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On a Nonlinear Elliptic Transmission Problem with Critical Growth*

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This paper is concerned with the existence of nontrivial solutions to a class of transmission problems with critical growth in a bounded domain on $\mathbb{R}^N$. Our approach is based on variational methods.

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1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$ and let $\Omega_1 \subset \Omega$ be a subdomain with smooth boundary Σ satisfying $\overline{\Omega_1} \subset \Omega$. Writing $\Gamma = \partial\Omega$ and $\Omega_2 = \Omega \backslash \overline{\Omega_1}$ we have $\Omega = \overline{\Omega_1} \bigcup \Omega_2$ and $\partial\Omega_2 = \Sigma \bigcup \Gamma$. We also denote η the unit normal vector on the boudary of Ω_2 pointing outward. This work is concerned with the existence of nontrivial solutions to a nonlinear elliptic transmission problem of the type

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega_1, \\ -\Delta v = |v|^{2^*-2}v & \text{in } \Omega_2, \\ v = 0 & \text{on } \Gamma, \\ u = v & \text{on } \Sigma \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} & \text{on } \Sigma. \end{cases}$$

The hypotheses on the continuous function $f : \overline{\Omega_1} \times \mathbb{R} \to \mathbb{R}$ are the following:

$$(f_1) \qquad \lim_{t\to 0} \frac{f(x,t)}{t} = 0,$$

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uniformly on $x \in \Omega_1$.

There exists $q \in (2, 2^*)$ verifying;

$$(f_2) \qquad \lim_{t \to +\infty} \frac{f(x,t)}{t^{q-1}} = 0,$$

uniformly on $x \in \Omega_1$ and the well known Ambrosetti-Rabinowitz superlinear condition, that is, there exists $\theta \in (2, 2^*)$ such that

$$(f_3) \qquad 0 < \theta F(x,t) = \theta \int_0^t f(x,s)ds \le tf(x,t),$$

for all $x \in \Omega_1$ and $t > 0$.

A typical example of a function satisfying the conditions $(f_1) - (f_3)$ is given by

$$f(x,t) = \sum_{i=1}^{k} C_i(x) t_+^{q_i - 1}$$

with $k \in \mathbb{N}$, $2 < q_i < 2^*$, $C_i \in L^\infty(\Omega_1)$ and $t_+ = \max\{t, 0\}$.

We state the main result of this paper.

Theorem 1.1. *Assume that conditions $(f_1) - (f_3)$ hold. Then, there exists $\lambda^* > 0$ such that, problem (P_λ) has a nontrivial solution for $\lambda \ge \lambda^*$.*

A problem similar to (P_λ) has been studied in [5] for equations of Kirchhoff type. There the nonlinearities are sublinear, in this way they are able to minimize the energy functional related to their problem. A problem similar to ours has been undertaken in [8], where the author uses the mountain pass theorem. In [8] there is a careful analysis of physical motivations concerning (P_λ). We refer to [3] for an account on problems with internal and exterior layers. Transmission problems involving wave equations have been studied in [6]. In this paper we consider nonlinearities of critical type, so we invoke the compactness principle [4] to take care of possible atoms appearing when one takes weak limits. As we shall see in the proof of Theorem 1.1, we just need to take care of these limits in Ω_2, the part of Ω where the critical nonlinearity acts. Lemmas 2.2 and 2.3 are devoted to show that our functional I satisfies the conditions of the mountain pas theorem. This allows us to find a convergent sequence (u_n, v_n) such that $I(u_n, v_n)$ converges to c^*, where I is the functional corresponding to problem (P_λ). We prove in Lemma 2.4 that the critical level c^* belongs to the interval $(0, \frac{1}{N} S^{N/2})$, where S is the Sobolev constant. In section 2 we define the space of functions setting.

2. The variational framework

We begin with some notations. For a given open set D, the standard norm in $L^p(D)$ is denoted by $|.|_{p,D}$. Our analysis takes place in the Sobolev space

$$E = \{(u,v) \in H^1(\Omega_1) \times H^1_\Gamma(\Omega_2); u = v \text{ on } \Sigma \text{ in the sense of traces}\},$$

where

$$H^1_\Gamma(\Omega_2) = \{v \in H^1(\Omega_2); v = 0 \text{ on } \Gamma \text{ in the sense of traces}\}.$$

The next lemma is important in our arguments and the proof can be seen in [5].

Lemma 2.1. *The set E is a closed subspace of $H^1(\Omega_1) \times H^1(\Omega_2)$ and*

$$\|(u,v)\|^2 = |\nabla u|^2_{2,\Omega_1} + |\nabla v|^2_{2,\Omega_2}$$

defines a norm in E equivalent to the standard norm of $H^1(\Omega_1) \times H^1(\Omega_2)$.

In view of conditions (f_1)–(f_3), the above problem has a variational structure. The associated functional

$$I(u,v) := \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\Omega_1} F(x,u)\ dx - \frac{1}{2^*}\int_{\Omega_2} |v|^{2^*}\ dx,$$

is well defined on E. Thus,

$$\begin{aligned} &I'(u,v)(\phi,\psi) \\ :=& \int_{\Omega_1} \nabla u \nabla \phi\ dx + \int_{\Omega_2} \nabla v \nabla \psi\ dx - \lambda \int_{\Omega_1} f(x,u)\phi\ dx - \int_{\Omega_2} |v|^{2^*-2} v\psi\ dx, \end{aligned}$$

for all $(\phi,\psi) \in E$. Hence, critical points of I are weak solutions of (P_λ).

In order to use Variational Methods, we first derive some results related to the Palais-Smale compactness condition.

We say that a sequence $(u_n, v_n) \in E$ is a Palais-Smale sequence for the functional I at level $d \in \mathbb{R}$ if

$$I(u_n, v_n) \to d \text{ and } \|I'(u_n, v_n)\| \to 0 \ \text{ in } E'.$$

If every Palais-Smale sequence of I has a strong convergent subsequence, then one says that I satisfies the Palais-Smale condition ($(PS)_d$ for short).

In the sequel, we prove that the functional I has the Mountain Pass Geometry. This fact is proved in the next lemmas:

Lemma 2.2. *Assume that conditions (f_1)–(f_2) hold. Then, there exist positive numbers ρ and α such that,*

$$I(u,v) \geq \alpha > 0, \quad \forall\ (u,v) \in E \text{ with } \|(u,v)\| = \rho$$

for all $\lambda > 0$.

Proof. It follows from (f_1)–(f_2) that

$$\int_{\Omega_1} F(x,u)\ dx \leq \int_{\Omega_1} \frac{\epsilon}{2}|u|^2\ dx + \int_{\Omega_1} \frac{C_\epsilon}{q}|u|^q\ dx$$

and by Sobolev Embedding Theorem we get

$$I(u,v) \geq C_1\|(u,v)\|^2 - \lambda C_2\|(u,v)\|^q - C_2\|(u,v)\|^{2^*}.$$

Since $2 < q < 2^*$, the result follows by choosing $\rho > 0$ small enough. □

Lemma 2.3. *Assume that conditions (f_1)–(f_3) hold. Then, there exists $(e_1, e_2) \in E$ with $I(e_1, e_2) < 0$ and $\|(e_1, e_2)\| > \rho$ for all $\lambda > 0$.*

Proof. Fix $(u_0, v_0) \in C_0^\infty(\Omega_1) \times C_0^\infty(\Omega_2) \setminus \{(0,0)\}$ with $u_0 \geq 0$ in Ω_1, $v_0 \geq 0$ in Ω_2 and $\|(u_0, v_0)\| = 1$. By (f_3) we get

$$I(t(u_0,v_0)) \leq \frac{t^2}{2} - t^\theta \lambda \int_{\Omega_1} |u_0|^\theta \, dx + C|\operatorname{support}(u_0)| - \frac{t^{2^*}}{2^*}\int_{\Omega_2} |v_0|^{2^*} \, dx.$$

Since $2 < \theta < 2^*$, the result follows by considering $(e_1 = t_* u_0, e_2 = t_* v_0)$ for some $t_* > 0$ large enough. □

Using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1], without (PS) condition (see [9, p. 12]), there exists a sequence $(u_n, v_n) \subset E$ satisfying

$$I(u_n, v_n) \to c_* \quad \text{and} \quad I'(u_n, v_n) \to 0,$$

where

$$c_* = \inf_{\gamma \in \Upsilon} \max_{t \in [0,1]} I(\gamma(t)) > 0,$$

and

$$\Upsilon := \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

In the following, we will use the number S given by

$$S := \inf_{w \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} |w|^{2^*} \mathrm{d}x\right)^{2/2^*}}$$

As in [2] and arguing as in [7], we are able to compare the minimax level c_* with S.

Lemma 2.4. *If the conditions (f_1)–(f_3) hold, then there exists $\lambda_* > 0$ such that c_* belongs to the interval $(0, \frac{1}{N}S^{N/2})$, for all $\lambda \geq \lambda_*$.*

Proof. If (e_1, e_2) are the functions given by Lemma 2.3, since I has the Mountain Pass Geometry, there exists a constant $\eta_\lambda > 0$, depending on λ, verifying $I(\eta_\lambda(e_1, e_2)) = \max_{\eta \geq 0} I(\eta(e_1, e_2))$. Hence, from (f_1)

$$\|(e_1, e_2)\|^2 \geq \eta_\lambda^{2^*-2} \int_{\Omega_2} |e_2|^{2^*} \, dx, \tag{1}$$

which implies that (η_λ) is bounded. Thus, there exists a sequence $\lambda_n \to +\infty$ and $\eta_0 \geq 0$ such that $\eta_{\lambda_n} \to \eta_0$ as $n \to +\infty$. Consequently, there is $K > 0$ such that

$$\eta_{\lambda_n}^2 \|(e_1, e_2)\|^2 \leq K \quad \forall n \in \mathbb{N},$$

hence

$$\lambda_n \int_{\Omega_1} f(x, \eta_{\lambda_n} e_1)\eta_{\lambda_n} e_1 \, dx + \eta_{\lambda_n}^{2^*} \int_{\Omega_2} |e_2|^{2^*} \, dx \leq K \quad \forall n \in \mathbb{N}. \tag{2}$$

If $\eta_0 > 0$, (2) leads to

$$\lim_{n\to\infty} \lambda_n \int_{\Omega_1} f(x, \eta_{\lambda_n} e_1)\eta_{\lambda_n} e_1 \, dx + \eta_{\lambda_n}^{2^*} \int_{\Omega_2} |e_2|^{2^*} \, dx = +\infty,$$

which is an absurd. Thus, we conclude that $\eta_0 = 0$. Now, we consider the path $\gamma_*(\eta) = \eta(e_1, e_2)$ for $\eta \in [0, 1]$, which belongs to $\gamma_* \in \Upsilon$ to get the following estimate

$$0 < c_* \leq \max_{\eta\in[0,1]} I(\gamma_*(\eta)) = I(\eta_\lambda(e_1, e_2)) \leq \frac{\eta_\lambda^2}{2} \|(e_1, e_2)\|^2.$$

In this way, if λ is large enough, we derive

$$\frac{\eta_\lambda^2}{2} \|(e_1, e_2)\|^2 < \frac{1}{N} S^{N/2},$$

which leads to

$$0 < c_* < \frac{1}{N} S^{N/2}.$$

□

3. Proof of Theorem 1.1

Proof. From Lemmas 2.2, 2.3 and 2.4, there exists a sequence $(u_n, v_n) \subset E$ verifying $I(u_n, v_n) \to c_*$ and $I'(u_n, v_n) \to 0$ with

$$c_* \in \left(0, \frac{1}{N} S^{N/2}\right),$$

for all $\lambda \geq \lambda_*$.

From (f_3), there is $C > 0$ such that

$$C + \|(u_n, v_n)\| \geq I(u_n, v_n) - \frac{1}{\theta} I'(u_n, v_n)(u_n, v_n) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|(u_n, v_n)\|^2,$$

$\forall n \in \mathbb{N}$.

Hence (u_n, v_n) is bounded in E and, up to a subsequence,

$$u_n \rightharpoonup u \ \text{ in } H^1(\Omega_1) \quad \text{and} \quad v_n \rightharpoonup v \ \text{ in } H^1_\Gamma(\Omega_2), \tag{3}$$

where $\rightharpoonup$ means weak convergence.

We claim that $\|(u_n, v_n)\|^2 \to \|(u,v)\|^2$ as $n \to \infty$, which will imply that

$$\|(u_n - u, v_n - v)\| \to 0.$$

By this claim and the fact that I is C^1, we obtain

$$I(u_n, v_n) \to I(u,v) \quad \text{and} \quad I'(u_n, v_n) \to I'(u,v)$$

and hence

$$I(u,v) = c_* > 0 \quad \text{and} \quad I'(u,v) = 0.$$

Thus, the result follows.

It order to prove the claim, taking a subsequence, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \mu, \qquad |\nabla v_n|^2 \rightharpoonup |\nabla v|^2 + \sigma \quad \text{and} \quad |v_n|^{2^*} \rightharpoonup |v|^{2^*} + \nu,$$

in the weak*-sense of measures.

Using the concentration compactness-principle due to Lions (cf. [4, Lemma 1.2]), we obtain at most a countable index set Λ, sequences $(x_i) \subset \Omega_1$, $(y_i) \subset \Omega_2$, $(\mu_i), (\sigma_i), (\nu_i) \subset [0, \infty)$, such that

$$\nu = \sum_{i \in \Lambda} \nu_i \delta_{y_i}, \qquad \mu \geq \sum_{i \in \Lambda} \mu_i \delta_{x_i}, \qquad \sigma \geq \sum_{i \in \Lambda} \sigma_i \delta_{y_i} \quad \text{and} \quad S\nu_i^{2/2^*} \leq \sigma_i, \tag{4}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega_1$ and δ_{y_i} is the Dirac mass at $y_i \in \Omega_2$.

Now, for every $\varrho > 0$, we set $\phi_\varrho(x) := \phi((x - x_i)/\varrho)$ where $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is such that $\phi \equiv 1$ on $B_1(0)$, $\phi \equiv 0$ on $\mathbb{R}^N \backslash B_2(0)$ and $|\nabla \phi|_\infty \leq 2$ and $\psi_\varrho(y) := \psi((y - y_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla \psi|_\infty \leq 2$.

Since $(\phi_\varrho u_n, \psi_\varrho v_n)$ is bounded in E, $I'(u_n, v_n)(\phi_\varrho u_n, \psi_\varrho v_n) \to 0$, that is,

$$\begin{aligned} &\int_{\Omega_1} \nabla u_n \cdot \nabla \phi_\varrho \, dx + \int_{\Omega_2} \nabla v_n \cdot \nabla \psi_\varrho \, dx \\ = \; &- \int_{\Omega_1} \phi_\varrho |\nabla u_n|^2 \, dx - \int_{\Omega_2} \psi_\varrho |\nabla v_n|^2 \, dx \\ &\quad + \lambda \int_{\Omega_1} f(x, \phi_\varrho u_n) \phi_\varrho u_n \, dx + \int_{\Omega_2} |\psi_\varrho v_n|^{2^*} \, dx + o_n(1). \end{aligned}$$

Since the support of ϕ_ρ is $B_{2\rho}(x_i)$, we obtain

$$\left| \int_{\Omega_1} \nabla u_n \nabla \phi_\rho \right| \leq \int_{B_{2\rho}(x_i)} |\nabla u_n| |\nabla \phi_\rho|.$$

By Hölder inequality and the fact that the sequence (u_n) is bounded, we have

$$\left| \int_{\Omega_1} \nabla u_n \nabla \phi_\rho \right| \leq C \left(\int_{B_{2\rho}(x_i)} |\nabla \phi_\rho| \right)^{1/2}.$$

By the Dominated Convergence Theorem $\int_{B_{2\rho}(x_i)} |\nabla\phi_\rho| \to 0$ as $\rho \to 0$, we obtain

$$\lim_{\varrho\to 0}\left[\lim_{n\to\infty}\int_{\Omega_1} \nabla u_n \cdot \nabla\phi_\varrho \, dx\right] = 0. \tag{5}$$

And a similar reasoning yields

$$\lim_{\varrho\to 0}\left[\lim_{n\to\infty}\int_{\Omega_2} \nabla v_n \cdot \nabla\psi_\varrho \, dx\right] = 0. \tag{6}$$

Moreover, since f has subcritical growth and ϕ_ϱ has compact support, we let $n \to \infty$ in the expression (6) to obtain

$$\int_{\Omega_2} \psi_\varrho \mathrm{d}\nu \geq \int_{\Omega_1} \psi_\varrho \mathrm{d}\mu + \int_{\Omega_2} \psi_\varrho \mathrm{d}\sigma.$$

Letting $\varrho \to 0$ we conclude that

$$\nu_i \geq \mu_i + \sigma_i \geq \sigma_i.$$

It follows from (4) that

$$\nu_i \geq S^{N/2} \quad \text{for every } i \in \Lambda. \tag{7}$$

Now we shall prove that the above expression (7) cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that $\nu_i \geq S^{N/2}$ for some $i \in \Lambda$. Since (u_n, v_n) is a $(PS)_{c_*}$, we have

$$c_* = I(u_n, v_n) - \frac{1}{2} I'(u_n, v_n)(u_n, v_n) + o_n(1).$$

Hence, from (f_3)

$$c_* \geq \frac{1}{N}\int_{\Omega_2} |v_n|^{2^*} \, dx + o_n(1) \geq \frac{1}{N}\int_{\Omega_2} \psi_\varrho |v_n|^{2^*} \, dx + o_n(1).$$

Letting $n \to \infty$, we get

$$c_* \geq \frac{1}{N}\sum_{i\in\Lambda} \psi_\varrho(y_i)\nu_i = \frac{1}{N}\sum_{i\in\Lambda} \nu_i \geq \frac{1}{N} S^{N/2},$$

which is in contradiction with Lemma 2.4. Thus Λ is empty and it follows that

$$\int_{\Omega_2} |v_n|^{2^*} \, dx \to \int_{\Omega_2} |v|^{2^*} \, dx.$$

Since

$$\int_{\Omega_1} f(x, u_n) u_n \, dx \to \int_{\Omega_1} f(x, u) u \, dx,$$

we conclude

$$\lim_{n\to\infty} \|(u_n, v_n)\|^2 = \lambda \int_{\Omega_1} f(x,u)u\ dx + \int_{\Omega_2} |v|^{2^*}\ dx.$$

On the other hand, from weak convergence

$$\|(u,v)\|^2 = \lambda \int_{\Omega_1} f(x,u)u\ dx + \int_{\Omega_2} |v|^{2^*}\ dx.$$

and the proof of claim is complete. □

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