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Common Fixed Point Theorems for Monotone Generalized Contractions Satisfying a Contractive Condition of Rational Type in Ordered Metric Spaces*

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The purpose of this paper is to present a fixed point theorem for monotone generalized contractions satisfying a contractive condition of rational type in the context of partially ordered metric spaces. Our results extend other recent results.

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1. Introduction and preliminaries

In [10] D. S. Jaggi proved the following fixed point theorem.

Theorem 1.1. *Let T be a continuous selfmap defined on a complete metric space (X,d). Suppose that T satisfies the following contractive condition*

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)\cdot d(y,Ty)}{d(x,y)} + \beta d(x,y),$$

for all $x,y \in X$, $x \neq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. Then T has a

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unique fixed point in X.

Recently, in [4] the authors obtain a version of Theorem 1.1 in the context of partially ordered metric spaces.

The main results of [4] can be summarized in the following theorem.

Theorem 1.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y),$$

for $x, y \in X$, with $x \geq y$, $x \neq y$ and $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$.

Suppose either

(i) T is continuous

or

(ii) if (x_n) is a nondecreasing sequence in X such that $x_n \to x$ then $x = \sup\{x_n\}$.

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$ then T has a fixed point.

Existence of fixed point in partially ordered sets has been considered recently in [4]–[22]. Tarski's theorem is used in [18] to show the existence of solutions for fuzzy equations and in [15] to prove existence theorems for fuzzy differential equations.

In [21], [8], [9], [5], [16], [17] some applications to matrix equations and to ordinary differential equations are presented. In [3], [12], [2] the authors prove some fixed point theorems for a mixed monotone mapping in a metric space endowed with a partial order and they apply their results to problems of existence and uniqueness of solutions for some boundary value problems.

In the fixed point theorems in partially ordered metric spaces the usual contractive condition is weakened but at the expense that the operator is monotone. The main idea appearing in [16], [21] combines the ideas in the classical contraction principle with those in the monotone iterative technique [2].

The main aim of this paper is to study some common fixed point theorems in the context of ordered metric spaces. Particularly, our results generalize Theorem 1.2.

In our study we use the altering distance functions which were introduced by Khan et al. in [11].

We recall the definition of this class of functions.

Definition 1.3. An altering distance function is a function $\Psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying:

(a) Ψ is continuous and non-decreasing

(b) $\Psi(t) = 0$ if and only if $t = 0$.

Examples of such functions are: $\Psi(t) = \lambda \cdot t$, with $\lambda \in (0, \infty)$; $\Psi(t) = \operatorname{arctg} t$; $\Psi(t) = \frac{t}{1+t}$.

2. Fixed point theorems

Firstly, we present the following definition.

Definition 2.1. Suppose that $(X, \leq)$ is a partially ordered set and $f, g : X \longrightarrow X$. We say that f is a g-nondecreasing if for any $x, y \in X$

$$g(x) \leq g(y) \Rightarrow f(x) \leq f(y).$$

Remark 2.2. Notice that if $g = Id_X$, where Id_X denotes the identity mapping on X, then Definition 2.1 coincides with the classical definition of nondecreasing function.

Remark 2.3. There exists g-nondecreasing functions which are not nondecreasing. For example, we consider $(\mathbb{R}, \leq)$ with the usual order in $\mathbb{R}$ and the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^4 + 3$ and $g(x) = x^4$. It is easily seen that f is a g-nondecreasing mapping and it is not nondecreasing.

Definition 2.4. Let $f, g : X \longrightarrow X$ be and $x_0 \in X$. We say that x_0 is a coincidence point of f and g if $f(x_0) = g(x_0)$.

Now, we present the main result of the paper.

Theorem 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f, g : X \longrightarrow X$ be two mappings satisfying:*

(a) *$f(X) \subset g(X)$, f is g-nondecreasing and $g(X)$ is sclosed.*
(b) *If $(x_n) \subset X$ is a nondecreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$ and $g(g(x)) \leq g(x)$.*
(c) *$\Psi(d(f(x), f(y))) \leq \Psi(M(x, y)) - \phi(M(x, y))$, for $g(x) \geq g(y)$ and $g(x) \neq g(y)$, where Ψ and ϕ are altering distance functions and*

$$M(x, y) = \max \left\{ \frac{d(g(x), f(x)) \cdot d(g(y), f(y))}{d(g(x), g(y))}, d(g(x), g(y)) \right\}.$$

(d) *There exists $x_0 \in X$ with $g(x_0) \leq f(x_0)$.*

Then f and g have a coincidence point. Further, if f and g conmute at their coincidence points then f and g have a common fixed point.

Proof. By (d), we take $x_0 \in X$ with $g(x_0) \leq f(x_0)$.

Since $f(X) \subset g(X)$ we can find $x_1 \in X$ such that $g(x_1) = f(x_0)$.

Again, as $f(X) \subset g(X)$ we can find $x_2 \in X$ so that $g(x_2) = f(x_1)$.

Continuing this process, we construct a sequence (x_n) in X such that

$$g(x_{n+1}) = f(x_n) \quad \text{for } n \geq 0. \tag{1}$$

For a better readability, we divide the proof in several steps.

Step 1: $(f(x_n))$ is a nondecreasing sequence. We will use the mathematical induction principle.

Since $g(x_0) \leq f(x_0)$ and $f(x_0) = g(x_1)$ we have $g(x_0) \leq g(x_1)$.

As f is g-nondecreasing, $f(x_0) \leq f(x_1)$. Thus, our claim is satisfied for $n = 0$.

Now, suppose that $f(x_{n-1}) \leq f(x_n)$.

By (1), $g(x_n) \leq g(x_{n+1})$, and using the g-nondecreasing character of f, we obtain $f(x_n) \leq f(x_{n+1})$.

This proves our claim.

Step 2: $\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$. In fact, if $d(f(x_n), f(x_{n+1})) = 0$, for some $n \geq 0$, taking into account (1), we obtain

$$g(x_{n+1}) = f(x_n) = f(x_{n+1})$$

and x_{n+1} is a coincidence point of f and g and this proves the first part of our theorem.

Suppose that $d(f(x_n), f(x_{n+1})) > 0$ for all $n \geq 0$.

By Step 1, $f(x_{n-1}) = g(x_n) < f(x_n) = g(x_{n+1})$ and using (c) and (1), we obtain

$$\begin{aligned}
&\Psi(d(f(x_n), f(x_{n+1}))\\
&\leq \Psi(M(x_n, x_{n+1})) - \phi(M(x_n, x_{n+1}))\\
&= \Psi\left(\max\left\{\frac{d(g(x_n), f(x_n)) \cdot d(g(x_{n+1}), f(x_{n+1}))}{d(g(x_n), g(x_{n+1}))}, d(g(x_n), g(x_{n+1}))\right\}\right)\\
&- \phi\left(\max\left\{\frac{d(g(x_n), f(x_n)) \cdot d(g(x_{n+1}), f(x_{n+1}))}{d(g(x_n), g(x_{n+1}))}, d(g(x_n), g(x_{n+1}))\right\}\right)\\
&= \Psi\left(\max\left\{\frac{d(f(x_{n-1}), f(x_n)) \cdot d(f(x_n), f(x_{n+1}))}{d(f(x_{n-1}), f(x_n))}, d(f(x_{n-1}), f(x_n))\right\}\right)\\
&- \phi\left(\max\left\{\frac{d(f(x_{n-1}), f(x_n)) \cdot d(f(x_n), f(x_{n+1}))}{d(f(x_{n-1}), f(x_n))}, d(f(x_{n-1}), f(x_n))\right\}\right)\\
&= \Psi(\max\{d(f(x_n), f(x_{n+1})), d(f(x_{n-1}), f(x_n))\}) \qquad (2)\\
&- \phi(\max\{d(f(x_n), f(x_{n+1})), d(f(x_{n-1}), f(x_n))\})
\end{aligned}$$

If $\max\{d(f(x_n), f(x_{n+1})), d(f(x_{n-1}), f(x_n))\} = d(f(x_n), f(x_{n+1}))$, then, from (2) and since $d(f(x_n), f(x_{n+1})) > 0$ and ϕ is an altering distance function, we obtain

$$\begin{aligned}
\Psi(d(f(x_n), f(x_{n+1}))) &\leq \Psi(d(f(x_n), f(x_{n+1}))) - \phi(d(f(x_n), f(x_{n+1})))\\
&< \Psi(d(f(x_n), f(x_{n+1}))),
\end{aligned}$$

which is a contradiction.

Therefore, $\max\{d(f(x_n), f(x_{n+1})), d(f(x_{n-1}), f(x_n))\} = d(f(x_{n-1}), f(x_n))$.

Consequently, $d(f(x_n), f(x_{n+1})) \leq d(f(x_{n-1}), f(x_n))$ for any $n \in \mathbb{N}$.

This says us that $(d(f(x_n), f(x_{n+1})))$ is a decreasing sequence of nonnegative real numbers and, thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = r.$$

Now, we will prove that $r = 0$.

In fact, from (2) we have

$$\Psi(d(f(x_n), f(x_{n+1}))) \leq \Psi(d(f(x_{n-1}), f(x_n))) - \phi(d(f(x_{n-1}), f(x_n)))$$

and letting $n \to \infty$ and taking into account the nonnegativity and continuity of Ψ and ϕ we have

$$\Psi(r) \leq \Psi(r) - \phi(r) \leq \Psi(r),$$

which implies that $\phi(r) = 0$. Since ϕ is an altering distance function, $r = 0$.

This finishes the proof of Step 2.

Step 3: $(f(x_n))$ is a Cauchy sequence. In contrary case, there exists $\varepsilon > 0$ such that we can find subsequences $(f(x_{m(k)}))$ and $(f(x_{n(k)}))$ of $(f(x_n))$ with $n(k) > m(k) > k$ satisfying

$$d(f(x_{n(k)}), f(x_{m(k)})) \geq \varepsilon. \tag{3}$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3). Hence

$$d(f(x_{n(k)-1}), f(x_{m(k)})) < \varepsilon. \tag{4}$$

By (3) y (4) and the triangular inequality, we have

$$\begin{aligned}
\varepsilon &\leq d(f(x_{n(k)}), f(x_{m(k)})) \\
&\leq d(f(x_{n(k)}), f(x_{n(k)-1})) + d(f(x_{n(k)-1}), f(x_{m(k)})) \\
&< d(f(x_{n(k)}), f(x_{n(k)-1})) + \varepsilon
\end{aligned}$$

Taking $k \to \infty$ and using Step 2, we get

$$\lim_{k\to\infty} d(f(x_{n(k)}), f(x_{m(k)})) = \varepsilon. \tag{5}$$

Again, the triangular inequality gives us

$$\begin{aligned}
\varepsilon &\leq d(f(x_{n(k)}), f(x_{m(k)})) \\
&\leq d(f(x_{n(k)}), f(x_{n(k)-1})) + d(f(x_{n(k)-1}), f(x_{m(k)-1})) \\
&\qquad + d(f(x_{m(k)-1}), f(x_{m(k)})) \\
&\leq d(f(x_{n(k)}), f(x_{n(k)-1})) + d(f(x_{n(k)-1}), f(x_{m(k)}) \\
&\qquad + d(f(x_{m(k)}), f(x_{m(k)-1})) + d(f(x_{m(k)-1}), f(x_{m(k)})) \\
&< d(f(x_{n(k)}), f(x_{n(k)-1})) + 2d(f(x_{m(k)}), f(x_{m(k)-1})) + \varepsilon
\end{aligned}$$

Letting $k \to \infty$ and using Step 2, we obtain

$$\lim_{k\to\infty} d(f(x_{n(k)-1}), f(x_{m(k)-1})) = \varepsilon. \tag{6}$$

Since $n(k) > m(k)$ and $(f(x_n))$ is a nondecreasing sequence

$$f(x_{n(k)-1}) = g(x_{n(k)}) \geq f(x_{m(k)-1}) = g(x_{m(k)}).$$

If $g(x_{n(k)}) = g(x_{m(k)})$ then the nondecreasing character of the sequence $(f(x_n))$ gives us

$$g(x_{n(k)}) = f(x_{n(k)-1}) \geq f(x_{m(k)}) = g(x_{m(k)+1}) \geq f(x_{m(k)-1}) = g(x_{m(k)}),$$

and, consequently,

$$f(x_{m(k)}) = f(x_{m(k)-1}) = g(x_{m(k)}).$$

Therefore, $x_{m(k)}$ is a coincidence point of f and g and this is the desired result.

Now, suppose that $g(x_{n(k)}) > g(x_{m(k)})$. By (c) we get

$$\begin{aligned}
&\Psi(d(f(x_{n(k)}), f(x_{m(k)}))) \leq \Psi(M(x_{n(k)}, x_{m(k)})) - \phi(M(x_{n(k)}, x_{m(k)}))\\
&= \Psi\left(\max\left\{\frac{d(g(x_{n(k)}), f(x_{n(k)}))\cdot d(g(x_{m(k)}), f(x_{m(k)}))}{d(g(x_{n(k)}), g(x_{m(k)}))}, d(g(x_{n(k)}), g(x_{m(k)}))\right\}\right)\\
&\quad - \phi\left(\max\left\{\frac{d(g(x_{n(k)}), f(x_{n(k)}))\cdot d(g(x_{m(k)}), f(x_{m(k)}))}{d(g(x_{n(k)}), g(x_{m(k)}))}, d(g(x_{n(k)}), g(x_{m(k)}))\right\}\right)\\
&= \Psi\left(\max\left\{\frac{d(f(x_{n(k)-1}), f(x_{n(k)}))\cdot d(f(x_{m(k)-1}), f(x_{m(k)}))}{d(f(x_{n(k)-1}), f(x_{m(k)-1}))},\right.\right.\\
&\qquad\qquad \left.\left. d(f(x_{n(k)-1}), f(x_{m(k)-1}))\right\}\right)\\
&\quad - \phi\left(\max\left\{\frac{d(f(x_{n(k)-1}), f(x_{n(k)}))\cdot d(f(x_{m(k)-1}), f(x_{m(k)}))}{d(f(x_{n(k)-1}), f(x_{m(k)-1}))},\right.\right.\\
&\qquad\qquad \left.\left. d(f(x_{n(k)-1}), f(x_{m(k)-1}))\right\}\right)
\end{aligned}$$

Letting $k \to \infty$ and using Step 2, (5) and (6) and the fact that Ψ and ϕ are altering distance functions, we have

$$\Psi(\varepsilon) \leq \Psi(\max\{0, \varepsilon\}) - \phi(\max\{0, \varepsilon\}) = \Psi(\varepsilon) - \phi(\varepsilon) < \Psi(\varepsilon),$$

which is a contradiction.

Therefore, $(f(x_n))$ is a Cauchy sequence.

This proves the Step 3.

Step 4: f and g have a coincidence point. In fact, since $(f(x_n))$ is a Cauchy sequence (thus, is a convergent sequence because (X, d) is a complete metric space), $f(x_n) = g(x_{n+1})$ for every $n \in \mathbb{N}$ and $g(X)$ is closed, we find $z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = g(z). \tag{7}$$

Now, we will prove that z is a coincidence point of f and g.

In fact, since $(f(x_n)) = (g(x_{n+1}))$ is a nondecreasing sequence (Step 1), taking into account (7) and assumption (b)

$$g(z) = \sup\{f(x_n) : n \in \mathbb{N}\} = \sup\{g(x_n) : n \in \mathbb{N}\}. \tag{8}$$

We distinguish two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $g(x_{n_0}) = g(z)$ and $(g(x_n))$ is nondecreasing. In this case, as $g(z) = \sup\{g(x_n) : n \in \mathbb{N}\}$, we have for $n > n_0$ that

$$g(x_n) \leq g(z) = g(x_{n_0}) \leq g(x_n),$$

or, equivalently, $g(x_n) = g(x_{n_0}) = g(z)$ for any $n > n_0$.

Particularly,

$$g(x_{n_0}) = g(x_{n_0+1}) = f(x_{n_0}),$$

and, thus, x_{n_0} is a coincidence point of f and g and this is the desired result.

Case 2: Suppose that $g(x_n) < g(z)$ for all $n \in \mathbb{N}$. In this case, since f is g-nondecreasing

$$f(x_n) \leq f(z) \quad \text{for all } n \in \mathbb{N}$$

and, taking into account (8)

$$g(z) \leq f(z).$$

Suppose that $g(z) < f(z)$.

Using a similar argument that in the proof of the Steps 1, 2 and 3, we find a sequence (y_n) satisfying:

(a) $(f(y_n))$ is nondecreasing.
Moreover, $f(z) \leq f(y_1)$.

(b) $f(y_n) = g(y_{n+1})$ for all $n \in \mathbb{N}$.

(c) There exists $z_1 \in X$ such that

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = g(z_1)$$

(d) $g(z_1) = \sup\{f(y_n) : n \in \mathbb{N}\} = \sup\{g(y_n) : n \in \mathbb{N}\}$.

Since $g(z) < f(z)$, taking into account (8), we get

$$g(x_n) \leq g(z) < f(z) \leq f(y_{n-1}) = g(y_n) \leq g(z_1), \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

Using assumption (c) with $x = y_n$ and $y = x_n$, we have

$$\begin{aligned}
&\Psi(d(f(y_n), f(x_n))) \\
&\leq \Psi(M(y_n, x_n)) - \phi(M(y_n, x_n)) \\
&= \Psi\left(\max\left\{\frac{d(g(y_n), f(y_n)) \cdot d(g(x_n), f(x_n))}{d(g(y_n), g(x_n))}, d(g(y_n), g(x_n))\right\}\right) \\
&\qquad - \phi\left(\max\left\{\frac{d(g(y_n), f(y_n)) \cdot d(g(x_n), f(x_n))}{d(g(y_n), g(x_n))}, d(g(y_n), g(x_n))\right\}\right)
\end{aligned}$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\begin{aligned}\Psi(d(g(z_1), g(z))) &\leq \Psi(d(g(z_1), g(z))) - \phi(d(g(z_1), g(z))) \\ &\leq \Psi(d(g(z_1), g(z)))\end{aligned}$$

and, using the fact that ϕ is an altering distance function,

$$\phi(d(g(z_1), g(z))) = 0$$

or, equivalently, $g(z_1) = g(z)$.

Finally, from (9), and, as $g(z_1) = g(z)$, we obtain $g(z) = f(z)$ which contradicts our assumption $g(z) < f(z)$.

Therefore, $f(z) = g(z)$.

This proves the Step 4.

In the sequel, we shall prove that f and g have a common fixed point assuming that f and g conmute at their coincidence points.

In fact, we take a coincidence point z of f and g, whose existence was proved in the first part of the theorem.

Then, $w = f(z) = g(z)$.

Since f and g conmute in z we have

$$f(w) = f(g(z)) = g(f(z)) = g(w).$$

Since $g(z) \leq g(g(z)) = g(w)$ (see, assumption (b) and Step 4) we can distinguish two cases.

Case 1: $g(z) = g(w)$. In this case, $w = f(z) = g(z) = g(w) = f(w)$ and w is the common fixed point of f and g.

This proves the desired result.

Case 2: $g(z) < g(w)$. In this case, by assumption (c) we get

$$\begin{aligned}&\Psi(d(f(w), f(z))) \\ &\leq \Psi(M(w,z)) - \phi(M(w,z)) \\ &= \Psi\left(\max\left\{\frac{d(g(w), f(w)) \cdot d(g(z), f(z))}{d(g(z), g(w))}, d(g(z), g(w))\right\}\right) \\ &\qquad - \phi\left(\max\left\{\frac{d(g(w), f(w)) \cdot d(g(z), f(z))}{d(g(z), g(w))}, d(g(z), g(w))\right\}\right) \\ &= \Psi(\max\{0, d(g(z), g(w))\}) - \phi(\max\{0, d(g(z), g(w))\}) \\ &< \Psi(d(g(z), g(w))) = \Psi(d(f(z), f(w))),\end{aligned}$$

which is a contradiction.

Therefore, $g(z) = g(w)$.

Consequently, Case 1 gives us the desired result. □

In the sequel, we present an example where it can be appreciated that assumptions in Theorem 2.5 do not guarantee uniqueness of the coincidence point.

Let $X = \{(1,0),(0,1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x,y) \le (z,t) \iff x \le z \text{ and } y \le t.$$

$(X, \le)$ is a partially ordered set whose different elements are not comparable. Besides, (X, d_2) is a complete metric space, where d_2 is the euclidean distance.

Put $f = g = Id_X$.

It is easily proved that assumption (a), (b) and (c) of Theorem 2.5 are satisfied. Moreover, $f(1,0) = g(1,0) = (1,0)$.

Notice that f and g have two coincidence points ($(1,0)$ and $(0,1)$).

The following result gives us a sufficient condition for the uniqueness of the coincidence point.

Theorem 2.6. *If we add the hypotheses that $g(X)$ is a totally ordered set and g is a injective mapping to the assumptions of Theorem* 2.5*, then we obtain uniqueness of the coincidence point of f and g. Moreover, if f and g commute at the coincidence points then f and g have a unique common fixed point.*

Proof. Suppose that f and g have two coincidence points y and z. Then

$$f(y) = g(y) = u$$
$$f(z) = g(z) = v.$$

As $g(X)$ is a totally ordered set and $g(y) = u$ and $g(z) = v$ are elements of $g(X)$, suppose that $g(y) \le g(z)$.

If $u = g(y) < g(z) = v$, by assumption (c) of Theorem 2.5, we have

$$\begin{aligned}
\Psi(d(u,v)) &= \Psi(d(f(y), f(z))) \le \Psi(M(y,z)) - \phi(M(y,z)) \\
&\le \Psi\left(\max\left\{\frac{d(g(y), f(y)) \cdot d(g(z), f(z))}{d(g(y), g(z))}, d(g(y), g(z))\right\}\right) \\
&\quad - \phi\left(\max\left\{\frac{d(g(y), f(y)) \cdot d(g(z), f(z))}{d(g(y), g(z))}, d(g(y), g(z))\right\}\right) \\
&= \Psi(\max\{0, d(u,v)\}) - \phi(\max\{0, d(u,v)\}) \\
&= \Psi(d(u,v)) - \phi(d(u,v)) < \Psi(d(u,v)),
\end{aligned}$$

which is a contradiction.

Therefore, $u = g(y) = g(z) = v$.

Finally, the injectivity of g gives us $y = z$.

This proves the first part of the theorem.

Now, suppose that z_1 and z_2 are two common fixed points of f and g.

Then, as

$$z_1 = f(z_1) = g(z_1)$$
$$z_2 = f(z_2) = g(z_2),$$

and, consequently, z_1 and z_2 are two coincidence points of f and g.

The uniqueness of the coincidence point above proved implies $z_1 = z_2$.

This finishes the proof. □

In the sequel, we present some consequences of Theorem 2.5.

Corollary 2.7. *Let $(X, \leq)$ be a partially orderd set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \longrightarrow X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose that*

$$\Psi(d(f(x), f(y)) \leq \Psi(N(x, y)) - \phi(N(x, y)), \quad \textit{for } x \geq y \textit{ and } x \neq y,$$

where Ψ and ϕ are altering distance functions and

$$N(x, y) = \max\left\{\frac{d(x, f(x)) \cdot d(y, f(y))}{d(x, y)}, d(x, y)\right\}.$$

Assume $(X, \leq)$ satisfies:

$$\textit{if } (x_n) \subset X \textit{ is a nondecreasing sequence such that } x_n \to x \textit{ then } x = \sup\{x_n\}. \tag{10}$$

Then f has at least a fixed point.

Proof. Applying Theorem 2.5 with $g = Id_X$ we obtain the result. □

Remark 2.8. Corollary 2.7 is valid if we replace condition (10) by the continuity of the operator f.

In fact, if $f(x_0) = x_0$ then we have the desired result.

Suppose $x_0 < f(x_0)$. Since f is a nondecreasing mapping we obtain that the sequence $x_{n+1} = f(x_n)$ is a nondecreasing mapping.

Using a similar argument that in the proof of Theorem 2.5, we get that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and (x_n) is a Cauchy sequence.

Thus, since (X, d) is a complete metric space, there exists $z \in X$ such that $x_n \to z$.

Since f is continuous,

$$x_{n+1} = f(x_n) \to f(z)$$

and the uniqueness of the limit gives us $f(z) = z$ (we omit the details).

In the sequel, we will prove that the uniqueness of the fixed point in Corollary 2.7 can be obtained using the following condition.

$$\text{For } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{11}$$

Theorem 2.9. *Adding* (11) *to the assumptions of Corollary* 2.7 *we obtain uniqueness of the fixed point.*

Proof. Suppose that f has two fixed points z and y.

We distinguish two cases.

Case 1: Suppose that z and y are comparable and $z \neq y$. Using the contractive condition we obtain

$$\begin{aligned}\Psi(d(z,y)) &= \Psi(d(f(z),f(y)) \leq \Psi(N(z,y)) - \phi(N(z,y)) \\ &= \Psi\left(\max\left\{\frac{d(z,f(z))\cdot d(y,f(y))}{d(z,y)}, d(z,y)\right\}\right) \\ &\qquad - \phi\left(\max\left\{\frac{d(z,f(z))\cdot d(y,f(y))}{d(z,y)}, d(z,y)\right\}\right) \\ &= \Psi(d(z,y)) - \phi(d(z,y)) \leq \Psi(d(z,y)).\end{aligned}$$

From the last inequality, we obtain $\phi(d(z,y)) = 0$ and, using the fact that ϕ is an altering distance function, $d(z,y) = 0$ and, therefore, $z = y$ which contradicts our assumption $z \neq y$.

Case 2: Suppose that z and y are not comparable. By (11), there exists x comparable to y and z.

The nondecreasing character of f implies that $f^n(x)$ is comparable to $f^n(z) = z$ and $f^n(y) = y$ for $n = 0, 1, 2, \ldots$

If there exists $n_0 > 1$ such that $f^{n_0}(x) = y$ then, as y is a fixed point, the sequence $\{f^n(x) : n \geq n_0\}$ is constant and equal to y and, consequently, $\lim_{n\to\infty} f^n(x) = y$.

Suppose that $f^n(x) \neq y$, for $n > 1$.

Using the contractive condition, we obtain for $n \geq 2$

$$\begin{aligned}&\Psi(d(f^n(x),y)) = \Psi(d(f^n(x),f^n(y))) \\ &\leq \Psi(N(f^{n-1}(x),f^{n-1}(y))) - \phi(N(f^{n-1}(x),f^{n-1}(y))) \\ &= \Psi\left(\max\left\{\frac{d(f^{n-1}(x),f^n(x))\cdot d(f^{n-1}(y),f^n(y))}{d(f^{n-1}(x),f^{n-1}(y))}, d(f^{n-1}(x),f^{n-1}(y))\right\}\right) \\ &\qquad - \phi\left(\max\left\{\frac{d(f^{n-1}(x),f^n(x))\cdot d(f^{n-1}(y),f^n(y))}{d(f^{n-1}(x),f^{n-1}(y))}, d(f^{n-1}(x),f^{n-1}(y))\right\}\right) \\ &\leq \Psi(d(f^{n-1}(x),f^{n-1}(y))) - \phi(d(f^{n-1}(x),f^{n-1}(y))) \\ &= \Psi(d(f^{n-1}(x),y)) - \phi(d(f^{n-1}(x),y)) \leq \Psi(d(f^{n-1}(x),y))\end{aligned} \tag{12}$$

The monotonicity of Ψ gives us

$$d(f^n(x),y) \leq d(f^{n-1}(x),y).$$

Hence, $\{d(f^n(x),y)\}$ is a decreasing sequence of nonnegative real numbers, and thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(f^n(x),y) = r$.

Now, we will prove that $r = 0$.

Letting $n \to \infty$ in (12) and, taking into account that Ψ and ϕ are continuous, we have

$$\Psi(r) \leq \Psi(r) - \phi(r) \leq \Psi(r)$$

and, thus, $\phi(r) = 0$. Since ϕ is an altering distance function, we have $r = 0$.

Therefore, $\lim_{n\to\infty} d(f^n(x), y) = 0$.

Using a similar argument, we can prove that $\lim_{n\to\infty} d(f^n(x), z) = 0$.

Finally, the uniqueness of the limit gives us $y = z$.

This finishes the proof. □

Corollary 2.10. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \longrightarrow X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose that*

$$d(f(x), f(y)) \leq k \cdot N(x, y) \quad \text{for } x \geq y \text{ and } x \neq y,$$

where $k \in [0, 1)$.

Assume that either f is continuous or $(X, \leq)$ satisfies (10).

Then f has at least a fixed point.

Proof. To apply Corollary 2.7 using as altering distance functions $\Psi(t) = t$ and $\phi(t) = (1 - k) \cdot t$. □

Remark 2.11. Theorem 1.2 is a consequence of Corollary 2.10 since if for $x, y \in X$ with $x \geq y$ and $x \neq y$ we have

$$\begin{aligned} &d(f(x), f(y)) \\ &\leq \alpha \cdot \frac{d(x, f(x)) \cdot d(y, f(y))}{d(x, y)} + \beta \cdot d(x, y), \quad \text{with } \alpha, \beta \in [0, 1) \text{ and } \alpha + \beta < 1 \end{aligned}$$

then

$$\begin{aligned} &d(f(x), f(y)) \\ &\leq (\alpha + \beta) \max\left\{\frac{d(x, f(x)) \cdot d(y, f(y))}{d(x, y)}, d(x, y)\right\} = (\alpha + \beta) \cdot N(x, y). \end{aligned}$$

Other consequence of Theorem 2.5 is the following common fixed point theorem with a contractive condition of integral type.

Corollary 2.12. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f, g : X \longrightarrow X$ be two mappings satisfying:*

(*a*) *$f(X) \subset g(X)$, f is g-nondecreasing and $g(X)$ is closed.*

(b) *If* $(x_n) \subset X$ *is a nondecreasing sequence in* X *such that* $x_n \to x$ *then* $x = \sup\{x_n\}$ *and* $g(g(x)) \leq g(x)$.

(c) $\int_0^{d(f(x),f(y))} \varphi(t)dt \leq k \int_0^{M(x,y)} \varphi(t)dt$ *for* $g(x) \geq g(y)$ *and* $g(x) \neq g(y)$, *where* $k \in [0,1)$ *and* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a Lebesgue-integrable function satisfying that* $\int_0^{\varepsilon} \varphi(t)dt > 0$ *for* $\varepsilon > 0$.

(d) *There exists* $x_0 \in X$ *with* $g(x_0) \leq f(x_0)$.

Then f *and* g *have a coincidence point. Further, if* f *and* g *conmute at their coincidence points then* f *and* g *have a common fixed point.*

Proof. It is easily proved that the function $\Psi : [0,\infty) \longrightarrow [0,\infty)$ given by $\Psi(t) = \int_0^t \varphi(s)ds$ is an altering distance function.

Using Theorem 2.5 with the functions $\Psi(t) = \int_0^t \varphi(s)ds$ and $\phi(t) = (1-k)\Psi(t)$ we obtain the desired result. □

If in Corollary 2.12, $g = Id_X$ we obtain the following fixed point theorem with a contractive condition of integral type.

Corollary 2.13. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $f : X \longrightarrow X$ *be a nondecreasing operator such that there exists* $x_0 \in X$ *with* $x_0 \leq f(x_0)$. *Suppose that*

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq k \cdot \int_0^{N(x,y)} \varphi(t)dt \quad \text{for } x \leq y \text{ and } x \neq y,$$

where $k \in [0,1)$ *and* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a Lebesgue-integrable function satisfying that* $\int_0^{\varepsilon} \varphi(t)dt > 0$ *for* $\varepsilon > 0$.

Assume that either f *is continuous or* $(X, \leq)$ *satisfies* (10).

Then f *has at least a fixed point.*

Finally, we present an example which can be studied by Corollary 2.7 and it cannot treated using Theorem 1.2.

This example appears in [13].

Example 2.14. Consider $X = [0, \frac{1}{2}]$ with the usual metric $d(x,y) = |x-y|$ for $x, y \in X$. (X, d) is a complete metric space.

In X we consider the order defined by

$$x, y \in X,\ x \preceq y \iff x, y \in \{0\} \cup \left\{\frac{1}{n} : n = 2, 3, \dots\right\} \text{ and } x \leq y,$$

where $\leq$ is the usual order in $\mathbb{R}$.

Let $f : X \longrightarrow X$ be the operator defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{n+1} & \text{if } x = \dfrac{1}{n}, n = 2, 3, \dots \\ \dfrac{1}{\sqrt{7}}, & \text{otherwise.} \end{cases}$$

It is easily proved that f is non-decreasing respect to $\preceq$ and that $(X, \preceq)$ satisfies condition (10).

Moreover, $0 \preceq f(0)$.

In the sequel we prove that f does not satisfy the contractive condition appearing in Theorem 1.2.

In fact, we take $y = 0$ and $x = \frac{1}{n}, n = 2, 3, \dots$

Obviously, $y \preceq x$ and $y \neq x$ and, moreover, if

$$d\left(f\left(\frac{1}{n}\right), f(0)\right) \leq \alpha \frac{d\left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right) \cdot d(0, f(0))}{d\left(\frac{1}{n}, 0\right)} + \beta d\left(\frac{1}{n}, 0\right)$$

for $n = 2, 3, \dots$

then we obtain

$$\frac{1}{n+1} \leq \alpha \cdot 0 + \beta \cdot \frac{1}{n}, \quad \text{for } n = 2, 3, \dots$$

or, equivalently

$$\frac{n}{n+1} \leq \beta, \quad \text{for } n = 2, 3, \dots.$$

Therefore, $\beta \geq 1$ and this implies that the contractive condition appearing in Theorem 1.2 is not satisfied.

On the other hand, if in Corollary 2.7 we take $\Psi(t) = t$ and $\phi(t) = t^3$, which are, obviously, altering distance functions, then we will prove that the operator f satisfies the contractive condition appearing in Corollary 2.7.

In fact, if $x, y \in X$ and $y \preceq x$ and $x \neq y$ then we have the following cases:

Case 1: $x = \frac{1}{n}$ $(n \geq 2)$ and $y = 0$.

Case 2: $x = \frac{1}{n}$ and $y = \frac{1}{m}$ with $m > n \geq 2$.

Case 1: $x = \frac{1}{n}$ and $y = 0$. In this case, we have

$$\begin{aligned}
&N\left(\frac{1}{n},0\right) - N\left(\frac{1}{n},0\right)^3 \\
&= \max\left\{\frac{d\left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right)\cdot d(0,f(0))}{d\left(\frac{1}{n},0\right)}, d\left(\frac{1}{n},0\right)\right\} \\
&\qquad - \left(\max\left\{\frac{d\left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right)\cdot d(0,f(0))}{d\left(\frac{1}{n},0\right)}, d\left(\frac{1}{n},0\right)\right\}\right)^3 \\
&= \frac{1}{n} - \frac{1}{n^3} \geq \frac{1}{n} - \frac{1}{n(n+1)} = \frac{1}{n+1} = d\left(f\left(\frac{1}{n}\right), f(0)\right)
\end{aligned}$$

In this case, our claim is satisfied.

Case 2: $x = \frac{1}{n}$ and $y = \frac{1}{m}$ with $m > n \geq 2$. In this case, we have

$$\begin{aligned}
N\left(\frac{1}{n},\frac{1}{m}\right) &= \max\left\{\frac{d\left(\frac{1}{n}, f\left(\frac{1}{n}\right)\right)\cdot d\left(\frac{1}{m}, f\left(\frac{1}{m}\right)\right)}{d\left(\frac{1}{n},\frac{1}{m}\right)}, d\left(\frac{1}{n},\frac{1}{m}\right)\right\} \\
&= \max\left\{\frac{\left|\frac{1}{n}-\frac{1}{n+1}\right|\cdot\left|\frac{1}{m}-\frac{1}{m+1}\right|}{\left|\frac{1}{n}-\frac{1}{m}\right|}, \left|\frac{1}{n}-\frac{1}{m}\right|\right\} \\
&= \max\left\{\frac{\frac{1}{n(n+1)}\cdot\frac{1}{m(m+1)}}{\frac{m-n}{m\cdot n}}, \frac{m-n}{m\cdot n}\right\} \\
&= \max\left\{\frac{1}{(n+1)(m+1)(m-n)}, \frac{m-n}{m\cdot n}\right\} \\
&= \frac{m-n}{m\cdot n} = \frac{1}{n} - \frac{1}{m}
\end{aligned}$$

We wish to prove that

$$d\left(f\left(\frac{1}{n}\right), f\left(\frac{1}{m}\right)\right) \leq N\left(\frac{1}{n},\frac{1}{m}\right) - \left(N\left(\frac{1}{n},\frac{1}{m}\right)\right)^3 \tag{13}$$

or, equivalently,

$$\left(\frac{1}{n+1} - \frac{1}{m+1}\right) \leq \left(\frac{1}{n} - \frac{1}{m}\right) - \left(\frac{1}{n} - \frac{1}{m}\right)^3$$

or,

$$\frac{m-n}{(n+1)(m+1)} \leq \frac{m-n}{m\cdot n} - \left(\frac{m-n}{m\cdot n}\right)^3$$

or,

$$\frac{(m-n)^2}{(m\cdot n)^3} \leq \frac{1}{m\cdot n} - \frac{1}{(n+1)(m+1)} = \frac{m+n+1}{m\cdot n\cdot(n+1)\cdot(m+1)}$$

or,

$$\left(\frac{1}{n}-\frac{1}{m}\right)^2=\frac{(m-n)^2}{(m\cdot n)^2}\leq\frac{m+n+1}{(n+1)(m+1)}. \tag{14}$$

Since

$$\left(\frac{1}{n}-\frac{1}{m}\right)^2<\frac{1}{n^2}<\frac{1}{n+1}<\frac{1}{n+1}\left(1+\frac{n}{m+1}\right)=\frac{m+n+1}{(n+1)(m+1)},$$

this proves (14), and, thus, the inequality (13) holds.

Therefore, the contractive condition appearing in Corollary 2.7 is also satisfied in this case.

By Corollary 2.7, f has at least a fixed point (obviously, $x_0=0$ and $x_1=\frac{1}{\sqrt{7}}$ are fixed points of f).

Notice that $(X,\preceq)$ does not satisfy condition (11) and we have not uniqueness of the fixed point of f.

On the other hand, if we consider the operator $\widetilde{f}:X\longrightarrow X$ defined by

$$\widetilde{f}=\begin{cases}0 & \text{if } x\in X-\{\frac{1}{n}:n\geq 2\}\\ \dfrac{1}{n+1} & \text{if } x=\frac{1}{n}, n=2,3,\ldots,\end{cases}$$

then using a similar argument to the previous one it is easily checked that $\widetilde{f}$ satisfies the assumptions of Corollary 2.7 and, consequently, $\widetilde{f}$ has a fixed point (notice that $x_0=0$ is the unique fixed point of $\widetilde{f}$).

On the other hand, $(X,\preceq)$ does not satisfy condition (11) and we have uniqueness of the fixed point of $\widetilde{f}$.

This example proves that condition (11) is a sufficient condition but not neccesary for the uniqueness of the fixed point.

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