Journal of Convex Analysis
Volume 20 (2013), No. 3, 753–761

Existence Theorems for Generalized Asymptotically Nonexpansive Mappings in Uniformly Convex Metric Spaces

Withun Phuengrattana*
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
withun_ph@yahoo.com

Suthep Suantai
Materials Science Research Center, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
suthep.s@cmu.ac.th

Received: May 19, 2011
Revised manuscript received: November 6, 2012

In this paper, we prove the fixed point property for generalized asymptotically nonexpansive mappings in uniformly convex metric spaces by using the concept of asymptotic center. We also discuss some fixed point theorems in uniformly convex metric spaces. Our results can be applied to obtain the existence of a fixed point of a generalized asymptotically nonexpansive mapping in both uniformly convex hyperbolic spaces and CAT(0) spaces.

Keywords: Generalized asymptotically nonexpansive mappings, uniformly convex metric spaces, uniformly convex hyperbolic spaces, CAT(0) spaces

1991 Mathematics Subject Classification: 47H09; 47H10

1. Introduction and Preliminaries

Let C be a nonempty closed subset of a metric space (X,d) and T be a mapping of C into itself. The set of all fixed points of T is denoted by $F(T)=\{x\in C : x=Tx\}$, and graph of T is denoted by $G(T)=\{(x,y)\in C\times C : y=Tx\}$. Clearly, the graph of a continuous mapping is always closed. A mapping T is called *a generalized asymptotically nonexpansive mapping* if there exist sequences $\{k_n\}\subset[1,\infty)$ and $\{s_n\}\subset[0,\infty)$ with $\lim_{n\to\infty}k_n=1$, $\lim_{n\to\infty}s_n=0$ such that

$$d\left(T^nx,T^ny\right)\le k_nd\left(x,y\right)+s_n \quad \text{for all } x,y\in C \text{ and } n\in\mathbb{N}. \tag{1}$$

In the case of $s_n=0$ for all $n\in\mathbb{N}$, a mapping T is called *an asymptotically nonexpansive mapping.* In particular, if $k_n=1$ and $s_n=0$ for all $n\in\mathbb{N}$, a

*The first author is supported by the Office of the Higher Education Commission and the Graduate School of Chiang Mai University, Thailand.

ISSN 0944-6532 / $ 2.50

mapping T reduces to *nonexpansive mapping*, that is, $d(Tx, Ty) \le d(x, y)$ for all $x, y \in C$. A mapping T is said to be *an asymptotically nonexpansive mapping in the intermediate sense* if it is uniformly continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(d(T^n x, T^n y) - d(x, y)\right) \le 0.$$

This mapping was first introduced in Banach spaces by Bruck et al. [1]. Obviously, if T is asymptotically nonexpansive mapping in the intermediate sense and if we put $H_n = \sup_{x,y\in C} \left(d(T^n x, T^n y) - d(x, y)\right) \vee 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} H_n = 0$ and

$$d(T^n x, T^n y) \le d(x, y) + H_n$$

for all $x, y \in C$. Therefore T is a generalized asymptotically nonexpansive mapping. It is clear that nonexpansive mappings, asymptotically nonexpansive mappings, and asymptotically nonexpansive mappings in the intermediate sense are all generalized asymptotically nonexpansive mappings. In 2007, Shahzad and Zegeye [2] introduced a class of generalized asymptotically quasi-nonexpansive mappings which is more general than the class of generalized asymptotically nonexpansive mappings. A mapping T is called *a generalized asymptotically quasi-nonexpansive mapping* if $F(T) \ne \emptyset$ and there exist sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ such that

$$d(T^n x, p) \le k_n d(x, p) + s_n \quad \text{for all } x \in C, p \in F(T) \text{ and } n \in \mathbb{N}. \tag{2}$$

It is clear that if T is a generalized asymptotically nonexpansive mapping with $F(T) \ne \emptyset$, then it is generalized asymptotically quasi-nonexpansive. In 2008, He et al. [3] introduced an implicit iteration process for finding a common fixed point of two generalized asymptotically nonexpansive mappings with the assumption that the common fixed point set of those two mappings is nonempty. However, there are no results concerning the existence of fixed points of generalized asymptotically nonexpansive mappings even in the Hilbert and Banach spaces setting. The next example shows that there is a generalized asymptotically nonexpansive mapping which is not asymptotically nonexpansive and its fixed point set is not necessarily closed.

Example 1.1. Define a mapping $T : \left[-\frac{2}{3}, \frac{2}{3}\right] \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ by

$$Tx = \begin{cases} x, & \text{if } x \in \left[-\dfrac{2}{3}, 0\right), \\ \dfrac{16}{81}, & \text{if } x = 0, \\ x^4, & \text{if } x \in \left(0, \dfrac{2}{3}\right]. \end{cases}$$

Then T is generalized asymptotically nonexpansive and $F(T)$ is not closed.

Proof. We will show that

$$|T^n x - T^n y| \le |x - y| + \left(\frac{2}{3}\right)^{4^n} \tag{3}$$

for all $x, y \in \left[-\frac{2}{3}, \frac{2}{3}\right]$ and $n \in \mathbb{N}$. Obviously, (3) holds if $x, y \in \left[-\frac{2}{3}, 0\right)$ or $x = y = 0$. Then, it suffices to consider the following 4 cases:

Case 1: $x, y \in \left(0, \frac{2}{3}\right]$. Then $|T^n x - T^n y| = \left|x^{4^n} - y^{4^n}\right| \leq \left(\frac{2}{3}\right)^{4^n}$.

Case 2: $x \in \left[-\frac{2}{3}, 0\right)$ and $y \in \left(0, \frac{2}{3}\right]$. Then $|T^n x - T^n y| = \left|x - y^{4^n}\right| = |x| + \left|y^{4^n}\right| \leq |x| + |y| = |x - y|$.

Case 3: $x \in \left[-\frac{2}{3}, 0\right)$ and $y = 0$. Then $|T^n x - T^n y| = \left|x - \left(\frac{2}{3}\right)^{4^n}\right| \leq |x - y| + \left(\frac{2}{3}\right)^{4^n}$.

Case 4: $x = 0$ and $y \in \left(0, \frac{2}{3}\right]$. Then $|T^n x - T^n y| = \left|\left(\frac{2}{3}\right)^{4^n} - y^{4^n}\right| \leq |x - y| + \left(\frac{2}{3}\right)^{4^n}$.

Therefore, inequality (3) holds. Then T is generalized asymptotically nonexpansive. It is easy to see that T is not asymptotically nonexpansive and $F(T) = \left[-\frac{2}{3}, 0\right)$ is not closed. $\square$

In 1970, Takahashi [4] introduced the concept of convex metric spaces by using the convex structure as follows: Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda) d(z, y)$$

for all $z \in X$. A metric space (X, d) together with a convex structure W is called a *convex metric space* which will be denoted by (X, d, W). A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. It is easy to see that open and closed balls are convex and the intersection of a family of convex subsets of X is also convex, see [4]. Clearly, a normed space and each of its convex subsets are convex metric space, but the converse does not hold.

A function f on a nonempty subset C of a convex metric space (X, d, W) is said to be *convex* if $f(W(x, y, \lambda)) \leq \lambda f(x) + (1 - \lambda) f(y)$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

In 1996, Shimizu and Takahashi [5] introduced the concept of uniform convexity in convex metric spaces and studied the properties of these spaces.

Definition 1.2. A convex metric space (X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0, 1]$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$ imply that

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq (1 - \delta(\varepsilon))\, r.$$

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces. Moreover, we known that if (X, d, W) is a uniformly convex metric space, then for each $x, y \in X$ and $\lambda \in [0, 1]$, there exists a unique element $z \in X$ such that $d(z, x) = \lambda d(x, y)$ and $d(z, y) = (1 - \lambda) d(x, y)$. The proof of this fact, we omit because it is similar to [16, Proposition 2.3].

The following lemma is very useful for our main results. It was proved by Shimizu and Takahashi [5, Theorem 1] in 1996.

Lemma 1.3. *Let (X, d, W) be a complete uniformly convex metric space. Then every decreasing sequence of nonempty bounded closed convex subsets of X has nonempty intersection.*

The famous fixed point theorem for mappings in Banach spaces has been first studied by Browder [6] and Göhde [7]; they proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space, then $F(T)$ is nonempty for any nonexpansive mapping $T : C \to C$. In 1972, Geobel and Kirk [8] extended that result to asymptotically nonexpansive mappings. Later, Kirk [9] obtained a similar result for complete CAT(0) spaces. Recently, Kohlenbach and Leuştean [10] generalized those results to complete uniformly convex hyperbolic spaces with monotone modulus of uniform convexity. Up to now, no researchers have studied the fixed point theorem for generalized asymptotically nonexpansive mappings in any spaces. In this paper, we prove the fixed point property for generalized asymptotically nonexpansive mappings in uniformly convex metric spaces. Our result can be applied to uniformly convex hyperbolic spaces and CAT(0) spaces.

2. Asymptotic Centers for Convex Metric Spaces

The notion of the asymptotic center can be introduced in the general setting of a convex metric space (X, d, W) as follows: Let C be a nonempty closed convex subset of X and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

Clearly, $r(\cdot, \{x_n\})$ is a continuous and convex function. The *asymptotic radius* of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf \{r(x, \{x_n\}) : x \in C\},$$

and the *asymptotic center* of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is clear that the asymptotic center $A(C, \{x_n\})$ is always closed and convex. It may either be empty, or consist of one or many points. The asymptotic center $A(C, \{x_n\})$ is singleton for uniformly convex hyperbolic spaces with monotone modulus of convexity [10], or CAT(0) spaces [11], or uniformly convex Banach spaces [12, 13].

The following lemma extends a result of uniqueness of the asymptotic center to convex metric spaces in the sense of Takahashi.

Lemma 2.1. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) and $\{x_n\}$ be a bounded sequence in X. Then $A(C, \{x_n\})$ is a singleton set.*

Proof. First, we show that $A\left(C,\{x_n\}\right)$ is nonempty. Let

$$C_k=\left\{y\in C: r\left(y,\{x_n\}\right)\leq r\left(C,\{x_n\}\right)+\frac{1}{k}\right\}\quad\text{for all } k\in\mathbb{N}.$$

Then C_k is a nonempty bounded closed convex subset of C and $C_{k+1}\subseteq C_k$ for all $k\in\mathbb{N}$. Since X is a complete uniformly convex metric space, it follows from Lemma 1.3 that $\bigcap_{k=1}^{\infty}C_k$ is nonempty. Next, we will show that $A\left(C,\{x_n\}\right)=\bigcap_{k=1}^{\infty}C_k$. Let $p\in A\left(C,\{x_n\}\right)$. It is obvious that $p\in\bigcap_{k=1}^{\infty}C_k$. Hence, $A\left(C,\{x_n\}\right)\subseteq\bigcap_{k=1}^{\infty}C_k$. Next, let $p\in\bigcap_{k=1}^{\infty}C_k$. Then

$$r\left(p,\{x_n\}\right)\leq r\left(C,\{x_n\}\right)+\frac{1}{k},\quad\text{for all } k\in\mathbb{N}.$$

Hence $r\left(p,\{x_n\}\right)\leq r\left(C,\{x_n\}\right)=\inf\{r\left(y,\{x_n\}\right): y\in C\}$. This implies that $r\left(p,\{x_n\}\right)=r\left(C,\{x_n\}\right)$. That is $p\in A\left(C,\{x_n\}\right)$. Therefore, $\bigcap_{k=1}^{\infty}C_k\subseteq A\left(C,\{x_n\}\right)$.

Finally, we show that $A\left(C,\{x_n\}\right)$ is singleton. To show this, suppose $y,z\in A\left(C,\{x_n\}\right)$ with $y\neq z$. Let $R=r\left(C,\{x_n\}\right)>0$. Then $r\left(y,\{x_n\}\right)=R$ and $r\left(z,\{x_n\}\right)=R$. Then for every $\varepsilon\in(0,1]$, there exists $N\in\mathbb{N}$ such that

$$d(y,x_n)\leq R+\varepsilon\quad\text{and}\quad d(z,x_n)\leq R+\varepsilon\ \text{ for all } n\geq N.$$

Moreover,

$$d(y,z)=\frac{d(y,z)}{R+\varepsilon}\left(R+\varepsilon\right)\geq\frac{d(y,z)}{R+1}\left(R+\varepsilon\right).$$

By the uniform convexity of X, we get that for each $n\geq N$,

$$d\left(W\left(y,z,\frac{1}{2}\right),x_n\right)\leq\left(1-\delta\left(\frac{d(y,z)}{R+1}\right)\right)\left(R+\varepsilon\right).$$

It follows that

$$r\left(W\left(y,z,\frac{1}{2}\right),\{x_n\}\right)\leq\left(1-\delta\left(\frac{d(y,z)}{R+1}\right)\right)\left(R+\varepsilon\right).$$

Since ε is arbitrary, we have

$$r\left(W\left(y,z,\frac{1}{2}\right),\{x_n\}\right)<R.$$

This is a contradiction. Hence $A\left(C,\{x_n\}\right)$ is singleton. $\square$

Remark 2.2. The similar result as Lemma 2.1 has been proved in [14] for a certain class of uniformly convex metric spaces with a monotone or lower semicontinuous from the right modulus of convexity. The proof of Lemma 2.1 is different from one in [14] and there is no any condition on the modulus of convexity.

3. Fixed Point Theorems

We now give the fixed point property for generalized asymptotically nonexpansive mappings in complete uniformly convex metric spaces.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a complete uniformly convex metric space (X, d, W). If $T : C \to C$ is a generalized asymptotically nonexpansive mapping whose graph $G(T)$ is closed, then T has a fixed point.*

Proof. Let $x \in C$ and $x_n = T^n x$. From Lemma 2.1, we know that $A(C, \{x_n\})$ is a singleton set. Let $A(C, \{x_n\}) = \{z\}$. Then $r(z, \{x_n\}) = r(C, \{x_n\}) = \inf\{r(y, \{x_n\}) : y \in C\}$. Let $R = r(z, \{x_n\}) = r(C, \{x_n\})$, we divide our proof into the following two cases:

Case (i): $R = 0$. Then $T^n x \to z$. Since

$$\begin{aligned} d(T^n z, z) &\le d(T^n z, T^n(T^n x)) + d\left(T^{2n} x, z\right) \\ &\le k_n d(z, T^n x) + s_n + d\left(T^{2n} x, z\right) \to 0 \end{aligned}$$

and $T(T^n z) = T^{n+1} z \to z$, it follows from the closedness of $G(T)$ that $Tz = z$.

Case (ii): $R > 0$. We will show that z is a fixed point of T. To show this, suppose not. Then $\{T^n z\}$ does not converge to z. So, there exists some $\varepsilon \in (0, 4]$ and a subsequence $\{n_i\}$ of positive integers such that

$$d(z, T^{n_i} z) \ge \frac{\varepsilon}{2} \quad \text{for all } i \in \mathbb{N}. \tag{4}$$

We can choose $\eta \in (0, 1]$ such that

$$1 - \delta\left(\frac{\varepsilon}{2(R+1)}\right) \le \frac{R-\eta}{R+\eta}. \tag{5}$$

From $\lim_{n\to\infty}\left(k_n\left(R + \frac{\eta}{2}\right) + s_n\right) = R + \frac{\eta}{2} < R + \eta$, we see that there exists $N_0 \in \mathbb{N}$ such that

$$k_n\left(R + \frac{\eta}{2}\right) + s_n < R + \eta \quad \text{for all } n \ge N_0. \tag{6}$$

Since $R = r(z, \{x_n\})$, there exists $N_1 \in \mathbb{N}$ such that

$$d(z, x_n) \le R + \frac{\eta}{2} \quad \text{for all } n \ge N_1. \tag{7}$$

Choose $i_0 \in \mathbb{N}$ such that $n_{i_0} > N_0$. Let $m \in \mathbb{N}$ be such that $m \ge N_1 + n_{i_0}$. Then, we obtain

$$\begin{aligned} d(x_m, T^{n_{i_0}} z) &= d\left(T^{n_{i_0}}\left(T^{m-n_{i_0}} x\right), T^{n_{i_0}} z\right) \\ &\le k_{n_{i_0}} d\left(T^{m-n_{i_0}} x, z\right) + s_{n_{i_0}} \\ &\le k_{n_{i_0}} d\left(x_{m-n_{i_0}}, z\right) + s_{n_{i_0}} \\ &\le k_{n_{i_0}}\left(R + \frac{\eta}{2}\right) + s_{n_{i_0}} \\ &< R + \eta, \end{aligned} \tag{8}$$

and

$$d\left(x_m, z\right) \leq R + \frac{\eta}{2} < R + \eta. \tag{9}$$

Further, by (4), we also have

$$d\left(z, T^{n_{i_0}} z\right) \geq (R+\eta) \frac{\varepsilon}{2(R+\eta)} \geq (R+\eta) \frac{\varepsilon}{2(R+1)}. \tag{10}$$

This combined with (8), (9) and the uniform convexity of X imply that for $m \geq N_1 + n_{i_0}$,

$$\begin{aligned} d\left(x_m, W\left(z, T^{n_{i_0}} z, \frac{1}{2}\right)\right) &\leq \left(1 - \delta\left(\frac{\varepsilon}{2(R+1)}\right)\right)(R+\eta) \\ &\leq R - \eta \\ &< R. \end{aligned}$$

Consequently, $r\left(W\left(z, T^{n_{i_0}} z, \frac{1}{2}\right), \{x_n\}\right) < r\left(C, \{x_n\}\right)$, which is a contradiction. Hence, z is a fixed point of T. □

Theorem 3.2. *Suppose that X, C, T are as in Theorem* 3.1. *Then $F(T)$ is nonempty closed and convex.*

Proof. By Theorem 3.1, $F(T)$ is nonempty. To show that $F(T)$ is closed, we let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$. By definition of T, we have

$$\begin{aligned} d\left(T^n x, x\right) &\leq d\left(T^n x, x_n\right) + d\left(x_n, x\right) \\ &\leq k_n d\left(x, x_n\right) + s_n + d\left(x_n, x\right) \to 0. \end{aligned}$$

That is $T^n x \to x$, and so $T\left(T^n x\right) = T^{n+1} x \to x$. By the closedness of $G(T)$, we have $Tx = x$. Hence $x \in F(T)$ so that $F(T)$ is closed.

Finally, we show that $F(T)$ is convex. Let x, y be two elements in $F(T)$ with $x \neq y$. We need to show that $z = W\left(x, y, \frac{1}{2}\right) \in F(T)$. By the definition of T, we obtain

$$d\left(x, T^n z\right) = d\left(T^n x, T^n z\right) \leq k_n d\left(x, z\right) + s_n = \frac{k_n}{2} d(x,y) + s_n$$

and

$$d\left(y, T^n z\right) = d\left(T^n y, T^n z\right) \leq k_n d\left(y, z\right) + s_n = \frac{k_n}{2} d(x,y) + s_n.$$

This implies that $\limsup_{n\to\infty} d\left(x, T^n z\right) \leq \frac{d(x,y)}{2}$ and $\limsup_{n\to\infty} d\left(y, T^n z\right) \leq \frac{d(x,y)}{2}$. Then, for each $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$T^n z \in \bar{B}\left(x, \frac{d(x,y)}{2} + \frac{1}{m}\right) \bigcap \bar{B}\left(y, \frac{d(x,y)}{2} + \frac{1}{m}\right) = D_m. \tag{11}$$

It is easy to see that $\{D_m\}$ is a decreasing sequence of nonempty bounded closed convex subsets of X. By Lemma 1.3, $\bigcap_{m\in\mathbb{N}} D_m \neq \emptyset$. We will show that $\lim_{m\to\infty} \operatorname{diam}(D_m) = 0$. Suppose $d = \lim_{m\to\infty} \operatorname{diam}(D_m) > 0$. Then, there exists $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1} \leq \frac{d}{2}$ and $d_m = \operatorname{diam}(D_m) > 0$ for each $m \geq N_1$. So, there exist $x_m, y_m \in D_m$ such that $d(x_m, y_m) \geq d_m - \frac{1}{m} \geq d - \frac{1}{m} \geq \frac{d}{2}$ for each $m \geq N_1$. Without loss of generality, we suppose $x \notin \bar{B}\left(y, \frac{d(x,y)}{2} + \frac{1}{m}\right)$ for each $m \geq N_1$. Let $r_m = d\left(x, \bar{B}\left(y, \frac{d(x,y)}{2} + \frac{1}{m}\right)\right)$ and $r = \lim_{m\to\infty} r_m$. Thus, we get $d(x_m, x) \leq r + \frac{1}{m}$, $d(y_m, x) \leq r + \frac{1}{m}$ and

$$d(x_m, y_m) \geq \left(r + \frac{1}{m}\right) \frac{d}{2\left(r + \frac{1}{m}\right)} \geq \left(r + \frac{1}{m}\right) \frac{d}{2\left(r + 1\right)}.$$

By the convexity of $\bar{B}\left(y, \frac{d(x,y)}{2} + \frac{1}{m}\right)$ and the uniform convexity of X, we have

$$r_m \leq d\left(x, W\left(x_m, y_m, \frac{1}{2}\right)\right) \leq \left(1 - \delta\left(\frac{d}{2\left(r+1\right)}\right)\right)\left(r + \frac{1}{m}\right).$$

Taking $m \to \infty$, we obtain $r \leq \left(1 - \delta\left(\frac{d}{2(r+1)}\right)\right) r < r$. This is a contradiction. Hence $\lim_{m\to\infty} \operatorname{diam}(D_m) = 0$. By (11), we must get $\lim_{n\to\infty} T^n z = z$. It follows from the closedness of $G(T)$ that $Tz = z$. Therefore, $F(T)$ is convex. □

In our next theorem, we show that the existence of a fixed point of a generalized asymptotically nonexpansive mapping in a uniformly convex metric space is equivalent to the existence of a bounded orbit at a point.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) and $T : C \to C$ be a generalized asymptotically nonexpansive mapping whose graph $G(T)$ is closed. Then $F(T) \neq \emptyset$ if and only if there exists an $x \in C$ such that $\{T^n x\}$ is bounded.*

Proof. The necessity is obvious. Conversely, assume that x is an element in C such that $\{T^n x\}$ is bounded. Let $x_n = T^n x$. By the same step of the proof as in Theorem 3.1, we can conclude that $F(T) \neq \emptyset$. □

Remark 3.4. (i) If T is a continuous generalized asymptotically nonexpansive mapping, then $G(T)$ is always closed. Therefore, Theorems 3.1, 3.2 and 3.3 are obtained for a class of continuous generalized asymptotically nonexpansive mappings.

(ii) The fixed point property for generalized asymptotically nonexpansive mappings in Hilbert spaces and uniformly convex Banach spaces are obtained by setting the convex structure $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

(iii) CAT(0) spaces and uniformly convex hyperbolic spaces are examples of uniformly convex metric spaces, see more details in [10, 15, 16]. Therefore, Lemma 2.1, Theorems 3.1, 3.2 and 3.3 are also obtained for these two classes of spaces.

Acknowledgements. The authors would like to thank the referees for valuable suggestions on the paper and thank the National Research University Project under Thailand's Office of the Higher Education Commission for financial support.

References

[1] R. Bruck, T. Kuczumow, S. Reich: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, Colloq. Math. 65 (2) (1993) 169–179.

[2] N. Shahzad, H. Zegeye: Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps, Appl. Math. Comput. 189 (2007) 1058–1065.

[3] L. He, L. Deng, J. Liu: Strong convergence theorem of implicit iteration process for generalized asymptotically nonexpansive mappings in Hilbert space, Int. J. Math. Math. Sci. 2008, Article ID 649510 (2008) 9 p.

[4] W. Takahashi: A convexity in metric space and nonexpansive mappings, Kōdai Math. Semin. Rep. 22 (1970) 142–149.

[5] T. Shimizu, W. Takahashi: Fixed points of multivalued mappings in certain convex metric spaces, Topol. Methods Nonlinear Anal. 8 (1996) 197–203.

[6] F. E. Browder: Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. USA 54 (1965) 1041–1044.

[7] D. Göhde: Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30 (1965) 251–258.

[8] K. Goebel, W. A. Kirk: A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171–174.

[9] W. A. Kirk: Geodesic geometry and fixed point theory II, in: International Conference on Fixed Point Theory and Applications (Valencia, 2003), J. García-Falset et al. (ed.), Yokohama Publ., Yokohama (2004) 113–142.

[10] U. Kohlenbach, L. Leuştean: Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, J. Eur. Math. Soc. 12(1) (2010) 71–92.

[11] S. Dhompongsa, W. A. Kirk, B. Sims: Fixed points of uniformly Lipschitzian mappings, Nonlinear Anal. 65 (2006) 762–772.

[12] K. Goebel, W. A. Kirk: Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge (1990).

[13] K. Goebel, S. Reich: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York (1984).

[14] R. Espínola, A. Fernández-León, B. Piątek: Fixed points of single- and set-valued mappings in uniformly convex metric spaces with no metric convexity, Fixed Point Theory Appl. 2010, Article ID 169837 (2010) 16 p.

[15] M. Bridson, A. Haefliger: Metric Spaces of Non-Positive Curvature, Springer, Berlin (1999).

[16] L. Leuştean: A quadratic rate of asymptotic regularity for CAT(0)-spaces, J. Math. Anal. Appl. 325(1) (2007) 386–399.