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On Support Points and Functionals of Unbounded Convex Sets

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Let K be a nonempty closed convex subset of a real Banach space of dimension at least two. Suppose that K does not contain any hyperplane. Then the set of all support points of K is pathwise connected and the set $\Sigma_1(K)$ of all norm-one support functionals of K is uncountable. This was proved for bounded K by L. Veselý and the author [3], and for general K by L. Veselý [8] using a parametric smooth variational principle. We present an alternative geometric proof of the general case in the spirit of [3].

Keywords: Convex set, support point, support functional, Bishop-Phelps theorem

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Introduction

Let K be a nonempty closed convex set in a real, at least two-dimensional Banach space X, with the topological dual X^*.

If $x \in K$ and $x^* \in X^* \setminus \{0\}$ are such that $x^*(x) = \sup x^*(K)$, we say that x is a *support point* of K and x^* is a *support functional* of K. Let $\mathrm{supp}(K)$ and $\Sigma_1(K)$ denote respectively the set of all support points of K and the set of all norm-one support functionals of K. Let $\Pi_1(K)$ denote the set of all norm-one linear functionals bounded above on K.

By the well-known Bishop-Phelps theorem [1], $\mathrm{supp}(K)$ is dense in the boundary of K, and $\Sigma_1(K)$ is dense in $\Pi_1(K)$. By the James theorem [5], $\Sigma_1(K) = S_{X^*}$ if and only if K is weakly compact.

In 1999, L. Zajíček asked (cf. [7]) whether the set $\Sigma_1(K)$ is uncountable provided K is bounded. In [3], the author and L. Veselý solved Zajíček's problem in the affirmative by proving that, for each bounded closed convex set K, the set $\Sigma_1(K)$ is uncountable in each nonempty relatively open subset of S_{X^*}. Moreover, in [3], we proved that, for each bounded closed convex set K, the set $\mathrm{supp}(K)$ is pathwise connected.

In [8], L. Veselý generalized the above results by using a parametric smooth vari-

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ational principle. In particular, he proved that, if K is nonempty closed convex and does not contain any hyperplane, then the set $\Sigma_1(K)$ is uncountable in each nonempty relatively open subset of $\Pi_1(K)$ and the set $\operatorname{supp}(K)$ is pathwise connected.

This paper contains an alternative geometric proof of these latter two results, consisting in an easy variation of the proof used in [3].

The paper is organized as follows. Section 1 contains some notations, definitions and auxiliary results. In Section 2, we prove Theorem 2.3 and Theorem 2.4 from which we derive the main results of this paper in Section 3.

1. Notations and preliminaries

Throughout all this paper, X denotes a real, at least two-dimensional Banach space with the topological dual X^*. We denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. For $x \in X$ and $r > 0$, we set $B(x,r) = x + rB_X = \{y \in X; \|y - x\| \leq r\}$. For $x, y \in X$, $[x, y]$ denotes the closed segment in X with endpoints x and y, and $(x, y) = [x, y] \setminus \{x, y\}$ is the corresponding "open" segment. For a bounded subset K of X, $x^* \in S_{X^*}$ and $\alpha > 0$, let $\mathcal{S}(K, x^*, \alpha) = \{x \in K;\ x^*x \geq \sup_K x^* - \alpha\}$ be the closed slice of K given by α and x^*.

For $x^* \in S_{X^*}$ and $\alpha \in (0, 1)$, we denote

$$C(\alpha, x^*) = \{x \in X; x^*(x) \geq \alpha\|x\|\}, \qquad C^\circ(\alpha, x^*) = \{x \in X; x^*(x) > \alpha\|x\|\}.$$

It is easy to see that $C(\alpha, x^*)$ is a nonempty closed convex cone, $C^\circ(\alpha, x^*)$ is a nonempty open convex cone, and $C^\circ(\alpha, x^*)$ coincides with $\operatorname{int} C(\alpha, x^*)$, the interior of $C(\alpha, x^*)$.

By $\mathcal{BCC}(X)$, we denote the metric space of all nonempty bounded closed convex subsets of X with the Hausdorff metric

$$h(A, B) = \max\left\{\sup_{a \in A} \operatorname{dist}(a, B),\ \sup_{b \in B} \operatorname{dist}(b, A)\right\}.$$

Let (Y, d) be a metric space. A multifunction $F : Y \to \mathcal{BCC}(X)$ is called:

(a) *Hausdorff upper semicontinuous* (H-u.s.c.) if $\forall y \in Y, \forall \varepsilon > 0, \exists \delta > 0$:

$$F(z) \subset F(y) + \varepsilon B_X \ \text{ whenever } z \in Y,\ d(z, y) < \delta;$$

(b) *Hausdorff lower semicontinuous* (H-l.s.c.) if $\forall y \in Y, \forall \varepsilon > 0, \exists \delta > 0$:

$$F(y) \subset F(z) + \varepsilon B_X \ \text{ whenever } z \in Y,\ d(z, y) < \delta;$$

(c) *Hausdorff continuous* (H-continuous) if F is both H-u.s.c. and H-l.s.c.;

(d) *(topologically) upper semicontinuous* (u.s.c.) if

$$F^{-1}(D) := \{y \in Y; F(y) \cap D \neq \emptyset\}$$

is closed whenever D is a closed subset of X;

(e) *(topologically) lower semicontinuous* (l.s.c.) if $F^{-1}(U)$ is open whenever U is an open subset of X.

It is easy to see that F is Hausdorff continuous if and only if it is continuous as a mapping between the metric spaces (Y, d) and $(\mathcal{BCC}(X), h)$. Moreover, the following two implications hold:

$$\text{H-l.s.c.} \Rightarrow \text{l.s.c.} \quad \text{and} \quad \text{u.s.c.} \Rightarrow \text{H-u.s.c.}$$

Lemma 1.1 ([3, Lemma 1.1]). *For $\alpha \in (0, 1/2)$ and $x^*, y^* \in S_{X^*}$, consider the following assertions:*

(*i*) $\|y^* - x^*\| \leq \alpha$;
(*ii*) $\inf y^* (C(\alpha, x^*)) = 0$;
(*iii*) $\|y^* - x^*\| \leq 2\alpha$.

Then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold.

Lemma 1.2 ([3, Lemma 1.2]). *Let $x^* \in S_{X^*}$, $\alpha \in (0, 1)$ and $R > 0$. Then the mapping $\Psi : [0, R] \to \mathcal{BCC}(X)$, given by $\Psi(t) = \{x \in C(\alpha, x^*);\ t \leq x^*(x) \leq R\}$, is Lipschitz.*

Lemma 1.3 ([2, Lemma 3.5.1]). *Let $K, B \in \mathcal{BCC}(X)$ and $\widetilde{K} = \overline{\text{conv}}\,(K \cup B)$. If $y^* \in X^*$ and $x \in \widetilde{K}$ are such that $y^*(x) = \sup y^*(\widetilde{K}) > \sup y^*(B)$, then $x \in K$.*

Proposition 1.4 ([4, Proposition 2.3]). *Let Y be a metric space. Let $F : Y \to \mathcal{BCC}(X)$ and $G : Y \to \mathcal{BCC}(X)$ be Hausdorff continuous multifunctions such that $F(y) \cap G(y)$ has nonempty interior for each $y \in Y$. Then the multifunction $F \cap G : Y \to \mathcal{BCC}(X)$, given by $(F \cap G)(y) = F(y) \cap G(y)$, $y \in Y$, is Hausdorff continuous.*

Proposition 1.5 ([4, Remark 3.10]). *Let Y be a metric space and $F : Y \to \mathcal{BCC}(X)$ be an H-l.s.c. multifunction such that, for every $y \in Y$, $F(y)$ has nonempty interior. Then there exists a continuous function $f : Y \to X$ such that $f(y) \in \operatorname{int} F(y)$, for each $y \in Y$.*

Lemma 1.6. *The map $\Theta : [-1, 1) \times S_{X^*} \to \mathcal{BCC}(X)$, defined by $\Theta(\alpha, x^*) = \mathcal{S}(B_X, x^*, 1 - \alpha)$, is continuous.*

Proof. *Step 1: for every $x_0^* \in S_{X^*}$ the map $\Theta(\cdot, x_0^*)$ is continuous.* Let $v \in X$ be such that $B_X \subset v + C(1/2, x_0^*)$. Then $\mathcal{S}(B_X, x_0^*, 1 - \gamma) = B_X \cap [v + C(1/2, x_0^*)] \cap \{\gamma \leq x_0^* \leq 1\} = B_X \cap [v + \Psi(\gamma)]$ where $\Psi(\gamma) = \{x \in C(1/2, x_0^*);\ \gamma - x_0^* v \leq x_0^* x \leq 1 - x_0^* v\}$. Now,

$$x_0^*(v) = \inf x_0^* (v + C(1/2, x_0^*)) \leq \inf x_0^*(B_X) = -1$$

implies that $\gamma - x_0^* v \geq 0$. By Lemma 1.2, the map Ψ is H-continuous. To conclude the proof of the first step just apply Proposition 1.4.

Step 2: for every $\alpha_0 \in [-1, 1)$ the map $\Theta(\alpha_0, \cdot)$ is $2/(1 - \alpha_0)$-Lipschitz. Let $x^*, y^* \in S_{X^*}$. Suppose $x \in \Theta(\alpha_0, x^*) \setminus \Theta(\alpha_0, y^*)$, then we have $y^*(x) \in [\alpha_0 - \|x^* - y^*\|, \alpha_0)$. Choose $\eta \in (0, 1 - \alpha_0)$ and $\nu \in \Theta(1 - \eta, y^*)$. Put $t = \frac{\alpha_0 - y^*(\nu)}{y^*(x) - y^*(\nu)}$ and $z = tx + (1 - t)\nu$.

Let us observe that $t \in [0,1]$, that $z \in B_X \cap \{y^* = \alpha_0\} \subset \Theta(\alpha_0, y^*)$ and that

$$\|x - z\| = \|x - \nu\|(1-t) \leq 2\tfrac{\alpha_0 - y^*(x)}{y^*(\nu) - y^*(x)} \leq \tfrac{2}{1-\alpha_0-\eta}\|x^* - y^*\|.$$

Letting $\eta \to 0$, we get $\operatorname{dist}(x, \Theta(\alpha_0, y^*)) \leq 2\|x^* - y^*\|/(1-\alpha_0)$. By the arbitrariness of $x \in \Theta(\alpha_0, x^*) \setminus \Theta(\alpha_0, y^*)$, we have

$$h\left(\Theta(\alpha_0, x^*), \Theta(\alpha_0, y^*)\right) \leq \tfrac{2}{1-\alpha_0}\|x^* - y^*\|.$$

Step 3. Consider $(\alpha_0, x_0^*), (\alpha, x^*) \in [-1, 1) \times S_{X^*}$. Step 2 ensures that:

$$\begin{aligned} h\left(\Theta(\alpha_0, x_0^*), \Theta(\alpha, x^*)\right) &\leq h\left(\Theta(\alpha_0, x_0^*), \Theta(\alpha_0, x^*)\right) + h\left(\Theta(\alpha_0, x^*), \Theta(\alpha, x^*)\right) \\ &\leq 2\|x_0^* - x^*\|/(1-\alpha_0) + h\left(\Theta(\alpha_0, x^*), \Theta(\alpha, x^*)\right). \end{aligned}$$

Let $(\alpha_0, x_0^*) \to (\alpha, x^*)$ and use Step 1. □

Lemma 1.7. *Let $x^* \in S_{X^*}$ and $\rho > 0$. Then there exists $z \in X$ such that*

$$y^*(u - b) \geq 1 \tag{1}$$

whenever $\|u\| \leq \rho$, $\|b - z\| \leq 1$, $y^* \in S_{X^*}$ and $\|y^* - x^*\| \leq 1/2$.

Proof. Choose $z_0 \in S_X$ such that $x^*(z_0) \leq -3/4$. Then $y^*(z_0) \leq -1/4$ whenever $\|y^* - x^*\| \leq 1/2$. Put $z = (4\rho + 8)z_0$, so:

$$y^*(u - b) = -y^*(z) + y^*(u) - y^*(b - z) \geq 1/4(4\rho + 8) - \rho - 1 = 1$$

for all $\|u\| \leq \rho$, $\|b - z\| \leq 1$ and $\|y^* - x^*\| \leq 1/2$. □

2. Touching a convex set by a cone

Let K be a nonempty closed convex set in X, $x \in K$ and $x^* \in X^* \setminus \{0\}$. If $x^*(x) = \sup x^*(K)$ then x is a *support point* of K and x^* is the corresponding *support functional*. Obviously, all positive multiples of a support functional of K are support functionals of K at the same point. Thus, sometimes, it is natural to consider only norm-one support functionals.

Definition 2.1. We denote by $\operatorname{supp}(K)$, $\Sigma_1(K)$ and $\Pi_1(K)$, respectively, the set of all support points of K, the set of all norm-one support functionals of K and the set of all norm-one linear functionals bounded above over K i.e.

$$\begin{aligned} \operatorname{supp}(K) &= \{x \in K;\ x^*(x) = \sup x^*(K) \text{ for some } x^* \in S_{X^*}\}, \\ \Sigma_1(K) &= \{x^* \in S_{X^*};\ x^*(x) = \sup x^*(K) \text{ for some } x \in K\}, \\ \Pi_1(K) &= \{x^* \in S_{X^*};\ \sup x^*(K) < \infty\}. \end{aligned}$$

Let us recall that, for $\alpha \in (0,1)$ and $x^* \in S_{X^*}$, we have defined the cone $C(\alpha, x^*)$ as the set $\{x \in X; x^*(x) \geq \alpha\|x\|\}$, whose interior is $C^\circ(\alpha, x^*) = \{x \in X; x^*(x) > \alpha\|x\|\}$.

Definition 2.2. Given a set $K \subset X$, a functional $x^* \in S_{X^*}$ and $\alpha \in (0,1)$, we shall denote by $\Omega_K(\alpha, x^*)$ the set of all points of K that can be touched by the cone $C^\circ(\alpha, x^*)$ from outside K, i.e.

$$\Omega_K(\alpha, x^*) = \{x \in K; K \cap [x + C^\circ(\alpha, x^*)] = \emptyset\}.$$

Let us observe that $\Omega_K(\xi, x^*) \subset \Omega_K(\alpha, x^*)$ if $\xi < \alpha$, and that $\Omega_K(\alpha, x^*)$ is contained in the boundary of K.

In the present section, we prove the following two theorems.

Theorem 2.3. *Let $K \in \mathcal{BCC}(X)$, $0 \in \operatorname{int} K$ and $\alpha \in (0,1)$. For $i = 1, 2$, let $x_i^* \in S_{X^*}$ be a support functional of K and $x_i \in K \cap C^\circ(\alpha, x_i^*)$ be the corresponding support point. Suppose $x_1^* \neq -x_2^*$ and define*

$$\varphi_t = \varphi(t) = \frac{(t-1)x_1^* + (2-t)x_2^*}{\|(t-1)x_1^* + (2-t)x_2^*\|} \quad t \in [1,2];$$

then there exists a continuous function $\gamma : [1,2] \to \operatorname{supp}(K)$ such that $\gamma(i) = x_i$ $(i = 1, 2)$ and $\gamma(t) \in \Omega_K(\alpha, \varphi_t) \cap C(\alpha, \varphi_t)$ whenever $t \in [1,2]$.

Theorem 2.4. *Let K be a closed convex subset of X such that $0 \in K$. For $i = 1, 2$, let $x_i^* \in S_{X^*}$ be a support functional of K and $x_i \in K$ be the corresponding support point. Suppose $\|x_1^* - x_2^*\| \leq 1/8$, $\alpha \in (0,1)$ and $x_i^* x_i > 0$ for $i = 1, 2$. Define $\varphi_t = \varphi(t)$ $(t \in [1,2])$ as in Theorem 2.3. Then there exists a continuous map $\gamma : [1,2] \to \operatorname{supp}(K)$ such that $\gamma(t) \in \Omega_K(\alpha, \varphi_t)$ whenever $t \in [1,2]$ and $\gamma(i) = x_i$ $(i = 1, 2)$.*

For the proofs of Theorem 2.3 and Theorem 2.4 we need some preliminary results.

Lemma 2.5. *Let $K \in \mathcal{BCC}(X)$, $\alpha \in (0,1)$ and $\gamma : S_{X^*} \to \operatorname{int} K$ be a continuous map. Then the map $F : x^* \mapsto K \cap [\gamma(x^*) + C(\alpha, x^*)]$ is H-continuous on S_{X^*}.*

Proof. Choose $r > 0$ such that $K \subset rB_X$, then

$$F(x^*) = K \cap [\gamma(x^*) + 2r(C(\alpha, x^*) \cap B_X)].$$

Let Θ be defined as in Lemma 1.6. Fix $x_1^*, x_2^* \in S_{X^*}$, it is easy to see that

$$h\left(B_X \cap C(\alpha, x_1^*), B_X \cap C(\alpha, x_2^*)\right) \leq h\left(\Theta(\alpha, x_1^*), \Theta(\alpha, x_2^*)\right).$$

Then, by the continuity of $\Theta(\alpha, \cdot)$, by the continuity of γ and by Proposition 1.4, F is H-continuous (note that $\operatorname{int} F(x^*) \neq \emptyset$ since γ takes values in $\operatorname{int} K$). $\square$

Lemma 2.6. *Let F be defined as in Lemma 2.5 and $s : S_{X^*} \to \mathbb{R}$ be a continuous map such that $s(x^*) < \sup_{F(x^*)} x^*$ whenever $x^* \in S_{X^*}$. If we define*

$$G(x^*) = \{x \in F(x^*);\ x^*(x) \geq s(x^*)\} \quad \forall x^* \in S_{X^*},$$

then G is H-continuous and takes values with non empty interior.

Proof. Since K is bounded, there exists $r > 0$ such that $K \subset rB_X$. Hence, we can assume, without any loss of generality, that $r > s(x^*) \geq -r$, for each $x^* \in S_{X^*}$. Let Θ be defined as in Lemma 1.6 and observe that

$$G(x^*) = F(x^*) \cap r\Theta(s(x^*)/r, x^*).$$

Now, F is continuous by Lemma 2.5, the map $x^* \mapsto \Theta(s(x^*)/r, x^*)$ is continuous by Lemma 1.6 and $\operatorname{int} G(x^*) \neq \emptyset$ whenever $x^* \in S_{X^*}$ $\big(s(x^*) < \sup_{F(x^*)} x^*\big)$. Proposition 1.4 concludes the proof. $\square$

Lemma 2.7. *Let K be a nonempty closed convex set in X, $\alpha \in (0,1)$ and let $S \subset \Pi_1(K)$ be a compact subset of S_{X^*}. Then there exist $\varepsilon, r > 0$ such that $K \cap [x + C(\alpha, x^*)] \subset rB_X$ whenever $(x, x^*) \in (\varepsilon B_X) \times S$.*

Proof. Define

$$\begin{aligned} E : X \times \Pi_1(K) &\to 2^X \\ (x, x^*) &\mapsto K \cap [x + C(\alpha, x^*)], \end{aligned}$$

and, for $(x, x^*) \in X \times \Pi_1(K)$, define $s(x, x^*) = \sup x^* (E(x, x^*))$. We claim that s is locally upper-bounded in $X \times \Pi_1(K)$. Assume the contrary, then there exists a sequence $\{(z_n, x_n^*)\}_{n \in \mathbb{N}} \subset X \times \Pi_1(K)$ converging to $(z, x^*) \in X \times \Pi_1(K)$ and such that $s(z_n, x_n^*) \to \infty$, i.e. there exists $x_n \in [z_n + C(\alpha, x_n^*)] \cap K$ such that $x_n^* x_n \to \infty$. Since $x_n^* \in S_{X^*}$ we have $\|x_n\| \to \infty$ and then $\|x_n - z_n\| \to \infty$. Since

$$\begin{aligned} x^* x_n &= (x^* - x_n^*)(x_n - z_n) + x_n^*(x_n - z_n) + x^*(z_n) \\ &\geq (\alpha - \|x^* - x_n^*\|)\|x_n - z_n\| + x^* z_n, \end{aligned}$$

$x^* x_n \to \infty$ as $n \to \infty$. A contradiction since $x_n \in K$ $\forall n \in \mathbb{N}$ and $x^* \in \Pi_1(K)$.

By our claim and by the compactness of S, there exist $\varepsilon, C_1 > 0$ such that $s(x, x^*) \leq C_1$ whenever $(x, x^*) \in (\varepsilon B_X) \times S$. Let us observe that, for $(x, x^*) \in (\varepsilon B_X) \times S$,

$$\operatorname{diam}([x + C(\alpha, x^*)] \cap K) \leq \frac{2}{\alpha}(C_1 - x^* x) \leq \frac{2}{\alpha}(C_1 + \varepsilon).$$

Then there exists $r > 0$ such that $K \cap [x + C(\alpha, x^*)] \subset rB_X$ whenever $(x, x^*) \in (\varepsilon B_X) \times S$. $\square$

Lemma 2.8. *Let $K_1 \in \mathcal{BCC}(X)$, $x^* \in S_{X^*}$. Then there exists $K_2 \in \mathcal{BCC}(X)$ such that*

$$K_1 \subset K_2, \quad \Omega_{K_1}(\alpha, z^*) = \Omega_{K_2}(\alpha, z^*), \quad \operatorname{int} K_2 \neq \emptyset$$

whenever $\alpha \in (0, 1/8)$, $z^* \in S_{X^*}$ and $\|z^* - x^*\| \leq 1/4$.

Proof. Suppose $K_1 \subset \rho B_X$ for some $\rho > 0$. By Lemma 1.7 there exists $z \in X$ such that, for $z^* \in S_{X^*}$ and $\|z^* - x^*\| \leq 1/4$, we have

$$y^*(u - b) \geq 1 \ \text{ whenever } \|u\| \leq \rho, \ \|b - z\| \leq 1, \ y^* \in S_{X^*} \text{ and } \|y^* - z^*\| \leq 2\alpha.$$

Put $B = z + B_X$ and $K_2 = \overline{\operatorname{conv}}\,(K_1 \cup B)$. Obviously $K_1 \subset K_2$ and $\operatorname{int} K_2 \neq \emptyset$. Fix $z^* \in S_{X^*}$ such that $\|z^* - x^*\| \leq 1/4$, then

$$\sup y^*(K_2) = \sup y^*(K_1) > \sup y^*(B) \quad \forall y^* \in S_{X^*},\ \|y^* - z^*\| \leq 2\alpha. \tag{2}$$

Fix $\alpha \in (0, 1/8)$ and let $x \in \Omega_{K_1}(\alpha, z^*)$. Then $x \in K_2$ and, by the Hahn-Banach theorem, there exists $y^* \in S_{X^*}$ such that $\sup y^*(K_1) = y^*(x) = \inf y^*(x + C(\alpha, z^*))$. This implies that $\inf y^*\,(C(\alpha, z^*)) = 0$. By Lemma 1.1, we have $\|y^* - z^*\| \leq 2\alpha$. By (2), $\sup y^*(K_2) = \sup y^*(K_1) = \inf y^*\,(x + C(\alpha, z^*))$. Consequently, K_2 and $x + C^\circ(\alpha, z^*)$ are disjoint which means that $x \in \Omega_{K_2}(\alpha, z^*)$.

Conversely, let $x \in \Omega_{K_2}(\alpha, z^*)$. Then, obviously, K_1 and $x + C^\circ(\alpha, z^*)$ are disjoint. We have to prove that $x \in K_1$. By the Hahn-Banach theorem, there exists $y^* \in S_{X^*}$ such that $\sup y^*(K_2) = y^*(x) = \inf y^*\,(x + C(\alpha, z^*))$. As above, this gives $\|y^* - z^*\| \leq 2\alpha$. Now, (2) and Lemma 1.3 imply that $x \in K_1$. □

Proof of Theorem 2.3. Let $\gamma_0 : [1, 2] \to \operatorname{int} K$ be defined by $\gamma_0(t) = 0$ $\forall t \in [1, 2]$. Now we proceed inductively. Suppose we already have a continuous curve $\gamma_{n-1} : [1, 2] \to \operatorname{int} K$ such that $x_i \in \gamma_{n-1}(i) + C^\circ(\alpha, x_i^*)$ for $i = 1, 2$. Define, for $t \in [1, 2]$,

$$\begin{aligned} F_n(t) &= [\gamma_{n-1}(t) + C(\alpha, \varphi_t)] \cap K, \\ G_n(t) &= \{x \in F_n(t);\ \varphi_t(x) \geq \sup_{F_n(t)} \varphi_t - 1/n\}. \end{aligned}$$

By Lemma 2.5 F_n is H-continuous, by Lemma 2.6 and by the continuity of the map $t \mapsto \sup_{F_n(t)} \varphi_t - 1/n$, G_n is H-continuous too. Moreover, since $\sup_{F_n(t)} \varphi_t - 1/n < \sup_{F_n(t)} \varphi_t$, G_n takes values with nonempty interior.

For $i = 1, 2$, define $\widetilde{G}_n(i) = G_n(i) \cap [x_i - C(\alpha, x_i^*)]$. Define $\widetilde{G}_n(t) = G_n(t)$, for each $t \in (1, 2)$. Obviously, $\widetilde{G}_n$ is H-l.s.c.. Moreover, since $(\gamma_{n-1}(i), x_i) \cap \operatorname{int} \widetilde{G}_n(i) \neq \emptyset$ for $i = 1, 2$, $\widetilde{G}_n$ takes values with non empty interior. Then, by Proposition 1.5, there exists a continuous curve $\gamma_n : [1, 2] \to K$ such that $\gamma_n(t) \in \operatorname{int} \widetilde{G}_n(t)$ $\forall t \in [1, 2]$. In particular $x_i \in \gamma_n(i) + C^\circ(\alpha, x_i^*)$.

This construction yields a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of continuous curves $\gamma_n : [1, 2] \to \operatorname{int} K$. Fix $t \in [1, 2]$ and $\overline{n} \in \mathbb{N}$. Since $F_n(t) \subset G_{n-1}(t)$ we have that, whenever $n > \overline{n}$, diam $F_n(t) \leq 2/(\alpha\overline{n})$. So, $\{\gamma_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy on $[1, 2]$ and X is complete. Let $\gamma = \lim_n \gamma_n$. Let us observe that $x_i \in F_n(i)$ $\forall n \in \mathbb{N}$, and hence that $\gamma(i) = x_i$, for $i = 1, 2$.

Since $\gamma_n(t) \in F_n(t) \subset F_1(t) \subset C(\alpha, \varphi(t))$ $\forall n \in \mathbb{N}$, $\gamma(t) \in C(\alpha, \varphi_t)$ whenever $t \in [1, 2]$. Moreover

$$\operatorname{diam}\,([\gamma(t) + C(\alpha, \varphi_t)] \cap K) \leq \operatorname{diam} F_n(t) \leq \frac{2}{\alpha(n-1)} \quad \forall n \in \mathbb{N},$$

hence $[\gamma(t) + C(\alpha, \varphi_t)] \cap K = \{\gamma(t)\}$ and so $\gamma(t) \in \Omega_K(\alpha, \varphi_t)$ whenever $t \in [1, 2]$. □

Proof of Theorem 2.4. We can suppose $\alpha \in (0, 1/8)$ and $x_i^* x_i > \alpha \|x_i\|$, $i = 1, 2$ ($\Omega_K(\alpha, x^*) \subset \Omega_K(\beta, x^*)$ if $\alpha < \beta$). By Lemma 2.7 and by the compactness of $\varphi([1, 2])$, there exist $r_0, \varepsilon > 0$ such that $[x + C(\alpha, x^*)] \cap K \subset r_0 B_X$, for each $(x, x^*) \in \varepsilon B_X \times \varphi([1, 2])$.

Let $K_1 = K \cap r_0 B_X$. By Lemma 2.8 there exists $K_2 \in \mathcal{BCC}(X)$ such that

$$K_1 \subset K_2, \quad \Omega_{K_1}(\alpha, x^*) = \Omega_{K_2}(\alpha, x^*), \quad \operatorname{int} K_2 \neq \emptyset$$

whenever $x^* \in \varphi([1, 2])$ and $\alpha \in (0, 1/8)$ (observe that $\|x_1^* - \varphi_t\| \leq 1/4$ for $t \in [1, 2]$). Let $x \in \varepsilon B_X \cap \operatorname{int} K_2$ be such that $x_i \in x + C^\circ(\alpha, x_i^*)$ for $i = 1, 2$. So, $x \in \operatorname{int} K_2$, $K_2 \in \mathcal{BCC}(X)$ and $x_i \in x + C^\circ(\alpha, x_i^*)$ $i = 1, 2$. Moreover, since $\Omega_{K_1}(\alpha, x^*) = \Omega_{K_2}(\alpha, x^*)$ $\forall x^* \in \varphi([1, 2])$ and $\forall \alpha \in (0, 1/8)$, $x_i^* x_i = \sup x_i^*(K_2)$ for $i = 1, 2$.

Using Theorem 2.3 (by a translation we can assume $x = 0$), we get a continuous curve $\gamma : [1, 2] \to K_2$ such that $\gamma(i) = x_i$, for $i = 1, 2$, and such that, for each $t \in [1, 2]$,

$$\gamma(t) \in \Omega_{K_2}(\alpha, \varphi_t) \cap [x + C(\alpha, \varphi_t)] = \Omega_{K_1}(\alpha, \varphi_t) \cap [x + C(\alpha, \varphi_t)].$$

Fix $t \in [1, 2]$. To conclude the proof we just have to prove that

$$\Omega_{K_1}(\alpha, \varphi_t) \cap [x + C(\alpha, \varphi_t)] \subset \Omega_K(\alpha, \varphi_t).$$

Suppose this is not the case and let $z \in \Omega_{K_1}(\alpha, \varphi_t) \cap [x + C(\alpha, \varphi_t)]$ be such that there exists $y \in [z + C^\circ(\alpha, \varphi_t)] \cap K$. Since

$$[z + C^\circ(\alpha, \varphi_t)] \cap K \subset [x + C^\circ(\alpha, \varphi_t)] \cap K \subset K_1,$$

then $y \in [z + C^\circ(\alpha, \varphi_t)] \cap K_1$. But $z \in \Omega_{K_1}(\alpha, \varphi_t)$, a contradiction. □

3. Main results

The following lemma is an easy consequence of Theorem 2.4

Lemma 3.1. *Let K be a nonempty closed convex subset of X and let $\alpha \in (0, 1)$. Suppose x_1^*, x_2^* are support functionals to K at $x_1, x_2 \in K$, respectively. If $\|x_1^* - x_2^*\| \leq \beta \leq 1/8$ then there exists a continuous curve $\gamma : [1, 2] \to \operatorname{supp}(K)$ such that $\gamma(i) = x_i$ ($i = 1, 2$), and, for every $t \in [1, 2]$, K is supported at $\gamma(t)$ by a functional $y_t^* \in \Sigma_1(K) \cap [x_1^* + 2(\beta + \alpha) B_{X^*}]$.*

Proof. By a translation we can suppose that $0 = (x_1 + x_2)/2$. If $x_1^*(x_1) = 0$ then $x_1^*(\lambda x_2 + (1 - \lambda) x_1) = 0 = \sup x_1^*(K)$ for each $\lambda \in [0, 1]$. In this case we can put $\gamma(1 + t) = \lambda x_2 + (1 - \lambda) x_1$, $t \in [0, 1]$. Proceed analogously if $x_2^*(x_2) = 0$.

Suppose that $x_i^* x_i > 0$, for $i = 1, 2$. Take φ and γ as in Theorem 2.4 and fix $t \in [1, 2]$. Since $\gamma(t) \in \Omega_K(\alpha, \varphi_t)$, K and $\gamma(t) + C^\circ(\alpha, \varphi_t)$ can be separated by some $y_t^* \in S_{X^*}$. Hence $\gamma(t) \in \operatorname{supp}(K)$ and the proof follows easily by Lemma 1.1 and the triangle inequality. □

Remark. Let us recall that, if K is a nonempty closed convex set in X, we have defined $\Pi_1(K) = \{f \in S_{X^*};\ \sup_K f < \infty\}$. It is easy to see that K does not contain any hyperplane iff $\Pi_1(K)$ is infinite and pathwise connected iff $\Pi_1(K)$ contains at least three elements.

Theorem 3.2. *For each $K \subset X$ closed convex not containing any hyperplane, the set* $\mathrm{supp}(K)$ *of its support points is pathwise connected.*

Proof. Let x and y be two points of $\mathrm{supp}(K)$. Let $x^*, y^* \in \Sigma_1(K)$ be the corresponding support functionals. By the Bishop-Phelps theorem, there exist finitely many functionals $x_0^*, x_1^*, \ldots, x_n^* \in \Sigma_1(K)$ such that $x_0^* = x^*$, $x_n^* = y^*$ and $\|x_{i-1}^* - x_i^*\| \leq 1/8$ whenever $1 \leq i \leq n$. Let $x_0, x_1, \ldots, x_n \in \mathrm{supp}(K)$ be corresponding support points such that $x_0 = x$, $x_n = y$. Now, put $\beta = 1/8$ and fix any $\alpha \in (0,1)$. Using Lemma 3.1, x_{i-1} and x_i can be connected by a continuous curve in $\mathrm{supp}(K)$ $(1 \leq i \leq n)$. □

Theorem 3.3. *For each $K \subset X$ closed convex not containing any hyperplane, the set $\Sigma_1(K)$ of its norm-one support functionals is uncountably dense in $\Pi_1(K)$, in the sense that each nonempty relatively open subset of $\Pi_1(K)$ contains uncountably many elements of $\Sigma_1(K)$.*

Proof. Suppose this is not the case, then there exist $\varepsilon \in (0, 1/8)$ and $x_1^* \in \Sigma_1(K)$ such that

$$B(x_1^*, 5\varepsilon) \cap \Sigma_1(K) = \{x_n^*\}_{n=1}^{\infty}, \tag{3}$$

where x_n^*'s are pairwise distinct and $\|x_1^* - x_2^*\| \leq \varepsilon$. Let $x_1, x_2 \in \mathrm{supp}(K)$ be support points corresponding to the support functionals x_1^*, x_2^*, respectively. By Lemma 3.1, with $\alpha = \beta = \varepsilon$, there exists a continuous map $\gamma : [1,2] \to \mathrm{supp}(K)$ such that $\gamma(i) = x_i$ $(i = 1,2)$ and K is supported at each $\gamma(t)$ by a support functional from $B(x_1^*, 4\varepsilon)$.

We claim that the nonempty, closed, convex sets $K_n = \{u \in K;\ x_n^*(u) = \sup x_n^*(K)\}$ $(n = 1, 2, \ldots)$ are pairwise disjoint. Indeed, if $u \in K_n \cap K_m$ with $n \neq m$, then each functional from the segment $[x_n^*, x_m^*]$ support K at u. Observe that $0 \notin [x_n^*, x_m^*]$ since $\|x_n^* - x_m^*\| \leq 1$. Hence, by normalizing the elements of $[x_n^*, x_m^*]$, we get a continuum in $\Sigma_1(K)$, containing x_n^* and x_m^*; but this contradicts (3). This proves our claim.

Now, $[1,2] = \bigcup_{n=1}^{\infty} \gamma^{-1}(K_n)$ is the disjoint union of countably many closed sets such that $\gamma^{-1}(K_1)$ and $\gamma^{-1}(K_2)$ are nonempty. But this is impossible by a theorem of Sierpinski (cf. [6, §47, III, Theorem 6]). □

Corollary 3.4. *For each $K \subset X$ closed convex not containing any hyperplane, the set $\Sigma_1(K)$ is uncountable.*

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[1]See also http://www.math.washington.edu/~phelps/supp.pdf