}{\partial x^2} &= \frac{x^2 + 2}{4} \sqrt[4]{\frac{1-y^2}{(1-x^2)^7}} > 0, \\ \frac{\partial^2 H(x, y)}{\partial x^2} \frac{\partial^2 H(x, y)}{\partial y^2} - \left(\frac{\partial^2 H(x, y)}{\partial x \partial y} \right)^2 &= \frac{x^2 + y^2 + 2}{8[(1-x^2)(1-y^2)]^{3/2}} > 0 \end{aligned}$$

on $(-1, 1) \times (-1, 1)$. It follows that H is convex, Fréchet differentiable and strictly convex on $\text{dom } \partial H = (-1, 1) \times (-1, 1)$, and $\text{dom } H^* = \mathbb{R}^2$. It follows that H is essentially strictly convex. Since H is not strictly convex ($H(x, 1) = 0$ for every $x \in [-1, 1]$), we have that H is not totally convex.

Let us provide some properties of essentially firmly subdifferentiable functions in terms of well-posedness and coercivity.

Proposition 3.13. *Let H be firmly subdifferentiable at a point of $\text{argmin } H$ (supposed to be nonempty). Then H is coercive and attains a strong minimum over X .*

Proof. Let $x \in \operatorname{argmin} H$ a point where H is firmly subdifferentiable. One has $0 \in \partial H(x)$ and there exists $\psi \in \Gamma_0$ such that

$$H(u) \geq H(x) + \psi(\|u - x\|), \quad \forall u \in X. \quad (5)$$

Let (x_n) be a minimizing sequence of H . By (5) we have that $\psi(\|x_n - x\|) \rightarrow 0$ and so $\|x_n - x\| \rightarrow 0$. One has also $\lim_{t \rightarrow \infty} \psi(t) = \infty$ and by (5) $\lim_{\|u\| \rightarrow \infty} H(u) = \infty$. $\square$

We now appeal to a dual interpretation of relation (4):

Lemma 3.14 ([20], Cor. 3.4.4). *Let $H \in \Gamma(X)$ and $(x, x^*) \in \partial H$. The statements below are equivalent:*

- (i) $\exists \psi \in \Gamma_0$ such that (4) holds,
- (ii) H^* is Fréchet differentiable at x^* .

We are now in a position to characterize the strongly adequate functions among the lsc ones:

Theorem 3.15. *Let $J \in F(X)$ be lsc. If J is strongly adequate then J is essentially firmly subdifferentiable. The converse holds for X reflexive.*

Proof. Assume J is strongly adequate. By Corollary 3.9 we have that $J \in \Gamma(X)$. Let $(x, x^*) \in \partial J$. Thus $x^* \in \operatorname{dom} \partial J^*$, and Proposition 3.7 says that J^* is Fréchet differentiable at x^* . By Lemma 3.14 there exists $\psi \in \Gamma_0$ such that (4) holds, meaning that J is firmly subdifferentiable at any point $x \in \operatorname{dom} \partial J$, thus essentially firmly subdifferentiable.

Assume J is essentially firmly subdifferentiable and X is reflexive. Let $x^* \in \operatorname{dom} \partial J^*$. Since X is reflexive there is $x \in X$ such that $(x, x^*) \in \partial J$. Since J is firmly subdifferentiable at x , Lemma 3.14 says that J^* is Fréchet differentiable at x^* . Consequently, J^* is essentially Fréchet differentiable and, by Proposition 3.7, J is strongly adequate. $\square$

Let us establish some links among some of the convexity notions quoted above.

Proposition 3.16. *Let $H \in \Gamma(X)$, and consider the following statements:*

- (i) H is totally convex at each point of $\operatorname{dom} \partial H$,
- (ii) H is essentially firmly subdifferentiable,
- (iii) H is essentially strongly convex,
- (iv) H is essentially strictly convex.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Moreover, if X is finite dimensional then (iv) $\Rightarrow$ (ii).

Proof. (i) $\Rightarrow$ (ii) was observed after Definition 3.10.

(ii) $\Rightarrow$ (iv). Let $x_0, x_1 \in X$ be such that $[x_0, x_1] \subset \operatorname{dom} \partial H$. Assume that H is not strictly convex on $[x_0, x_1]$; then $x_0 \neq x_1$ and there exists $\lambda \in (0, 1)$ such that

$H(x_\lambda) = (1 - \lambda)H(x_0) + \lambda H(x_1)$, where $x_\lambda := (1 - \lambda)x_0 + \lambda x_1$. Take $x^* \in \partial H(x_\lambda)$ and $\psi \in \Gamma_0$ for which (4) holds. Then

$$H(x_i) \geq H(x_\lambda) + \langle x_i - x_\lambda, x^* \rangle + \psi(\|x_i - x_\lambda\|) \quad (i \in \{0, 1\}).$$

Multiplying both terms of this inequality by $1 - \lambda > 0$ for $i = 0$ and by $\lambda > 0$ for $i = 1$, then adding side by side we get

$$H(x_\lambda) = (1 - \lambda)H(x_0) + \lambda H(x_1) \geq H(x_\lambda) + (1 - \lambda)\psi(\|x_0 - x_\lambda\|) + \lambda\psi(\|x_1 - x_\lambda\|).$$

It follows that $\psi(\|x_0 - x_\lambda\|) = \psi(\|x_1 - x_\lambda\|) = 0$, whence the contradiction $x_0 = x_1 (= x_\lambda)$.

Let now $x^* \in \text{dom}(\partial H)^{-1}$. Then there exists $x \in \text{dom} \partial H$ such that $x^* \in \partial H(x)$. By (ii), there exists $\psi \in \Gamma_0$ such that (4) holds. Using Lemma 3.14 we obtain that H^* is Fréchet differentiable at x^* , and so $x^* \in \text{int}(\text{dom } H^*)$; hence ∂H^* is bounded on a neighborhood V of x^* . Because $(\partial H)^{-1}(u^*) \subset \partial H^*(u^*)$ for every $u^* \in X^*$, we obtain that $(\partial H)^{-1}$ is bounded on V .

(ii) $\Rightarrow$ (iii) follows immediately from (ii) $\Rightarrow$ (iv) and the fact that $\partial H(x) \neq \emptyset$ for every $x \in \text{dom} \partial H$.

(iv) $\Rightarrow$ (ii). Let $\dim X < \infty$. By [17, Th. 2.6.3] (or [4, Th. 5.4]) we know that H^* is essentially smooth. Since X is finite dimensional this amounts to say that H^* is essentially Fréchet differentiable, and, by Corollary 3.19, that H is essentially firmly subdifferentiable. $\square$

We have seen in Example 3.12 that, for $\dim X < \infty$, there exist essentially strictly convex functions which are not totally convex. A natural question is if (iv) $\Rightarrow$ (i) in Corollary 3.16 for $\dim X < \infty$. The answer is negative, as shown in the following example.

Example 3.17. Let $H : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ be defined by $H(x, y) := -\sqrt{(1 - x^2)(1 - y^2)}$ for $(x, y) \in [-1, 1] \times [-1, 1]$, $H(x, y) := \infty$ otherwise. Then $\text{dom } \partial H = [(-1, 1) \times (-1, 1)] \cup \{-1, 1\} \times \{-1, 1\}$, H is strictly convex on every segment included in $\text{dom } \partial H$ and $\text{dom } H^* = \mathbb{R}^2$. It follows that H is essentially strictly convex. Since H is not strictly convex ($H(x, 1) = 0$ for every $x \in [-1, 1]$), we have that H is not totally convex at each point of $\text{dom } \partial H$.

Remark 3.18. Proposition 3.16 provides in particular a significant improvement of [16, Prop. 2.1].

By juxtaposition of Theorem 3.15 and Proposition 3.7 we obtain:

Corollary 3.19. *Let $J \in F(X)$ be lsc. If J^* is essentially Fréchet differentiable, then J is essentially firmly subdifferentiable. The converse holds for X reflexive.*

Remark 3.20. According to Proposition 3.16, the first part of Corollary 3.19 improves [18, Th. 3].

Remark 3.21. According to Remark 3.6 and Corollary 3.19, a cofinite function $H \in \Gamma(X)$ (that is H^* is finite-valued) with X reflexive is essentially firmly subdifferentiable iff H^* is essentially Fréchet differentiable on X^* . For instance, the square of the norm of a reflexive Banach space is essentially firmly subdifferentiable iff the square of the dual norm is Fréchet differentiable on X^* (see Section 5).

We end this section by a more complete result concerning the reflexive case. It includes [18, Th. 4] and a part of [18, Prop. 2].

Proposition 3.22. *Let X be a reflexive Banach space and $H \in F(X)$. The following assertions are equivalent:*

- (i) H is strongly adequate,
- (ii) H^* is essentially Fréchet differentiable,
- (iii) H is essentially firmly subdifferentiable,
- (iv) H is essentially strongly convex.

Proof. (i) $\Leftrightarrow$ (ii) is established in Theorem 3.15, and (ii) $\Leftrightarrow$ (iii) in Corollary 3.19. By Proposition 3.16 one has (iii) $\Rightarrow$ (iv). The equivalence (ii) $\Leftrightarrow$ (iv) is [18, Th. 4].

We give a proof of (iv) $\Rightarrow$ (ii) for reader's convenience. By Definition 3.8 we have that H is essentially strictly convex, and so, by [4, Th. 5.4], H^* is essentially smooth. By [4, Th. 5.6] we obtain that $\text{dom } \partial H^* = \text{int}(\text{dom } H^*) \neq \emptyset$ and H^* is Gâteaux differentiable on $\text{int}(\text{dom } H^*)$. Let $\bar{x}^* \in \text{int}(\text{dom } H^*)$ and let us show that H^* is Fréchet differentiable at $\bar{x}^*$. For this we apply [20, Th. 3.3.2]. Set $\bar{x} := \nabla H^*(\bar{x}^*) \in \text{dom } \partial H$ and take $((x_n, x_n^*))_{n \geq 1} \subset \partial H$ with $x_n^* \rightarrow \bar{x}^*$. By [20, Th. 3.3.2] applied for the Gâteaux bornology we have that $x_n \rightarrow^w \bar{x}$. Since $H(\bar{x}) \geq H(x_n) + \langle \bar{x} - x_n, x_n^* \rangle$ for every $n \geq 1$, we obtain that $H(\bar{x}) \geq \limsup H(x_n)$, and so $H(x_n) \rightarrow H(\bar{x})$ because H is weakly lsc. Because H is essentially strongly convex, there exist $\tilde{x}^* \in \partial H(\bar{x})$ and $\psi \in \Gamma_0$ such that $H(x) \geq H(\bar{x}) + \langle x - \bar{x}, \tilde{x}^* \rangle + \psi(\|x - \bar{x}\|)$ for every $x \in X$, and so $H(x_n) \geq H(\bar{x}) + \langle x_n - \bar{x}, \tilde{x}^* \rangle + \psi(\|x_n - \bar{x}\|)$ for every $n \geq 1$. Taking the lim sup in both terms we get $\limsup \psi(\|x_n - \bar{x}\|) \leq 0$, and so $\|x_n - \bar{x}\| \rightarrow 0$. Applying now [20, Th. 3.3.2] for the Fréchet bornology we obtain that H^* is Fréchet differentiable at $\bar{x}^*$. $\square$

4. Relative projections on closed sets

Given $f \in F(X)$, S a closed subset of the Banach space X such that

$$S \cap \text{dom } f \neq \emptyset,$$

and $x^* \in X^*$, let us consider the problem:

$$P_S(f, x^*) : \min (f(x) - \langle x, x^* \rangle) \quad \text{for } x \in S.$$

Such problems have been studied in [9] under the name relative projection (of x^* on S modulo f). They are natural generalizations of the Bregman projections

and generalized projections defined and studied by Alber ([1], [6], ...). In [9] the function f is assumed to be convex. We don't retain this assumption here, and just assume that $f \in F(X)$. For instance, taking $f := -\frac{1}{2} \|\cdot\|^2$ on the Hilbert space $(X, \|\cdot\|)$ and $S \subset X$ a bounded subset, the problem $P_S(f, x^*)$ consists of finding the farthest points of S from $-x^* \in X^* = X$. Taking $f := \frac{1}{2} \|\cdot\|^2$ and $S \subset X$, still in the Hilbert space setting, the problem $P_S(f, x^*)$ consists in finding the best approximation of $x^* \in X^* = X$ by elements of S .

Definition 4.1. We will say that S is *f-strongly Tchebychev* if for every $x^* \in X^*$ the problem $P_S(f, x^*)$ admits a strong minimum; in other words if any minimizing sequence of $P_S(f, x^*)$ norm-converges toward a (necessarily unique) solution of $P_S(f, x^*)$.

Denoting by ι_S the indicator function of S , it is clear that if S is *f-strongly Tchebychev*, then $f + \iota_S$ is strongly adequate; conversely, if $f + \iota_S$ is strongly adequate and cofinite, then S is *f-strongly Tchebychev*. We can state:

Theorem 4.2. *Let X be a Banach space, $f \in F(X)$, f lsc, and $S \subset X$ closed satisfying $S \cap \text{dom } f \neq \emptyset$. If S is *f-strongly Tchebychev* then $f + \iota_S$ is essentially firmly subdifferentiable and $S \cap \text{dom } f$ is convex.*

*Conversely, if X is reflexive, S is convex, and f is essentially firmly subdifferentiable, finite and continuous at a point of S , then $f + \iota_S$ is strongly adequate; moreover, if $f + \iota_S$ is cofinite then S is *f-strongly Tchebychev*.*

Proof. Assume S is *f-strongly Tchebychev*. Then $J := f + \iota_S$ is strongly adequate and, by Theorem 3.15, J is essentially firmly subdifferentiable. In particular, J is convex and so $\text{dom } J = S \cap \text{dom } f$ is convex.

Conversely, let us first notice that, by [15, Prop. 10.d] or [20, Th. 2.8.7(iii)], one has $\partial(f + \iota_S)(x) = \partial f(x) + \partial \iota_S(x)$ for all $x \in X$. We thus have $\text{dom } \partial J = S \cap \text{dom } \partial f$. Now for any $x \in \text{dom } \partial J$, any $x^* \in \partial J(x)$, there exist $u^* \in \partial f(x)$ and $v^* \in N(S, x)$ such that $x^* = u^* + v^*$. Since f is firmly subdifferentiable at x , there exists $\psi \in \Gamma_0$ such that, for any $u \in X$, $f(u) \geq f(x) + \langle u - x, u^* \rangle + \psi(\|u - x\|)$, and thus $J(u) \geq J(x) + \langle u - x, x^* \rangle + \psi(\|u - x\|)$, that means J is firmly subdifferentiable at each $x \in \text{dom } \partial J$. By the second part of Theorem 3.15 we infer that J is strongly adequate. When $f + \iota_S$ is cofinite this means that S is *f-strongly Tchebychev*. $\square$

Corollary 4.3. *Let X be a Banach space and S a nonempty closed bounded subset of X . The following statements are equivalent:*

- (i) *the function $\frac{1}{2} \|\cdot\|^2 + x^*$ attains a strong maximum over S for any $x^* \in X^*$,*
- (ii) *S is a singleton.*

Proof. It is clear that (ii) $\Rightarrow$ (i). Conversely, (i) says that the function $J := -\frac{1}{2} \|\cdot\|^2 + \iota_S$ is strongly adequate, hence convex by Corollary 3.9. It follows that S is convex. Because J is strongly adequate, there exists $x_0 \in S$ such that $J(x_0) < J(x)$ for every $x \in S \setminus \{x_0\}$. Assume that $S \neq \{x_0\}$, and take $x_1 \in S \setminus \{x_0\}$. Then

$x_\lambda := (1 - \lambda)x_0 + \lambda x_1 \in S \setminus \{x_0\}$ and

$$-\frac{1}{2} \|x_\lambda\|^2 = J(x_\lambda) \leq (1 - \lambda)J(x_0) + \lambda J(x_1) = -(1 - \lambda)\frac{1}{2} \|x_0\|^2 - \lambda\frac{1}{2} \|x_1\|^2$$

for every $\lambda \in [0, 1]$. Since $\|x_\lambda\|^2 \leq (1 - \lambda) \|x_0\|^2 + \lambda \|x_1\|^2$ for $\lambda \in [0, 1]$ with strict inequality for $\lambda \in (0, 1)$ and $\|x_0\| \neq \|x_1\|$, we obtain that $\|x_0\| = \|x_1\|$, and so we get the contradiction $J(x_0) = J(x_1)$. Hence S is necessarily a singleton. $\square$

Remark 4.4. In the Hilbert space setting, Corollary 4.3 gives the equivalence between the next two statements (involving farthest points to the nonempty closed bounded set $S \subset X$):

- (i) for any $x^* \in X^* = X$, the function $x \mapsto \|x^* - x\|$ attains a strong maximum over S ,
- (ii) S is a singleton.

5. Variational characterizations of closed convex sets in E-spaces

Let us recall that a Banach space X is said to be an E-space if X is rotund and every weakly closed set in X is approximately compact. Such spaces, introduced in [11], admit several characterizations. For instance, the theorem in [13, p. 146] says that X is an E-space iff X is reflexive, rotund, and any weakly convergent sequence within the unit sphere of X is norm convergent. Anderson's Theorem (see [13, p. 149]) says that the Banach space X is an E-space iff the square of the dual norm $\|\cdot\|_*^2$ is Fréchet differentiable on X^* . In the light of the previous results we thus can obtain other characterizations for the E-spaces.

Proposition 5.1. *For any Banach space X , the statements below are equivalent:*

- (i) For any $x^* \in X^*$, the function $\frac{1}{2} \|\cdot\|^2 - x^*$ attains a strong minimum over X ,
- (ii) X is an E-space.

Proof. Setting $J := \frac{1}{2} \|\cdot\|^2$, one has $J^* = \frac{1}{2} \|\cdot\|_*^2$; the equivalence between (i) and (ii) is obtained by Proposition 3.7 and Remark 3.6 using Anderson's Theorem mentioned above. $\square$

In [16, Th. 3.3], the E-spaces are characterized among the reflexive Banach spaces as the locally totally convex spaces (that are the reflexive Banach spaces whose square of the norm is totally convex at any point). Below we characterize the E-spaces in terms of the firm subdifferentiability of the square of the norm:

Proposition 5.2. *Given a Banach space $(X, \|\cdot\|)$, let us consider the following statements:*

- (i) X is an E-space,
- (ii) $\|\cdot\|^2$ is totally convex on X ,
- (iii) $\|\cdot\|^2$ is essentially firmly subdifferentiable.

Then we have $(i) \Rightarrow (ii) \Rightarrow (iii)$, and, if X is reflexive, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Proof. (i) $\Rightarrow$ (ii) is proved in [16, Th. 3.3], and (ii) $\Rightarrow$ (iii) has been quoted in the comments after Definition 3.10.

Assume now that X is reflexive and (iii) holds. By Corollary 3.19 $(\frac{1}{2}\|\cdot\|^2)^* = \frac{1}{2}\|\cdot\|_*^2$ is essentially Fréchet differentiable, and this amounts to the Fréchet differentiability of $\|\cdot\|_*^2$ on X^* ; using Anderson's Theorem mentioned above we obtain that X is an E-space. $\square$

In order to obtain new variational characterizations of the closed convex sets in an E-space let us recall that, given a lsc function $I \in F(X)$, the problem

$$P(I) : \quad \text{minimize } I(x) \quad \text{for } x \in X$$

is said to be Tykhonov well posed (TWP) if I attains a strong minimum over X (i.e. any minimizing sequence is norm-convergent, see e.g. [10]). Several characterizations of the E-spaces in terms of TWP problems have been established (see [10, Th. II.2] or [13, Th. 2, p. 150]): we know that the Banach space X is an E-space iff for any nonempty closed convex set K in X the problem $P(\|\cdot\| + \iota_K)$ or, equivalently, $P(\frac{1}{2}\|\cdot\|^2 + \iota_K)$, is TWP. By [10, Th. II.2], the Banach space X is an E-space iff for any $x^* \in X^* \setminus \{0\}$ the problem $P(x^* + \iota_{B(X)})$, where $B(X)$ denotes the closed unit ball of X , is TWP. The same theorem says that, denoting $S(X)$ the unit sphere of X , X is an E-space iff the problem $P(x^* + \iota_{S(X)})$ is TWP for any $x^* \in X^* \setminus \{0\}$. Our Proposition 5.1 provides another characterization of such spaces: the Banach space X is an E-space iff for any $x^* \in X^*$ the problem $P(\frac{1}{2}\|\cdot\|^2 + x^*)$ is TWP.

To end this paper let us go back to f -strongly Tchebychev sets (see Definition 4.1) in the case when $f = \frac{1}{2}\|\cdot\|^2$. In this situation a nonempty closed S in X is f -strongly Tchebychev iff the problem $P(\frac{1}{2}\|\cdot\|^2 + \iota_S - x^*)$ is TWP for every $x^* \in X^*$. If the underlying Banach space is a Hilbert space this amounts to say that the problem

$$\text{minimize } \|x^* - u\| \quad \text{for } u \in S$$

is TWP for any $x^* \in X^* = X$.

Proposition 5.3. *Let S be a nonempty closed set in a Banach space X . Assume that for any $x^* \in X^*$ the problem*

$$\text{minimize } \frac{1}{2}\|x\|^2 - \langle x, x^* \rangle \quad \text{for } x \in S$$

is TWP. Then S is convex. If X is an E-space, the converse holds.

Proof. The first part follows from the first part Theorem 4.2 applied to $\frac{1}{2}\|\cdot\|^2$. Assume now that X is an E-space. By the first part of Proposition 5.2 we know that $\frac{1}{2}\|\cdot\|^2$ is essentially firmly subdifferentiable and we conclude the proof with the second part of Theorem 4.2. $\square$

References

- [1] Y. I. Alber: Young-Fenchel transformation and some new characteristics of Banach spaces, in: *Functions Spaces* (Edwardsville, 2006), K. Jarosz (ed.), *Contemp. Math.* 435, Amer. Math. Soc., Providence (2007) 1–19.
- [2] E. Asplund: Fréchet differentiability of convex functions, *Acta Math.* 121 (1968) 31–47.
- [3] E. Asplund, R. T. Rockafellar: Gradients of convex functions, *Trans. Amer. Math. Soc.* 139 (1969) 443–467.
- [4] H. H. Bauschke, J. M. Borwein, P. L. Combettes: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, *Commun. Contemp. Math.* 3 (2001) 615–647.
- [5] H. H. Bauschke, J. M. Borwein, P. L. Combettes: Bregman monotone optimization algorithms, *SIAM J. Control Optim.* 42 (2003) 596–636.
- [6] L. M. Bregman: The relaxation method for finding common points of convex sets and its application to the solution of problems in convex programming, *USSR Comput. Math. Math. Phys.* 7 (1967) 200–217.
- [7] D. Butnariu, A. N. Iusem: *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, *Applied Optimization* 40, Kluwer, Dordrecht (2000).
- [8] D. Butnariu, A. N. Iusem, C. Zălinescu: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Analysis* 10 (2003) 35–61.
- [9] D. Butnariu, E. Resmerita: Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.* 2006 (2006) Art. ID 84919, 39 p.
- [10] A. L. Dontchev, T. Zolezzi: *Well-Posed Optimization Problems*, Springer, Berlin (1993).
- [11] K. Fan, I. Glicksberg: Some geometric properties of the spheres in a normed linear space, *Duke Math. J.* 25 (1998) 553–568.
- [12] J. R. Giles: *Convex Analysis with Application in the Differentiation of Convex Functions*, Pitman, Boston (1982).
- [13] R. B. Holmes: *A Course on Optimization and Best Approximation*, *Lect. Notes Math.* 257, Springer, Berlin (1972).
- [14] M. Lassonde: Asplund spaces, Stegall variational principle and the RNP, *Set-Valued Var. Anal.* 17 (2009) 183–193.
- [15] J. J. Moreau: *Fonctionnelles Convexes*, Collège de France, Paris (1967).
- [16] E. Resmerita: On total convexity, Bregman projections and stability in Banach spaces, *J. Convex Analysis* 11 (2004) 1–16.
- [17] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).
- [18] T. Strömberg: Duality between Fréchet differentiability and strong convexity, *Positivity* 15 (2011) 527–536.
- [19] M. Volle, J. B. Hiriart-Urruty: A characterization of essentially strictly convex functions in reflexive Banach spaces, *Nonlinear Anal.* 75 (2012) 1617–1622.

- [20] C. Zălinescu: *Convex Analysis in General Vector Spaces*, World Scientific, River Edge (2002).