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Separation by Convex Interpolation Families*

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A set of continuous functions defined on an interval I is called an n-parameter Beckenbach family, if each n points of $I\times\mathbb{R}$ (with pairwise distinct first coordinates) can be interpolated by a unique element of the set. The aim of the present note is to characterize those pairs of real valued functions that can be separated by a member of a given convex Beckenbach family of order n. The key idea of the proof is to identify the family with $\mathbb{R}^n$ via a suitable homeomorphism. Then, the classical Helly Theorem guarantees the existence of a proper separator.

Keywords: Interpolation families, Haar and Chebyshev systems, Separation theorems, Helly's Theorem

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1. Introduction

Sandwich theorems play a crucial role especially in the field of Convex Analysis [12], [17]. For example, if a convex and a concave function is given such that the convex one is "above" the concave, then there exists an affine function between them. Of course, the assumptions on convexity/concavity are *sufficient* but not *necessary* for the existence of an affine separator.

Assume that I is an interval and $f,g\colon I\to\mathbb{R}$ are such that can be separated by an affine function $h\colon I\to\mathbb{R}$. Then, as direct calculations show, for all elements x,y of I and $\lambda\in[0,1]$, the following inequalities hold:

$$\begin{aligned} f\big(\lambda x+(1-\lambda)y\big) &\le \lambda g(x)+(1-\lambda)g(y),\\ g\big(\lambda x+(1-\lambda)y\big) &\ge \lambda f(x)+(1-\lambda)f(y). \end{aligned}\tag{1}$$

Surprisingly, due to a result of Nikodem and Wąsowicz [14], these inequalities are

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not merely a *consequence* of the existence of an affine separator, but *characterize* the existence of such separator.

Separation problems of this spirit were studied intensively by several authors and in several contexts. The polynomial case is due to Wąsowicz [20] and to Balaj and Wąsowicz [2]. Their approach is based on some selection principles of Behrend and Nikodem [4] and of Balaj and Nikodem [1]. The separation problem was solved by Nikodem and Páles [13] for so-called two parameter interpolation families. When the parameters of the family is *arbitrary* but the structure is *linear*, the characterization was presented by Bessenyei and Páles [5].

The present note makes an attempt for a common view of the previous two papers, placing their results into the context of an arbitrary parameter but not necessarily linear family. More precisely, our aim is to characterize those pairs of functions that can be separated by a member of a given (convex-closed) interpolation family:

Definition. Let H be a real subset of at least n elements. A set $\mathcal{F}_n(H)$ of real functions is called an n-parameter *interpolation family* over H, if its members are defined on H and, for all points $(x_1, y_1), \dots, (x_n, y_n)$ of $H \times \mathbb{R}$ (with pairwise distinct first coordinates) there exists exactly one element φ of $\mathcal{F}_n(H)$ such that

$$\varphi(x_1) = y_1, \dots, \varphi(x_n) = y_n.$$

An n-parameter interpolation family is said to be a *Beckenbach family*, if its members are continuous. A Beckenbach family of order n is denoted by $\mathcal{B}_n(H)$.

Throughout this paper, the members of an interpolation family $\mathcal{F}_n(H)$ are termed briefly *generalized lines*. The most important subclass of interpolation families can be obtained via Haar systems. Later we shall prove that a linear interpolation family coincides with the linear hull of a suitable Haar system.

Definition. Let H be a real subset of at least n elements and let $\omega_1, \dots, \omega_n \colon H \to \mathbb{R}$ be given functions. We say that $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is a (positive) *Haar system* on H if, for all elements $x_1 < \dots < x_n$ of H,

$$\begin{vmatrix} \boldsymbol{\omega}(x_1) & \dots & \boldsymbol{\omega}(x_n) \end{vmatrix} := \begin{vmatrix} \omega_1(x_1) & \dots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_1) & \dots & \omega_n(x_n) \end{vmatrix} \overset{(>)}{\neq} 0.$$

Under a *Chebyshev system* we mean a Haar system of continuous functions.

The most important example for a Haar system of parameter n is the polynomial system, that is, monomials up to degree $(n-1)$. Indeed, the corresponding determinant in this case is a Vandermonde determinant and hence nonvanishing. In fact, the polynomial system is also a Chebyshev system.

Obviously, the set of all linear combinations of a Haar system is an interpolation family. Hence, for our convenience, the terminology "Haar system" is also used for this linear span. In fact, identifying the vectors of functions with their linear hull is not misleading and is widely accepted in the technical literature. Let us mention that Haar and Chebyshev systems play an important role, sometimes indirectly, in numerous fields of mathematics; the book of Karlin and Studden [10] contains a rich material and bibliography of the topics for the interested reader.

The particular importance of different kind of interpolation families is that they induce (generalized) convexity notions. Not claiming completeness, we quote here the works of Beckenbach [3], Hopf [9], Popoviciu [15] and Tornheim [18]. For further details, consult the introduction of [6].

The main result of the paper characterizes those pairs of real functions that can be separated by a generalized line belonging to a Beckenbach family which is supposed to be closed under convex combinations. The characterization is given in terms of systems of inequalities. The proof is based on the geometric feature of interpolation families and does not require preliminary selection methods. The key tool of the proof is the classical Helly Theorem, one of the most important result in Convex and Combinatorial Geometry [19]. However, in the lack of linear structure, it cannot be applied *directly*. Therefore, we shall make a "detour" according to the figure below:

$$\begin{tikzcd}
\big(\mathcal{B}_n(I),d\big) \arrow[r, "\Phi"] \arrow[d] & \big(\mathbb{R}^n,\|\cdot\|\big) \arrow[d] \\
\bigcap_{x\in I}\mathcal{K}(x)\neq\emptyset & \bigcap_{x\in I}K(x)\neq\emptyset \arrow[l, "\Phi^{-1}"']
\end{tikzcd}$$

It turns out that a Beckenbach family can be considered as a metric space $\big(\mathcal{B}_n(I),d\big)$. Moreover, this metric space is Φ-homeomorphic to the Euclidean space $\big(\mathbb{R}^n,\|\cdot\|\big)$. Denoting the set of generalized lines that separate the given functions at point x by $\mathcal{K}(x)$, one should check that $\bigcap_{x\in I}\mathcal{K}(x)$ is nonempty. Instead of this, we prove that the Φ-image of the intersection, denoted by $\bigcap_{x\in I}K(x)$, is nonempty. To do this, we conclude that the sets $K(x)$ are such subsets of $\mathbb{R}^n$ that fulfill the conditions of Helly's Theorem. Checking most of the conditions is quite simple; the only difficulty is to prove that each $(n+1)$ member of the collection $\{K(x)\mid x\in I\}$ has a nonempty intersection. This property can be verified via constructing a homotopy based on the systems of those inequalities that are involved in our main (characterization) theorem.

2. Auxiliary Lemmas

The next three lemmas subsume the most important auxiliary results that are used in proving the main theorem. The first one states that an interpolation family can be metrized in a quite natural way. Let us emphasize that, in the rest of this note, every topological notion about interpolation families is interpreted within this framework.

Lemma 2.1. *Let $\mathcal{F}_n(H)$ be an interpolation family over the set H of at least n elements and let $x_1,\dots,x_n$ be pairwise distinct elements of H. Then, $\big(\mathcal{F}_n(H),d\big)$*

is a complete metric space where

$$d(\varphi,\psi) := \max\{|\varphi(x_1)-\psi(x_1)|,\dots,|\varphi(x_n)-\psi(x_n)|\}.$$

Moreover, $\varphi_m \to \varphi$ *if and only if* $\varphi_m \to \varphi$ *pointwise on* H.

Proof. The properties $d(\varphi,\psi)\ge 0$, $d(\varphi,\psi)=d(\psi,\varphi)$ and $d(\varphi,\varphi)=0$ are trivial. Assume $d(\varphi,\psi)=0$. Then, φ and ψ coincide at n pairwise distinct points of H and hence, due to the unique interpolation property, $\varphi=\psi$. The triangle inequality is a direct consequence of the properties of absolute value and of the maximum functions. The completeness of the metric space follows from the completeness of $\mathbb{R}^n$. Indeed, take a Cauchy-sequence (φ_m) in $\mathcal{F}_n(H)$. Then, $\varphi_m(x_k)$ is a Cauchy-sequence in $\mathbb{R}$ for all $k=1,\dots,n$. The completeness of the reals implies that there exist values y_k such that $\varphi_m(x_k)\to y_k$ as $m\to\infty$. Let φ be the unique element of $\mathcal{F}_n(H)$ which is determined by the interpolation properties

$$\varphi(x_k)=y_k \quad (k=1,\dots,n).$$

According to the construction, $\varphi_m\to\varphi$ resulting in the completeness of $\big(\mathcal{F}_n(H),d\big)$. Finally, the unique interpolation property implies that $\varphi_m\to\varphi$ pointwise on H if $\varphi_m\to\varphi$. ☐

Lemma 2.2. *If* $\mathcal{F}_n(H)$ *is an interpolation family over the set* H *of at least* n *elements and* $x_1,\dots,x_n$ *are pairwise distinct elements of* H*, then the mapping* $\Phi\colon \mathcal{F}_n(H)\to\mathbb{R}^n$ *given by* $\Phi(\varphi)=\big(\varphi(x_1),\dots,\varphi(x_n)\big)$ *is a homeomorphism.*

Proof. The bijectivity of Φ is a straightforward consequence of the unique interpolation property, while the continuity of Φ follows immediately from the definition of the metrics. For the continuity of Φ^{-1}, take a sequence $y_m=(y_{m1},\dots,y_{mn})$ in $\mathbb{R}^n$ that tends to $y=(y_1,\dots,y_n)$ as $m\to\infty$. If $\varphi_m:=\Phi^{-1}(y_m)$ and $\varphi:=\Phi^{-1}(y)$, then, by definition, for all $k=1,\dots,n$,

$$\varphi_m(x_k)=y_{mk}, \qquad \varphi(x_k)=y_k.$$

Therefore $\varphi_m\to\varphi$. This means that $\Phi^{-1}(y_m)\to\Phi^{-1}(y)$ as $m\to\infty$, showing the desired continuity. ☐

Lemma 2.3. *Let* H *be a real subset of at least* n *elements. An interpolation family* $\mathcal{F}_n(H)$ *is linear if and only if there exists a Haar system* $\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$ *such that* $\mathcal{F}_n(H)=\mathrm{lin}(\boldsymbol{\omega})$*. Moreover, there exist nonlinear interpolation families that are convex.*

Proof. We shall concentrate only for the implication $(i)\Rightarrow(ii)$ since the other one is trivial. Fix pairwise distinct elements $x_1,\dots,x_n$ of H and define $\omega_k\colon H\to\mathbb{R}$ of $\mathcal{F}_n(H)$ via the interpolation properties

$$\omega_k(x_l)=\delta_{kl},$$

where δ_{kl} stands for the Kronecker delta symbol. Our aim is to prove that $\boldsymbol{\omega}:=(\omega_1,\dots,\omega_n)$ is a Haar system fulfilling $\mathcal{F}_n(H)=\mathrm{lin}(\boldsymbol{\omega})$. Let $\varphi\in\mathcal{F}_n(H)$ be arbitrary and assume that $\varphi(x_l) = \alpha_l$. Since $\mathcal{F}_n(H)$ is closed under linear combinations, the function $\omega := \alpha_1\omega_1 + \cdots + \alpha_n\omega_n$ belongs to $\mathcal{F}_n(H)$. On the other hand,

$$\omega(x_l) = \sum_{j=1}^{n} \alpha_j\omega_j(x_l) = \sum_{j=1}^{n} \alpha_j\delta_{jl} = \alpha_l = \varphi(x_l).$$

Consequently, due to the unique interpolation property, $\omega = \varphi$. That is, $\mathcal{F}_n(H) = \mathrm{lin}(\boldsymbol{\omega})$. Finally, we have to check, that $\boldsymbol{\omega}$ is a Haar system, indeed. Suppose indirectly, that there exist pairwise distinct elements $t_1, \dots, t_n$ of H such that

$$0 = \begin{vmatrix} \omega_1(t_1) & \dots & \omega_n(t_1) \\ \vdots & \ddots & \vdots \\ \omega_1(t_n) & \dots & \omega_n(t_n) \end{vmatrix}.$$

In this case, there exist nontrivial coefficients $\beta_1, \dots, \beta_n$ such that $\psi = \beta_1\omega_1 + \cdots + \beta_n\omega_n$ vanishes at $t_1, \dots, t_n$. Note that $\psi \not\equiv 0$ since $\psi(x_l) = \beta_l$ and there exists an index l such that $\beta_l \neq 0$. Then, the points $(t_1, 0); \dots; (t_n, 0)$ can be interpolated by ψ and by the zero function, which is a contradiction.

For the second statement, take an arbitrary set H^* of at least $(n + 1)$ elements and a Haar system $\mathcal{F}_{n+1}(H^*)$ over this set. Fix $x^* \in H^*$, define $H := H^* \setminus \{x^*\}$, and denote those elements of $\mathcal{F}_{n+1}(H^*)$ that take value 1 at x^* by $\mathcal{F}_n(H)$. Then, $\mathcal{F}_n(H)$ is a convex interpolation family of parameter n over H. On the other hand, $\mathcal{F}_n(H)$ has no linear structure since it is not closed under multiplication by -1. (In more simple: polynomials p of degree at most n on $[0, 1]$ fulfilling $p(-1) = 1$ form a convex but nonlinear Beckenbach family of parameter n.) □

In fact, the second statement of Lemma 2.1 can considerably generalized: Due to a result of Tornheim [18], uniform convergence on compact subintervals of the domain is equivalent to the convergence of n pairwise distinct points among generalized lines belonging to a n parameter Beckenbach family.

According to Lemma 2.3, linear interpolation families are linear hulls. However, the counterpart of the lemma shows, that this is not the case if we assume only convex-closedness. This observation guarantees that our main result is a meaningful generalization of the earlier (linear) ones.

3. The Main Result

Theorem 3.1. *Let $\mathcal{B}_n(I)$ be a convex Beckenbach family over the real interval I and $f, g \colon I \to \mathbb{R}$ be given functions. Then the following statements are equivalent:*

(*i*) *there exists $h \in \mathcal{B}_n(I)$ such that $f \le h \le g$;*

(*ii*) *for all $u \le x_1 < \cdots < x_n \le v$ of I, we have the inequalities*

$$\varphi_1(v) \ge f(v), \qquad \psi_1(v) \le g(v); \tag{2}$$

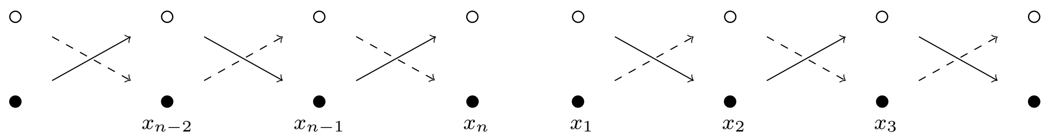
and

$$\varphi_2(u) \ge f(u), \qquad \psi_2(u) \le g(u), \tag{3}$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{B}_n(I)$ *are determined by the interpolation properties*

$$\begin{aligned}
&\varphi_1(x_k) = g(x_k), && \psi_1(x_k) = f(x_k), && n-k \in \{0,\dots,n-1\} \cap 2\mathbb{Z};\\
&\varphi_1(x_k) = f(x_k), && \psi_1(x_k) = g(x_k), && n-k \in \{0,\dots,n-1\} \cap \big(2\mathbb{Z}+1\big);\\
&\varphi_2(x_k) = g(x_k), && \psi_2(x_k) = f(x_k), && k \in \{1,\dots,n\} \cap \big(2\mathbb{Z}+1\big);\\
&\varphi_2(x_k) = f(x_k), && \psi_2(x_k) = g(x_k), && k \in \{1,\dots,n\} \cap 2\mathbb{Z}.
\end{aligned}$$

The meaning of the interpolation properties appearing in the theorem can be illustrated in the following expressive way. For simplicity, the symbols "$\circ$" and "$\bullet$" stand for the values of g and f, respectively. Then, φ_1 and φ_2 are obtained by "$\rightarrow$", while ψ_1 and ψ_2 are obtained by "$\dashrightarrow$".



Proof. For simplicity, denote φ_1 by φ. To verify $(i) \Rightarrow (ii)$, assume indirectly that $\varphi(v) < f(v)$. Let $\varepsilon > 0$ be arbitrary, and consider the generalized line φ_ε determined by the interpolation properties

$$\varphi_\varepsilon(x_k) = \varphi(x_k) + (-1)^{n-k}\varepsilon \quad (k = 1,\dots,n.)$$

By Bolzano's Theorem, and by the inequalities $f \le h \le g$, there exists $\xi_k \in \,]x_k, x_{k+1}[$ for all $k = 1,\dots,n-1$ such that $h(\xi_k) = \varphi_\varepsilon(\xi_k)$. Then, $h(t) \ne \varphi_\varepsilon(t)$ if $t \in I \setminus \{\xi_1,\dots,\xi_{n-1}\}$. Moreover, using the conventions $\xi_0 := \inf I$ and $\xi_n := \sup I$, for all $x \in]\xi_{k-1}, \xi_k[$ and $k = 1,\dots,n$ we have the inequalities

$$(-1)^{n-k}\big(\varphi_\varepsilon(x) - h(x)\big) > 0.$$

In particular, $h(v) < \varphi_\varepsilon(v)$. Since $\varphi_\varepsilon \to \varphi$ pointwise as $\varepsilon \to 0$, there exists some positive ε_0 such that $\varphi(v) < \varphi_{\varepsilon_0}(v) < f(v)$. Consequently $h(v) < f(v)$ follows, which contradicts to $f \le h$. The further inequalities can be proved via similar arguments.

For the converse implication, first we check that $f \le g$ holds. Let v be an element of I differing from $\inf(I)$, and fix the elements $x_1 < \dots < x_n := v$. Then, according to the definition of φ and applying the corresponding inequality of the second assertion,

$$f(v) \le \varphi(v) = g(v)$$

follows. The case when $u = \inf(I)$ belongs to the interval, both the inequalities of (3) lead to $f(u) \le g(u)$, which means that f is majorized by g on the whole interval I. In particular, the subset $\mathcal{K}(x)$ of $\mathcal{B}_n(I)$, defined by

$$\mathcal{K}(x) := \{\omega \in \mathcal{B}_n(I) \mid f(x) \le \omega(x) \le g(x)\}$$

is non-empty for all elements $x \in I$. Let $t_1,\dots,t_n$ be fixed and pairwise distinct elements of I, and consider the homeomorphism $\Phi\colon \mathcal{B}_n(I) \to \mathbb{R}^n$ given by

$$\Phi(\omega) := \big(\omega(t_1),\dots,\omega(t_n)\big).$$

Then, $K(x) := \Phi\big(\mathcal{K}(x)\big) \neq \emptyset$ for all $x \in I$. Let $y_1 = (y_{11}, \dots, y_{1n})$ and $y_2 = (y_{21}, \dots, y_{2n})$ be elements of $K(x)$ and $\lambda \in [0,1]$. Then, there exist generalized lines ω_1 and ω_2 that belong to $\mathcal{K}(x)$ and fulfill the properties

$$\omega_1(t_k) = y_{1k}, \qquad \omega_2(t_k) = y_{2k}.$$

Define $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. Since $\mathcal{B}_n(I)$ is convex, ω is a generalized line that belongs to $\mathcal{K}(x)$. On the other hand, $\omega(t_k) = \lambda y_{1k} + (1-\lambda)y_{2k}$ by construction, showing that $\Phi(\omega) = \lambda y_1 + (1-\lambda)y_2$. In other words, $K(x)$ is convex.

Let $y_m = (y_{m1}, \dots, y_{mn})$ be a sequence in $K(x)$ that tends to $y = (y_1, \dots, y_n)$. Then, there exist a sequence (ω_m) in $\mathcal{K}(x)$ and an element ω of $\mathcal{B}_n(I)$ such that

$$\omega_m(t_k) = y_{mk}, \qquad \omega(t_k) = y_k.$$

Since $\omega_m \to \omega$ at the points of the set $\{t_1, \dots, t_n\}$, $\omega_m \to \omega$ on the whole interval I as $m \to \infty$. In particular, $\lim_{m\to\infty} \omega_m(x) = \omega(x)$. On the other hand, the property $\omega_m \in \mathcal{K}(x)$ implies

$$f(x) \le \omega_m(x) \le g(x);$$

passing to infinity, $\omega \in \mathcal{K}(x)$ follows. That is, the set $K(x)$ is closed.

Let $x_1 < \dots < x_n$ be elements of I and take the homeomorphism $\Psi\colon \mathcal{B}_n(I) \to \mathbb{R}^n$ defined by the usual way

$$\Psi(\omega) := \big(\omega(x_1), \dots, \omega(x_n)\big).$$

The closed set $\Psi\big(\mathcal{K}(x_1) \cap \ldots \cap \mathcal{K}(x_n)\big)$ is bounded evidently in the maximum norm of $\mathbb{R}^n$ and hence, by the Heine–Borel Theorem, is compact. Since Φ is a homeomorphism,

$$\begin{aligned} K(x_1) \cap \ldots \cap K(x_n) &= \Phi\big(\mathcal{K}(x_1)\big) \cap \ldots \cap \Phi\big(\mathcal{K}(x_n)\big) \\ &= \Phi\big(\mathcal{K}(x_1) \cap \ldots \cap \mathcal{K}(x_n)\big) \\ &= \big(\Phi \circ \Psi^{-1}\big)\Big(\Psi\big(\mathcal{K}(x_1) \cap \ldots \cap \mathcal{K}(x_n)\big)\Big). \end{aligned}$$

Applying the continuity of $\Phi \circ \Psi^{-1}$ and the compactness of the argument in the latter term, we get that each n member subcollection of the family $\{K(x) \mid x \in I\}$ has a compact intersection.

Finally we show, that each $(n+1)$ member subcollection of the family $\{K(x) \mid x \in I\}$ has a nonempty intersection. Clearly, it is enough to check the analogous property for the members of the family $\{\mathcal{K}(x) \mid x \in I\}$. Take elements $x_1 < \dots < x_{n+1} =: v$ of I and define the generalized lines $\varphi := \varphi_1$ and $\psi := \psi_1$ as in assertion *(ii)*. If $f(v) \le \varphi(v) \le g(v)$ or $f(v) \le \psi(v) \le g(v)$ holds, then $h = \varphi$ or $h = \psi$ belongs to the intersection $\mathcal{K}(x_1) \cap \ldots \cap \mathcal{K}(x_{n+1})$. In the opposite case, due to the inequalities (2), $\varphi(v) > g(v)$ and $\psi(v) < f(v)$ follow. Define the homotopy $H\colon [0,1] \times I \to \mathbb{R}$ as the convex combination of φ and ψ:

$$H(t,x) := (1-t)\varphi(x) + t\psi(x).$$

The convexity of $\mathcal{B}_n(I)$ guarantees that $H(t,\cdot)$ is a generalized line for all $t \in [0,1]$. By Bolzano's Theorem, there exists $t_0 \in]0,1[$ such that $f(v) \le H(t_0, v) \le g(v)$. In this case, $h = H(t_0,\cdot)$ is a generalized line belonging to the intersection in question.

To sum up the aboves, $\{K(x) \mid x \in I\}$ is such a collection of nonempty, convex, closed subsets of $\mathbb{R}^n$, in which there exist finite many sets of compact intersection and each $(n+1)$ member subcollection has a nonempty intersection. Helly's Theorem guarantees, that the entire intersection $\bigcap\{K(x) \mid x \in I\}$ is also nonempty. Therefore, there exists an element h of $\bigcap\{\mathcal{K}(x) \mid x \in I\}$, which, by definition of the sets $\mathcal{K}(x)$, is a proper separator for f and g. □

As it turns out from the proof, inequalities (3) guarantee the order of f and g at the left endpoint of the domain if the endpoint belongs to I. Similarly, the order at the right endpoint (if it makes sense) is determined by (2). Therefore, in case of open intervals, only one of the pairs of inequalities is needed.

4. Applications and Closing Remarks

If the underlying Beckenbach family is a Chebyshev system, then the main result reduces to the next corollary. Its statement remains true also for Haar systems [5].

Corollary 4.1. *Let I be a real interval, $f, g\colon I \to \mathbb{R}$ and $\boldsymbol{\omega} := (\omega_1, \dots, \omega_n)$ is a positive Chebyshev system over I. Then, there exists an $\boldsymbol{\omega}$-affine function $h\colon I \to \mathbb{R}$ such that $f \le h \le g$ if and only if, for all elements $x_1 \le \dots \le x_{n+1}$ of I the next inequalities are satisfied:*

$$\begin{vmatrix} \dots & \boldsymbol{\omega}(x_{n-2}) & \boldsymbol{\omega}(x_{n-1}) & \boldsymbol{\omega}(x_n) & \boldsymbol{\omega}(x_{n+1}) \\ \dots & g(x_{n-2}) & f(x_{n-1}) & g(x_n) & f(x_{n+1}) \end{vmatrix} \le 0;$$

$$\begin{vmatrix} \dots & \boldsymbol{\omega}(x_{n-2}) & \boldsymbol{\omega}(x_{n-1}) & \boldsymbol{\omega}(x_n) & \boldsymbol{\omega}(x_{n+1}) \\ \dots & f(x_{n-2}) & g(x_{n-1}) & f(x_n) & g(x_{n+1}) \end{vmatrix} \ge 0.$$

Hint. Fix points $x_1 < \ldots < x_n$ of I and let $x_{n+1} := v \ge x_n$ be arbitrary. For our convenience, denote the generalized line φ_1 of the main result by φ. Using the representation $\varphi = \alpha_1\omega_1 + \dots + \alpha_n\omega_n$ and the interpolation properties $\varphi(x_n) = g(x_n), \varphi(x_{n-1}) = f(x_{n-1}), \dots$ reduce to a system of inhomogeneous linear equations with respect to $\alpha_1, \dots, \alpha_n$. Applying Cramer's Rule,

$$\alpha_k = (-1)^{n-k} \frac{D_{f,g}(x_1, \dots, x_n)}{D(x_1, \dots, x_n)}$$

where $D(x_1, \dots, x_n)$ is the determinant built from the columns $\boldsymbol{\omega}(x_1), \dots, \boldsymbol{\omega}(x_n)$, and the other determinant $D_{f,g}(x_1, \dots, x_n)$ is obtained from this, replacing its last row by $\dots, f(x_{n-1}), g(x_n)$. Then, rearranging the inequality $f(x_{n+1}) \le \varphi(x_{n+1})$ and applying the expansion theorem of determinants, we get the first inequality of the corollary. If the base points are not pairwise distinct, the inequality holds obviously. The proof of the other case is similar. □

Two direct consequences of Corollary 4.1 are as follow. The first one is the polynomial setting, the main result of [2] and [20]. For technical convenience, we use the next concepts: if points $x_0 \le \cdots \le x_n$ are fixed elements of an interval I, then denote the Vandermonde determinants built on the system $\{x_0, \dots, x_n\} \setminus \{x_k\}$ by $V_k(x_0, \dots, x_n)$. Further, denote the sets $(2\mathbb{Z}) \cap [0, n]$ and $(2\mathbb{Z}+1) \cap [0, n]$ by $N_0(n)$ and $N_1(n)$, respectively.

Corollary 4.2. *Let I be an interval and $f, g\colon I \to \mathbb{R}$ be given functions. Then, there exists a polynomial h of degree at most $(n-1)$ satisfying $f \le h \le g$ if and only if, for all elements $x_0 \le \cdots \le x_n$ of I, the following inequalities hold:*

$$\sum_{k \in N_0(n)} f(x_{n-k}) V_{n-k}(x_0, \dots, x_n) \le \sum_{k \in N_1(n)} g(x_{n-k}) V_{n-k}(x_0, \dots, x_n),$$

$$\sum_{k \in N_0(n)} g(x_{n-k}) V_{n-k}(x_0, \dots, x_n) \le \sum_{k \in N_1(n)} f(x_{n-k}) V_{n-k}(x_0, \dots, x_n).$$

The other direct consequence of Corollary 4.1 (and of course, also of Corollary 4.2) is the main result of [14], which is one of the motivations of the present note.

Corollary 4.3. *Let I be an interval and $f, g\colon I \to \mathbb{R}$ be given functions. Then, there exists an affine function $h\colon I \to \mathbb{R}$ if and only if, for all elements x, y of I and $\lambda \in [0, 1]$, the inequalities* (1) *hold.*

Some open problems can be posed in connection our main result. If we have two parameters, then the separation can be characterized under no further assumption on the underlying Beckenbach family. (For details, consult [13].) Hence the question arises, quite evidently, if the convex-closedness in our main result is redundant or not.

In fact, the main result of [5] holds for Haar systems, that is, for such linear interpolation families whose members are not supposed to be necessarily continuous. This suggest that the main result of this note could be studied within the framework of interpolation families instead of Beckenbach families.

In many cases that have been studied (see [5], [13], [14]), the existence of an affine separator (between the given functions) turns out to be equivalent to the simultaneous existence of convex/concave separators. The method followed here does not allow an analogous statement in the main result. At the moment, the question remains open whether this is the situation or not.

Finally let us mention a historical remark about the Helly Theorem. Helly obtained this result in 1913, but, because of some political reason, he was unable to published it until 1923 [8]. By this time, two alternative proofs had appeared: the first one was due to Radon [16] from 1921, while the second one was obtained by Kőnig [11] in 1922. For precise details, we refer to the work [7].

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