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Uniform Convexity of Geodesic Ptolemy Spaces

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Dedicated to Art Kirk on the occasion of his 75th birthday.

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It has been recently proved that balls in a proper geodesic Ptolemy space are strictly convex. In this work we further study Ptolemy spaces in this direction. In particular we prove the uniform convexity of geodesic Ptolemy spaces when assuming a uniform continuity condition on a midpoint map and use this property to deduce different geometrical and convexity related facts about such spaces. We also show that a nice modulus of uniform convexity can be found in such a setting.

1. Introduction

In normed spaces there are many natural convexity notions that play essential roles in the study of a wide range of important problems. One of such significant properties is the uniform convexity which deals with the rotundity of balls. Here we study convex properties of the metric for some particular geodesic spaces, namely geodesic Ptolemy spaces, with emphasis on the uniform convexity of such spaces.

A metric space (X,d) is called a Ptolemy metric space if

$$d(x,z)d(y,p) \le d(x,y)d(z,p) + d(x,p)d(y,z) \quad \text{for every } x,y,z,p \in X.$$

The above relation is known as the Ptolemy inequality.

In the case of normed spaces, the Ptolemy inequality has a great impact on the regularity of the space. Namely, it was proved in [13] that a normed space is an inner product space if and only if it is a Ptolemy space.

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The Ptolemy inequality was used by Foertsch and Schroeder in [8] to study the boundary at infinity of CAT(-1) spaces. It was shown that the boundary of a CAT(-1) space endowed with a Bourdon or a Hamenstädt metric is a Ptolemy space. At the same time, a condition for equality to hold in the Ptolemy inequality was also provided. This property led to the study of the relation between Gromov hyperbolic spaces and CAT(-1) spaces (see [9] for details). Motivated by these results, Ptolemy metric spaces were further investigated in [7] where a characterization of CAT(0) spaces in terms of the Ptolemy inequality is given.

CAT(0) spaces are Ptolemy spaces (see for instance [3] for a justification), but a geodesic Ptolemy space is not necessarily uniquely geodesic and thus cannot satisfy the CAT(0) condition. Foertsch, Lytchak and Schroeder proved in [7] that every Ptolemy metric space can be isometrically embedded into a complete geodesic Ptolemy space. Using this property, the authors easily built examples of geodesic Ptolemy spaces which are not uniquely geodesic. However, it was shown in the same paper that a proper geodesic Ptolemy space is uniquely geodesic, where the properness assumption may be replaced by the existence of a continuous midpoint map. Naturally, the authors raised then the still open question of whether a proper geodesic Ptolemy space (or a geodesic Ptolemy space with a continuous midpoint map) is CAT(0).

The relation between CAT(0) spaces and Ptolemy spaces in the framework of Riemannian or Finsler manifolds was studied in [3] where it was shown that a complete Riemannian manifold is a Ptolemy space if and only if it is CAT(0) and that a Ptolemy Finsler manifold is Riemannian.

A direct consequence of the CAT(0) condition is the fact that CAT(0) spaces are Busemann convex, but being Busemann convex is a weaker property than being CAT(0). However, it was proved in [7] that the Busemann convexity implies the CAT(0) condition in the setting of a Ptolemy space. More precisely, it was shown that a metric space is CAT(0) if and only if it is Busemann convex and Ptolemy.

Furthermore, Foertsch and Schroeder showed in [8] that proper geodesic Ptolemy spaces are, in particular, strictly convex. As a consequence of this result the authors proved that the metric projection onto a closed and convex subset is a singlevalued and continuous mapping. It is still open whether this projection is nonexpansive as it is the case in CAT(0) spaces.

The regularity of geodesic Ptolemy spaces was further studied in [6]. By imposing another convexity-like property (namely the uniform continuity condition of the midpoint map) it was proved that certain properties which were so far known to hold in uniformly convex geodesic spaces that admit a modulus of uniform convexity which satisfies certain conditions (see [5, 11]) also hold in geodesic Ptolemy spaces. More precisely, it was proved that, in a complete geodesic Ptolemy space with a uniformly continuous midpoint map, every decreasing sequence of nonempty, bounded, closed and convex sets has a nonempty intersection and the asymptotic center of a bounded sequence is a singleton. These properties were then applied to metric fixed point theory. In the same paper the question was raised whether geodesic Ptolemy spaces with a uniformly continuous midpoint map are uniformly convex and, in such a case, what properties can a modulus of convexity satisfy.

Here we provide an affirmative answer to the above mentioned question and we revisit results on the geometry of proper Ptolemy spaces given in [8] for the non proper case. In Section 2 we introduce some preliminary notions and results that are used in the sequel. In Section 3 we give some continuity notions for midpoint maps in metric spaces and study their relation to Busemann convexity. Section 4 contains the main results of this work. We prove that every geodesic Ptolemy space which admits a uniformly continuous midpoint map is uniformly convex. As a consequence we show that in a bounded complete Ptolemy space with a uniformly continuous midpoint map the metric projection on a closed and convex subset is a singlevalued and uniformly continuous mapping extending a result proved in [8]. Using a strengthened version of the uniform continuity of the midpoint map (the strong uniform continuity) we conclude that every geodesic Ptolemy space admitting such a midpoint map is uniformly convex and we can find a modulus of uniform convexity that does not depend on the radius of balls.

2. Preliminaries

Let (X, d) be a metric space. A *geodesic path* from x to y is a mapping $c : [0, l] \to X$, where $[0, l] \subseteq \mathbb{R}$, such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a *geodesic segment* which joins x and y. Notice that the geodesic segment from x to y is not necessarily unique. If no confusion arises, we use $[x, y]$ to denote a geodesic segment joining x and y. (X, d) is a *(uniquely) geodesic space* if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$, and we write $z = (1 - t)x + ty$ for simplicity. This, too, may not be unique. A *subset of X is convex* if it contains any geodesic segment that joins every two points of it. More details about geodesic metric spaces can be found in [2, 4].

The *metric* $d : X \times X \to \mathbb{R}$ is *convex* if for any $x, y, z \in X$ one has

$$d(x, (1 - t)y + tz) \leq (1 - t)d(x, y) + td(x, z) \quad \text{for all } t \in [0, 1].$$

In a geodesic Ptolemy space the metric is convex. This can be easily seen by applying the Ptolemy inequality for the points $x, y, (1 - t)y + tz, z$.

The geodesic space (X, d) is *Busemann convex* if given any pair of geodesic paths $c_1 : [0, l_1] \to X$ and $c_2 : [0, l_2] \to X$ with $c_1(0) = c_2(0)$ one has

$$d(c_1(tl_1), c_2(tl_2)) \leq td(c_1(l_1), c_2(l_2)) \quad \text{for all } t \in [0, 1].$$

Applying a simple reasoning we notice that in the definition of Busemann convexity we can renounce to the condition $c_1(0) = c_2(0)$. Then,

$$d(c_1(tl_1), c_2(tl_2)) \leq (1 - t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)) \quad \text{for all } t \in [0, 1].$$

It is also clear that a Busemann convex space is uniquely geodesic and with convex metric.

X is said to be *uniformly convex* (see [10]) if for any $r > 0$ and $\epsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for $a, x, y \in X$,

$$\left.\begin{array}{l} d(x,a) \le r \\ d(y,a) \le r \\ d(x,y) \ge \epsilon r \end{array}\right\} \Rightarrow d\left(\frac{1}{2}x + \frac{1}{2}y, a\right) \le (1-\delta)r. \tag{1}$$

From this definition, it is easy to see that uniformly convex metric spaces are uniquely geodesic. A mapping $\delta : (0,\infty) \times (0,2] \to (0,1]$ providing such a $\delta = \delta(r,\epsilon)$ for a given $r > 0$ and $\epsilon \in (0,2]$ is called a *modulus of uniform convexity.* In this paper we will mainly refer to this definition of uniform convexity. However, this notion can also be defined depending on the center of the ball. This definition was used in [12]. X is said to be *pointwise uniformly convex* if for every $a \in X$, $r > 0$ and $\epsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for $x, y \in X$, (1) holds. Then, the *modulus of pointwise uniform convexity* is a mapping $\delta : X \times (0,\infty) \times (0,2] \to (0,1]$ providing such a $\delta = \delta(a, r, \epsilon)$ for a given $a \in X$, $r > 0$ and $\epsilon \in (0,2]$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ consists of three points x_1, x_2 and x_3 in X (the *vertices* of the triangle) and three geodesic segments corresponding to each pair of points (the *edges* of the triangle). For the geodesic triangle $\Delta{=}\Delta(x_1,x_2,x_3)$, a *comparison triangle* is a triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d(x_i, x_j) = d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1,2,3\}$. Such a triangle always exists and is unique up to isometry.

A geodesic triangle Δ satisfies the *CAT*(0) *inequality* if for every comparison triangle $\bar{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x,y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the comparison points of x and y, i.e., if $x = (1-t)x_i + tx_j$ then $\bar{x} = (1-t)\bar{x}_i + t\bar{x}_j$ for $i, j \in \{1,2,3\}$.

A geodesic space is a *CAT*(0) *space* if every geodesic triangle satisfies the CAT(0) inequality. CAT(0) spaces are uniformly convex and they admit a modulus of uniform convexity which does not depend on the radius of balls. More precisely, the modulus of uniform convexity of a Hilbert space $\delta : (0,2] \to (0,1]$,

$$\delta(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$$

acts as a modulus of uniform convexity in any CAT(0) space.

If X is a metric space and C a nonempty subset of X, we define the *distance of a point* $x \in X$ *to* C by $\mathrm{dist}(x, C) = \inf\{d(x,c) : c \in C\}$. The *metric projection* (or *nearest point mapping*) P_C onto C is the mapping

$$P_C(x) = \{c \in C : d(c,x) = \mathrm{dist}(x,C)\}, \quad \text{for every } x \in X.$$

3. Midpoint maps in metric spaces

Let (X, d) be a metric space. A midpoint map is a map $m : X \times X \to X$ such that for every $x, y \in X$

$$d(x, m(x, y)) = d(y, m(x, y)) = \frac{1}{2} d(x, y).$$

Obviously, one can define a midpoint map in each geodesic space. However, it is also worth noticing that if the metric space X is complete and admits a midpoint map, then the space is geodesic.

Let $m : X \times X \to X$ be a midpoint map. Recall that m is *continuous* if for every $x, y, x_n, y_n \in X$, where $n \in \mathbb{N}$, such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ we have that

$$\lim_{n\to\infty} d(m(x_n, y_n), m(x, y)) = 0.$$

We give next two strengthened continuity notions for midpoint maps that we will use in this work. We say that m is *uniformly continuous* if for every $x_n, x'_n, y_n, y'_n \in X$, where $n \in \mathbb{N}$, with $\lim_{n\to\infty} d(x_n, x'_n) = 0$ and $\lim_{n\to\infty} d(y_n, y'_n) = 0$ we have that

$$\lim_{n\to\infty} d(m(x_n, y_n), m(x'_n, y'_n)) = 0.$$

Similarly, m is called *strong uniformly continuous* if for every $\alpha_n \geq 0$ and $x_n, x'_n, y_n, y'_n \in X$, where $n \in \mathbb{N}$, with $\lim_{n\to\infty} \alpha_n d(x_n, x'_n) = 0$ and $\lim_{n\to\infty} \alpha_n d(y_n, y'_n) = 0$ we have that

$$\lim_{n\to\infty} \alpha_n d(m(x_n, y_n), m(x'_n, y'_n)) = 0.$$

It is not hard to see that in the context of a uniquely geodesic space, properness implies the continuity of the only midpoint map that can be defined, while compactness yields its uniform continuity. We also have the following.

Proposition 3.1.

(i) *Every Busemann convex geodesic space has a unique midpoint map which is strong uniformly continuous. In addition, a strong uniformly continuous midpoint map is also uniformly continuous and a uniformly continuous midpoint map is continuous.*

(ii) *In a compact space, a continuous midpoint map is also uniformly continuous.*

(iii) *There exist uniquely geodesic spaces that are not Busemann convex, but the midpoint map is strong uniformly continuous.*

Proof. Item (i) is obvious. To prove (ii), let $m : X \times X \to X$ be a continuous midpoint map and $(x_n), (x'_n), (y_n), (y'_n)$ be sequences in X with $\lim_{n\to\infty} d(x_n, x'_n) = 0$ and $\lim_{n\to\infty} d(y_n, y'_n) = 0$. Take $(d(m(x_k, y_k), m(x'_k, y'_k)))$ a convergent subsequence of $(d(m(x_n, y_n), m(x'_n, y'_n)))$. We claim that

$$\lim_{k\to\infty} d(m(x_k, y_k), m(x'_k, y'_k)) = 0.$$

From this the conclusion follows. Since X is compact there exist the sequences $(x_s) \subseteq (x_k)$ and $(x'_s) \subseteq (x'_k)$ which converge to some $x \in X$, $(y_s) \subseteq (y_k)$ and $(y'_s) \subseteq (y'_k)$ which converge to some $y \in X$. Then,

$$\lim_{s\to\infty} d(m(x_s, y_s), m(x, y)) = 0 \quad \text{and} \quad \lim_{s\to\infty} d(m(x'_s, y'_s), m(x, y)) = 0.$$

Thus, $\lim_{s\to\infty} d(m(x_s, y_s), m(x'_s, y'_s)) = 0$. This proves our claim.

Items (i) and (ii) can be easily proved. We prove (iii) by using [6, Example 3.2] which shows that there exist uniquely geodesic spaces that are not Busemann convex, but the only midpoint map one can define is uniformly continuous. With a slightly different argument, we show that in this particular space the midpoint map is not only uniformly continuous, but strong uniformly continuous. Let X be the positive octant of the spherical space $(\mathbb{S}^2, d)$. For $n \in \mathbb{N}$, suppose $\alpha_n \geq 0$, $x_n, x'_n, y_n, y'_n \in X$ with

$$\lim_{n\to\infty} \alpha_n d(x_n, x'_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \alpha_n d(y_n, y'_n) = 0.$$

Denote

$$m_n = \frac{1}{2}x_n + \frac{1}{2}y_n, m'_n = \frac{1}{2}x'_n + \frac{1}{2}y'_n \quad \text{and} \quad m''_n = \frac{1}{2}x'_n + \frac{1}{2}y_n.$$

Since $d\left(m_n, m'_n\right) \leq d\left(m_n, m''_n\right) + d\left(m''_n, m'_n\right)$, it is enough to prove that

$$\lim_{n\to\infty} \alpha_n d(m_n, m''_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \alpha_n d(m'_n, m''_n) = 0.$$

Knowing that $\mathrm{diam}(X) = \frac{\pi}{2}$ we have that

$$d(x_n, m_n) \leq \frac{\pi}{4} \quad \text{and} \quad d(x'_n, m''_n) \leq \frac{\pi}{4},$$

and so $4 \cos d(x_n, m_n) \cos d(x'_n, m''_n) \geq 2$. Because

$$\cos d(m_n, m''_n) \geq 1 - \frac{1 - \cos d(x_n, x'_n)}{4 \cos d(x_n, m_n) \cos d(x'_n, m''_n)}, \tag{2}$$

it follows that

$$\cos d(m_n, m''_n) \geq 1 - \frac{1 - \cos d(x_n, x'_n)}{2} = \frac{1 + \cos d(x_n, x'_n)}{2} \geq \cos d(x_n, x'_n). \tag{3}$$

But this yields that $d(m_n, m''_n) \leq d(x_n, x'_n)$ which implies that

$$\lim_{n\to\infty} \alpha_n d(m_n, m''_n) = 0.$$

Similarly, $\lim_{n\to\infty} \alpha_n d(m'_n, m''_n) = 0$. $\square$

4. Uniform convexity of geodesic Ptolemy spaces

We begin this section by proving that a geodesic Ptolemy space with a uniformly continuous midpoint map is uniformly convex. This fact was raised as a question in [6].

Theorem 4.1. *Let X be a geodesic Ptolemy space with a uniformly continuous midpoint map. Then X is uniformly convex.*

Proof. We argue by contradiction. Suppose there exist $r > 0$ and $\epsilon \in (0, 2]$ such that for every $\delta \in (0, 1]$ there exist $a, x, y \in X$ with

$$d(a, x) \leq r, d(a, y) \leq r, d(x, y) \geq \epsilon r \quad \text{and} \quad d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) > (1 - \delta)r.$$

Let $\delta_n = \frac{1}{n}$ for $n \geq 3$. Then there exist $a_n, x_n, y_n \in X$ such that

$$d(a_n, x_n) \leq r, d(a_n, y_n) \leq r, d(x_n, y_n) \geq \epsilon r$$
$$\text{and} \quad d\left(a_n, \frac{1}{2}x_n + \frac{1}{2}y_n\right) > \left(1 - \frac{1}{n}\right) r.$$

Let $x'_n \in [a_n, x_n]$ with $d(x_n, x'_n) = \frac{2}{n} d(a_n, x_n)$ and $d(a_n, x'_n) = \left(1 - \frac{2}{n}\right) d(a_n, x_n)$. Then,

$$d(x_n, x'_n) \leq \frac{2}{n} r \quad \text{and} \quad d(a_n, x'_n) \leq \left(1 - \frac{2}{n}\right) r.$$

Similarly, one can take $y'_n \in [a_n, y_n]$ with

$$d(y_n, y'_n) \leq \frac{2}{n} r \quad \text{and} \quad d(a_n, y'_n) \leq \left(1 - \frac{2}{n}\right) r.$$

Let $l_n = d(x_n, y_n)$, $l'_n = d(x'_n, y'_n)$, $b_n = d\left(\frac{1}{2}x'_n + \frac{1}{2}y'_n, x_n\right)$, $b'_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x'_n\right)$, $c_n = d\left(\frac{1}{2}x'_n + \frac{1}{2}y'_n, y_n\right)$, $c'_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, y'_n\right)$, $m_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x'_n + \frac{1}{2}y'_n\right)$. Since X has a uniformly continuous midpoint map it follows that $\lim_{n\to\infty} m_n = 0$. By the triangle inequality,

$$l_n \leq b_n + c_n. \tag{4}$$

Using the Ptolemy inequality,

$$\begin{aligned} &d\left(a_n, \frac{1}{2}x_n + \frac{1}{2}y_n\right) d(x'_n, y'_n) \\ &\leq d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x'_n\right) d(a_n, y'_n) + d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, y'_n\right) d(a_n, x'_n). \end{aligned}$$

Hence,

$$b'_n + c'_n \geq \frac{1 - 1/n}{1 - 2/n} l'_n = l'_n + \frac{1/n}{1 - 2/n} l'_n. \tag{5}$$

Applying again the Ptolemy inequality we have that

$$d\left(\frac{1}{2}x'_n + \frac{1}{2}y'_n, x_n\right) d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x'_n\right)$$
$$\leq d(x_n, x'_n) d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x'_n + \frac{1}{2}y'_n\right) + \frac{1}{4}d(x_n, y_n)d(x'_n, y'_n).$$

Thus,

$$b_n b'_n \leq \frac{2}{n} r m_n + \frac{1}{4} l_n l'_n. \tag{6}$$

Likewise,

$$c_n c'_n \leq \frac{2}{n} r m_n + \frac{1}{4} l_n l'_n. \tag{7}$$

We also know that

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \leq d(x'_n, y'_n) + \frac{4}{n} r$$

and

$$d(x_n, y_n) \geq d(x'_n, y'_n) - d(x'_n, x_n) - d(y_n, y'_n) \geq d(x'_n, y'_n) - \frac{4}{n} r,$$

so $|l_n - l'_n| \leq \frac{4}{n} r$. Similarly, we obtain that $\left|b'_n - \frac{l_n}{2}\right| \leq \frac{2}{n} r$, $\left|b_n - \frac{l_n}{2}\right| \leq \frac{4}{n} r$, $\left|c'_n - \frac{l_n}{2}\right| \leq \frac{2}{n} r$ and $\left|c_n - \frac{l_n}{2}\right| \leq \frac{4}{n} r$.

Write $l'_n = l_n + d_n \frac{r}{n}$, where $d_n \in [-4, 4]$. We may assume that (d_n) is convergent (otherwise choose a convergence subsequence). Let $d = \lim_{n\to\infty} d_n$ and set

$$S = \left\{(\lambda_n) \subseteq \mathbb{R} : \lim_{n\to\infty} n\lambda_n = 0\right\}.$$

Then,

$$l'_n = l_n + d\frac{r}{n} + (d_n - d)\frac{r}{n} = l_n + d\frac{r}{n} + \gamma_n,$$

where $(\gamma_n) \in S$. Applying the same reasoning, there exist $b, b', c, c' \in [-4, 4]$ and (α_n), (α'_n), (β_n), $(\beta'_n) \in S$ such that

$$b_n = \frac{l_n}{2} + b\frac{r}{n} + \alpha_n, \qquad b'_n = \frac{l_n}{2} + b'\frac{r}{n} + \alpha'_n,$$
$$c_n = \frac{l_n}{2} + c\frac{r}{n} + \beta_n \quad \text{and} \quad c'_n = \frac{l_n}{2} + c'\frac{r}{n} + \beta'_n.$$

From (4) it follows that $b + c \geq 0$. Using (5) we have that

$$l_n + (b' + c')\frac{r}{n} + \alpha'_n + \beta'_n \geq l_n + d\frac{r}{n} + \gamma_n + \frac{1/n}{1 - 2/n}\left(l_n + d\frac{r}{n} + \gamma_n\right),$$

and so

$$b' + c' + (\alpha'_n + \beta'_n)\frac{n}{r} \geq d + \gamma_n \frac{n}{r} + \frac{1}{r(1 - 2/n)}\left(\epsilon r + d\frac{r}{n} + \gamma_n\right).$$

Taking the limit when $n \to \infty$ we obtain that $b' + c' \geq d + \epsilon$.
Adding (6) and (7) we have that

$$\frac{l_n}{2}(b + b' + c + c') \leq \frac{l_n}{2} d + \psi_n, \tag{8}$$

where

$$\psi_n = \frac{l_n}{2}\frac{n}{r}(\gamma_n - \alpha_n - \alpha_n' - \beta_n - \beta_n') + (4m_n - b'\alpha_n - b\alpha_n' - c\beta_n' - c'\beta_n) \\ - \frac{r}{n}(bb' + cc') - \frac{n}{r}(\alpha_n\alpha_n' + \beta_n\beta_n').$$

Dividing by l_n in (8) and then taking the limit when $n \to \infty$ it follows that $b + b' + c + c' \leq d$ (since $\epsilon r \leq l_n \leq 2r$). But this implies $\epsilon \leq 0$ which is false. □

Remark 4.2. It is easy to see that proper geodesic Ptolemy spaces are pointwise uniformly convex since they are strictly convex (see [8, Theorem 1.5]). If we replace the properness assumption by compactness then the strict convexity will imply uniform convexity. This can also be proved by applying the above result after noticing that a compact geodesic Ptolemy space is uniquely geodesic (see [7, Theorem 1.2]) and so the midpoint map is uniformly continuous.

We give next in the setting of complete geodsic Ptolemy spaces with a uniformly continuous midpoint map a result on the existence of unique balls of minimal radius that include a bounded set. A similar result was proved in [8, Corollary 2.4] in proper geodesic Ptolemy spaces when considering the set to be compact.

Corollary 4.3. *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map and consider $K \subseteq X$ a nonempty and bounded set. Then there exists a unique closed ball of minimal radius $\widetilde{B}(p, r)$ centered at $p \in X$ with radius $r \geq 0$ such that $K \subseteq \widetilde{B}(p, r)$.*

Proof. Denote $r = \inf\{s > 0 :$ there exists $p_s \in X$ such that $K \subseteq \widetilde{B}(p_s, s)\}$. We may suppose $r > 0$ because otherwise the conclusion is obvious. For $i \in \mathbb{N}$, let

$$C_i = \bigcap_{x \in K} \widetilde{B}\left(x, r + \frac{1}{i}\right).$$

Then, (C_i) is a decreasing sequence of nonempty, closed and bounded sets. The convexity of the metric yields that C_i is also convex. By [6, Theorem 3.3] it follows that there exists $p \in X$ such that $p \in \cap_{i \in \mathbb{N}} C_i$. This yields that $K \subseteq \widetilde{B}(p, r)$.
Suppose there exists $q \in X$, $q \neq p$ such that $K \subseteq \widetilde{B}(q, r)$. Let $\delta : (0, \infty) \times (0, 2] \to (0, 1]$ be a modulus of uniform convexity for X and denote $\delta^* = \delta\left(r, \frac{d(p,q)}{r}\right)$. Since for each $x \in K$, $d(x, p) \leq r$ and $d(x, q) \leq r$ we have that

$$d\left(x, \frac{1}{2}p + \frac{1}{2}q\right) \leq (1 - \delta^*)\, r.$$

This implies that

$$K \subseteq \widetilde{B}\left(\frac{1}{2}p + \frac{1}{2}q, (1-\delta^*)\, r\right),$$

which contradicts the minimality of r. This ends the proof. □

In order to understand how close geodesic Ptolemy spaces with a uniformly continuous midpoint map fall to CAT(0) spaces, it would be interesting to see what can be said about the regularity of a modulus of uniform convexity. A first natural question is whether there exists a modulus which does not depend on the radius of balls as it was pointed out in Section 2 to be the case for CAT(0) spaces. The next result follows from the proof of Theorem 4.1.

Corollary 4.4. *Let X be a geodesic Ptolemy space with a uniformly continuous midpoint map. Then for every $\epsilon \in (0,2]$ and every $R \geq 1$ there exists $\delta_R(\epsilon) \in (0,1]$ such that it is a modulus of uniform convexity for any ball of radius r with $\frac{1}{R} \leq r \leq R$.*

Proof. Suppose to the contrary that there exist $\epsilon \in (0,2]$ and $R \geq 1$ such that for $\delta_n = \frac{1}{n}$ $(n \geq 3)$ there exist $\frac{1}{R} < r_n < R$ and $a_n, x_n, y_n \in X$ with $d(a_n, x_n) \leq r_n$, $d(a_n, y_n) \leq r_n$, $d(x_n, y_n) \geq \epsilon r_n$ and $d\left(a_n, \frac{1}{2}x_n + \frac{1}{2}y_n\right) > \left(1 - \frac{1}{n}\right) r_n$. Following the same steps as in Theorem 4.1 it is not hard to observe that the proof works if $\lim_{n\to\infty} r_n \neq 0$ and $\lim_{n\to\infty} r_n \neq \infty$. But these situations do not arise since $\frac{1}{R} < r_n < R$. □

We extend next a result of [8, Corollary 2.5] where it is proved that for a proper geodesic Ptolemy space the metric projection onto closed and convex subsets is a singlevalued and continuous mapping. In the setting of bounded and complete geodesic Ptolemy spaces with a uniformly continuous midpoint map we can conclude that the metric projection is uniformly continuous. Note that for proving the uniform continuity of the metric projection, the boundedness assumption is natural even in the setting of uniformly convex Banach spaces (see [1, Lemma 2.5]).

Proposition 4.5. *Let X be a bounded complete geodesic Ptolemy space with a uniformly continuous midpoint map and let $C \subseteq X$ be nonempty, closed and convex. Then the metric projection P_C is a singlevalued and uniformly continuous mapping.*

Proof. We prove first that for $x \in X$, $P_C(x) \neq \emptyset$. Let $\alpha = \operatorname{dist}(x, C)$ and for $p \in \mathbb{N}$ consider the sets

$$C_p = \left\{c \in C : d(x,c) \leq \alpha + \frac{1}{p}\right\}.$$

It is easy to see that (C_p) is a descending sequence of nonempty, bounded and closed sets. The convexity of the metric assures that these sets are also convex. By [6, Theorem 3.3], it follows that there exists $c_1 \in C$ such that $c_1 \in \bigcap_{p\in\mathbb{N}} C_p$.

This yields that $d(x, c_1) = \alpha$ which shows that $P_C(x) \neq \emptyset$.
Applying Theorem 4.1 we know that X is uniformly convex and so it follows that for every $x \in X$, $P_C(x)$ is a singleton. Denote this unique point in $P_C(x)$ by $p(x)$. Suppose there exists $R > 0$ such that for any $n \in \mathbb{N}$ there exist $x_n, y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$ and $d(p(x_n), p(y_n)) \geq R$. Let $n_0 \in \mathbb{N}$ so that $d(x_n, y_n) < \frac{R}{4}$ for $n \geq n_0$. Then,

$$\begin{aligned} d(p(x_n), p(y_n)) &\leq d(p(x_n), x_n) + d(x_n, y_n) + d(y_n, p(y_n)) \\ &\leq d(p(x_n), x_n) + d(x_n, y_n) + d(y_n, p(x_n)) \\ &\leq 2(d(p(x_n), x_n) + d(x_n, y_n)). \end{aligned}$$

Hence, for $n \geq n_0$,

$$d(x_n, p(x_n)) \geq \frac{R}{4}. \tag{9}$$

Since X is bounded, there exists $M \geq \frac{R}{2}$ such that $d(x_n, p(x_n)) \leq M$ for every $n \in \mathbb{N}$. Denote $R' = \max\left\{\frac{4}{R}, M + 2\right\}$ and $\epsilon = \frac{R}{M+2}$. Because

$$\begin{aligned} d(x_n, p(y_n)) &\leq d(x_n, y_n) + d(y_n, p(y_n)) \\ &\leq d(x_n, y_n) + d(y_n, p(x_n)) \\ &\leq d(x_n, p(x_n)) + 2d(x_n, y_n) \end{aligned}$$

and

$$d(p(x_n), p(y_n)) \geq \epsilon(d(x_n, p(x_n)) + 2d(x_n, y_n)),$$

by Corollary 4.4, we obtain that

$$d(x_n, p(x_n)) < d\left(x_n, \frac{1}{2}p(x_n) + \frac{1}{2}p(y_n)\right) \leq (1 - \delta_{R'}(\epsilon))\left(d(x_n, p(x_n)) + \frac{2}{n}\right),$$

where $\delta_{R'}(\epsilon)$ is as in Corollary 4.4. Denoting the limit superior $l = \limsup_{n\to\infty} d(x_n, p(x_n)) \leq M$ we have that $l = 0$. But this contradicts (9). □

Our last result gives a condition under which we can guarantee that for a Ptolemy space there exists a modulus of uniform convexity which only depends on the value of separation between two points inside a ball and which is independent of the radius of the ball.

Theorem 4.6. *Let X be a geodesic Ptolemy space with a strong uniformly continuous midpoint map. Then X is uniformly convex and we can find a modulus of uniform convexity that does not depend on the radius of balls.*

Proof. We solely point out how one can take the points in the construction used in the proof and why we can show in this setting that there exists a modulus of uniform convexity that does not depend on the radius.
Suppose there exists $\epsilon \in (0, 2]$ such that for every $\delta \in (0, 1]$ there exist $r > 0$ and $a, x, y \in X$ with $d(a, x) \leq r$, $d(a, y) \leq r$, $d(x, y) \geq \epsilon r$ and $d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) > (1-\delta)r$. Let $\delta_n = \frac{1}{n}$ for $n \geq 3$. Then there exist $r_n > 0$ and $a_n, x_n, y_n \in X$ such that

$d(a_n, x_n) \le r_n$, $d(a_n, y_n) \le r_n$, $d(x_n, y_n) \ge \epsilon r_n$ and $d\left(a_n, \frac{1}{2}x_n + \frac{1}{2}y_n\right) > \left(1 - \frac{1}{n}\right) r_n$. Let $x_n' \in [a_n, x_n]$ with $d(x_n, x_n') \le \frac{2}{n} r_n$ and $d(a_n, x_n') \le \left(1 - \frac{2}{n}\right) r_n$. Similarly, take $y_n \in [a_n, y_n]$ with $d(y_n, y_n') \le \frac{2}{n} r_n$ and $d(a_n, y_n') \le \left(1 - \frac{2}{n}\right) r_n$.
Let again $m_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x_n' + \frac{1}{2}y_n'\right)$. Since X has a strong uniformly continuous midpoint map it follows that $\lim_{n\to\infty} \frac{m_n}{r_n} = 0$. Moreover, $\epsilon r_n \le l_n \le 2r_n$ for every $n \ge 3$. Now the rest of the proof follows in the same manner as the one of Theorem 4.1. $\square$

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