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On the Moduli and Characteristic of Monotonicity in Orlicz-Lorentz Function Spaces

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In this paper we calculate the characteristic of monotonicity of Orlicz-Lorentz function spaces $\Lambda_{\Phi,\omega}$. Since degenerate Orlicz functions φ and degenerate weight functions ω are also admitted, the investigations concern the most possible wide class of Orlicz-Lorentz function spaces. These results concern both cases - an infinite and a finite non-atomic measure space, although in case of the finite measure the results are much more interesting. Let us recall that calculating of the characteristic of monotonicity of a Banach lattice is of great interest because of the result of Betiuk-Pilarska and Prus [2] stating that if a Banach lattice X has this characteristic strictly smaller then 1 and X is weakly orthogonal, then it has the weak fixed point property (see [29]).

Keywords: Banach lattice, Köthe space, Orlicz-Lorentz space, Luxemburg norm, modulus of monotonicity, characteristic of monotonicity, strict monotonicity, uniform monotonicity, weak fixed point property, weak orthogonality

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1. Introduction

Let us denote $S_+(X) = S(X) \cap X_+$, where $S(X)$ is the unit sphere of a real Banach lattice X (for its definition see [3], [28] and [36]) and X_+ is the positive cone of X. In the whole paper we consider only real Banach lattices.

A Banach lattice X is said to be strictly monotone if for all $x, y \in X_+$ such that $y \le x$ and $y \ne x$, we have $\|y\| < \|x\|$. A Banach lattice X is said to be uniformly monotone if for any $\varepsilon \in (0,1)$ there is $\delta(\varepsilon) \in (0,1)$ such that $\|x - y\| \le 1 - \delta(\varepsilon)$ whenever $x, y \in X_+$, $y \le x$, $\|x\| = 1$ and $\|y\| \ge \varepsilon$ (see [3]).

Given any Banach lattice X, the function $\delta_{m,X} : [0,1] \to [0,1]$ defined by

$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : x, y \in X_+, y \le x, \|x\| = 1, \|y\| \ge \varepsilon\} \tag{1}$$

is called the lower modulus of monotonicity of X. It is known (see [16]) that

$$\begin{aligned} \delta_{m,X}(\varepsilon) &= \inf\{1 - \|x - y\| : x, y \in X_+, y \le x, \|x\| = 1, \|y\| = \varepsilon\} \\ &= 1 - \sup\{\|x - y\| : x, y \in X_+, y \le x, \|x\| = 1, \|y\| = \varepsilon\} \\ &= 1 - \sup\{\|x - y\| : x, y \in X_+, y \le x, \|x\| = 1, \|y\| \ge \varepsilon\}. \end{aligned}$$

The lower modulus of monotonicity $\delta_{m,X}$ is a convex function on the interval $[0,1]$ (see [32]), so $\delta_{m,X}$ is non-decreasing on $[0,1]$ and continuous on the interval $[0,1)$. It is also clear that $\delta_{m,X}(\varepsilon) \le \varepsilon$ for any $\varepsilon \in [0,1]$. Obviously, X is uniformly monotone if and only if $\delta_{m,X}(\varepsilon) > 0$ for every $\varepsilon \in (0,1]$. Moreover, it is easy to see that a Banach lattice X is strictly monotone if and only if $\delta_{m,X}(1) = 1$.

Given any Banach lattice X, the number $\varepsilon_{0,m}(X) \in [0,1]$ defined by

$$\varepsilon_{0,m}(X) = \sup\{\varepsilon \in [0,1] \colon \delta_{m,X}(\varepsilon) = 0\} = \inf\{\varepsilon \in (0,1] \colon \delta_{m,X}(\varepsilon) > 0\}$$

(where $\inf \emptyset := 1$) is called the characteristic of monotonicity of X. Obviously, a Banach lattice X is uniformly monotone if and only if $\varepsilon_{0,m}(X) = 0$. In [11, Theorem 2.1] it has been shown that

$$\varepsilon_{0,m}(X) = 1 - \lim_{\varepsilon \to 1^-} \delta_{m,X}(\varepsilon) \ge 1 - \delta_{m,X}(1). \tag{2}$$

It is worth noticing that there exist Banach lattices such that the inequality in this formula is sharp (see [11, Examples 2.3 and 2.4].)

In [11] another modulus of monotonicity and the characteristic of monotonicity generated itself for Köthe spaces were introduced (see (3)). As we will see in case of Orlicz-Lorentz spaces, using the new formula for the characteristic of monotonicity of Köthe spaces it should be easier to calculate this coefficient in some classes of Köthe spaces.

Denote by (T, Σ, μ) a positive, complete and σ-finite measure space and by $L^0 = L^0(T, \Sigma, \mu)$ the space of all (equivalence classes of) real-valued and $\Sigma-$measurable functions defined on T. For two functions $x, y \in L^0$ we write $x \le y$ if $x(t) \le y(t)$ $\mu-$a.e. in T. By $E = (E, \le, \|\cdot\|_E)$ we denote a Köthe space over the measure space (T, Σ, μ), that is, E is a Banach subspace of L^0 which satisfies the following conditions (see [28] and [36]):

(i) if $|x| \le |y|$, $y \in E$ and $x \in L^0$, then $x \in E$ and $\|x\|_E \le \|y\|_E$,
(ii) there exists a function $x \in E$ which is strictly positive $\mu-$a.e. in T.

It is obvious that any Köthe space E is a Banach lattice. Let us define for E the modulus $\widehat{\delta}_{m,E} : [0,1] \to [0,1]$ by the formula

$$\widehat{\delta}_{m,E}(\varepsilon) = \inf\{1 - \|x - x\chi_A\|_E : x \in X_+, \|x\|_E = 1, A \in \Sigma, \|x\chi_A\|_E \ge \varepsilon\}.$$

The characteristic of monotonicity $\widehat{\varepsilon}_{0,m}(E)$ corresponding to the modulus $\widehat{\delta}_{m,E}$ is defined by

$$\widehat{\varepsilon}_{0,m}(E) = \sup\left\{\varepsilon \in [0,1] : \widehat{\delta}_{m,E}(\varepsilon) = 0\right\} = \inf\left\{\varepsilon \in [0,1] : \widehat{\delta}_{m,E}(\varepsilon) > 0\right\}$$

(where $\inf \emptyset := 1$). By [11, Corollary 2.9], we have

$$\varepsilon_{0,m}(E) = \widehat{\varepsilon}_{0,m}(E) = \lim_{\varepsilon \to 1^-} \sup\left\{\|x\chi_{A'}\|_E : x \in S_+(E), A \in \Sigma, \|x\chi_A\|_E \ge \varepsilon\right\}. \quad (3)$$

For more informations on the monotonicity properties and the coefficient of monotonicity we refer to [5, 6, 11, 16, 17, 18, 21, 23, 31, 33].

Now we will recall the definition of Orlicz-Lorentz function spaces. These spaces were introduced by A. Kamińska (see [24], [25] and [26]) at the beginning of 1990s. Her investigations gave an impulse to further investigations of the spaces and their results have been published among others in the papers [19, 20, 22, 34, 35, 13, 14, 27, 8, 9, 10, 12].

Let in the following $L^0 = L^0([0,\gamma))$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0,\gamma)$, where $\gamma \le \infty$. Denoting the Lebesgue measure by m, for any $x \in L^0$ we define its distribution function $\mu_x : [0,+\infty) \to [0,\gamma]$ by

$$\mu_x(\lambda) = m\{t \in [0,\gamma) : |x(t)| > \lambda\}$$

(see [1], [30] and [36]) and its nonincreasing rearrangement $x^* : [0,\gamma) \to [0,\infty]$ as

$$x^*(t) = \inf\{\lambda \ge 0 : \mu_x(\lambda) \le t\}$$

(under the convention $\inf \emptyset = \infty$). We say that two functions $x, y \in L^0$ are equimeasurable if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \ge 0$. It is obvious that equimeasurability of x and y gives the equality $x^* = y^*$.

Let (R_1, Σ_1, μ_1) and (R_2, Σ_2, μ_2) be complete σ-finite measure spaces. A map σ from R_1 into R_2 is called a measure preserving transformation if for any Σ_2-measurable subset A of R_2, the set $\sigma^{-1}(A) = \{t \in R_1 : \sigma(t) \in A\}$ is a Σ_1-measurable subset of R_1 and $\mu_1(\sigma^{-1}(A)) = \mu_2(A)$ (see [1]). It is well known that a measure preserving transformation induces equimeasurability, that is, if σ is a measure preserving transformation, then x and $x \circ \sigma$ are equimeasurable functions. The converse is false (see [1]).

Recall that a Köthe space E, where $E \subset L^0$, is called a symmetric space if E is rearrangement invariant which means that if $x \in E$, $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\| = \|y\|$ (see [7]). For basic properties of symmetric spaces we refer to [1], [30] and [36].

In the whole paper Φ denotes an Orlicz function (see [4, 37, 38]), that is, $\Phi : [-\infty, \infty] \to [0, \infty]$ (our definition is extended from R into R^e by assuming $\Phi(-\infty) = \Phi(\infty) = \infty$) and Φ is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ (that is, in particular, $\lim_{u \to (b(\Phi))^-} \Phi(u) = \Phi(b(\Phi))$, where $b(\Phi) = \sup\{u \geq 0\colon \Phi(u) < \infty\}$) and not identically equal to zero on $(-\infty, \infty)$.

We say that an Orlicz function Φ satisfies condition Δ_2 for all $u \in \mathbb{R}_+$ (respectively, at infinity) if there is $K > 0$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \mathbb{R}$ (respectively, for all $u \in \mathbb{R}$ satisfying $|u| \geq u_0$ with some $u_0 > 0$ such that $\Phi(u_0) < \infty$). We write then $\Phi \in \Delta_2(\mathbb{R}_+)$ ($\Phi \in \Delta_2(\infty)$), respectively.

In the following we will use the well known parameter $a(\Phi)$ for the Orlicz function Φ defined by $a(\Phi) := \sup\{u > 0\colon \Phi(u) = 0\}$.

Let $\omega : [0, \gamma) \to R_+$ be a non-increasing and locally integrable function (not identically 0), called a weight function. We say that a weight function ω is regular if there exists $\eta > 0$ such that

$$\int_0^{2t} \omega(s)ds \geq (1 + \eta) \int_0^t \omega(s)ds$$

for any $t \in [0, \gamma/2)$ (see [26, 18]). Note that if the weight function ω is regular, then $\int_0^\infty \omega(t)dt = \infty$ if $\gamma = \infty$ and $\gamma_0 := \sup\{t \in [0, \gamma) : \omega(t) > 0\} > \gamma/2$ whenever $\gamma < \infty$.

Given any Orlicz function Φ and a weight function ω, we define on L^0 the convex modular

$$\mathrm{I}_{\Phi,\omega}(x) = \int_0^\gamma \Phi(x^*(t))\omega(t)dt,$$

(see [24, 19]), and the Orlicz-Lorentz space

$$\Lambda_{\Phi,\omega} = \Lambda_{\Phi,\omega}([0, \gamma)) = \{x \in L^0 : \mathrm{I}_{\Phi,\omega}(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

(see [24, 19]), which becomes a Banach symmetric space under the Luxemburg norm

$$\|x\|_{\Lambda_{\Phi,\omega}} = \inf\{\lambda > 0 : \mathrm{I}_{\Phi,\omega}(x/\lambda) \leq 1\}.$$

In our investigations we will apply the lemmas that are presented below. We omitted their proofs because using convexity of the modular $I_{\Phi,\omega}$, Lemmas 1.2 and 1.3 can be proved analogously as their analogues in Orlicz spaces and the first lemma has been already proved by A Kamińska.

Lemma 1.1 ([24], Lemma 3.2). *Assume that $|x(t)| < |y(t)|$ for $t \in A \subset [0, \gamma)$, where $\mu(A) > 0$ and $|x(t)| \leq |y(t)|$ for m-a.e. $t \in [0, \gamma)$. If $\mu_x(\lambda) < \infty$ for any $\lambda > 0$, then $x^*(t) < y^*(t)$ for $t \in B$, where $B \subset [0, \gamma)$ has a positive measure.*

Lemma 1.2 (cf. [11], Lemma 3.1). *Let Φ be an Orlicz function with $a(\Phi) > 0$ and satisfying condition $\Delta_2(\infty)$ and let $c \in (a(\Phi), +\infty)$. Then for any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that if $x \in \Lambda_{\Phi,\omega}$, $|x(t)| \geq c$ for m-a.e. $t \in [0,\gamma)$ and $\mathrm{I}_{\Phi,\omega}(x) \leq \delta(\varepsilon)$, then $\|x\|_{\Lambda_{\Phi,\omega}} \leq \varepsilon$.*

Lemma 1.3 (cf. [16], Lemma 4). *Let $\gamma < \infty$ and $\Phi \in \Delta_2(\infty)$. Then for any $\varepsilon \in (0,1)$ there exists $p(\varepsilon) \in (0,1)$ such that $\|x\|_{\Lambda_{\Phi,\omega}} \leq 1 - p(\varepsilon)$ whenever $\mathrm{I}_{\Phi,\omega}(x_n) \leq 1 - \varepsilon$.*

2. Results

We start with the following

Theorem 2.1. *Let $\gamma = \infty$. If the Orlicz function Φ satisfies condition $\Delta_2(\mathbb{R})$ and the weight function ω is regular, then $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 0$. In the opposite case, $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$.*

Proof. It is well known that if $\Phi \in \Delta_2(\mathbb{R})$ and ω is regular, then the space $\Lambda_{\Phi,\omega}$ is uniformly monotone (see [19, 15]), whence $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 0$.

Notice also that if $\Phi \notin \Delta_2(\mathbb{R})$ or $\int_0^\infty \omega(t)dt < \infty$, then $\Lambda_{\Phi,\omega}$ contains an order linearly isometric copy of ℓ^∞ (see [24]). Consequently, $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 0$ and $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$.

Assume now that $\Phi \in \Delta_2(\mathbb{R})$ and $\int_0^\infty \omega(t)dt = \infty$, but ω is not regular. Then there exists a sequence $(t_n)_{n=1}^\infty$ of positive numbers such that

$$0 < \int_0^{2t_n} \omega(t)dt \leq \left(1 + \frac{1}{n}\right) \int_0^{t_n} \omega(t)dt. \tag{4}$$

Since the Orlicz function Φ satisfies condition $\Delta_2(\mathbb{R})$, Φ takes only finite falues and, in consequence, we can find a sequence of positive numbers $(u_n)_{n=1}^\infty$ such that

$$\int_0^{2t_n} \Phi(u_n)\omega(t)dt = 1$$

for any $n \in \mathbb{N}$. Defining $x_n := u_n\chi_{[0,2t_n)}$ and $y_n := u_n\chi_{[0,t_n)}$, we get $0 \leq y_n \leq x_n$, $\|x_n\|_{\Lambda_{\Phi,\omega}} = 1$ and, by inequality (4), $\frac{n}{n+1} \leq \mathrm{I}_{\Phi,\omega}(y_n) \leq \|y_n\|_{\Lambda_{\Phi,\omega}} \leq 1$ for all $n \in \mathbb{N}$. Since $(x_n - y_n)^* = y_n^*$, we have also that $\frac{n}{n+1} \leq \|x_n - y_n\|_{\Lambda_{\Phi,\omega}} \leq 1$ for any $n \in \mathbb{N}$. Therefore, $\delta_{m,\Lambda_{\Phi,\omega}}(\varepsilon) = 0$ for any $\varepsilon \in [0,1)$, whence $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$. $\square$

Remark 2.2. It is worth noticing here that if $\Phi \in \Delta_2(\mathbb{R})$ and $\int_0^\infty \omega(t)dt = \infty$, then $\Lambda_{\Phi,\omega}$ is strictly monotone (even if ω is not regular) (see [19, 15]), whence $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1$.

Theorem 2.3. *Suppose that $0 < \gamma < \infty$, Φ is an Orlicz function satisfying condition $\Delta_2(\infty)$ and define $\gamma_0 := \sup\{t \in [0,\gamma) : \omega(t) > 0\}$, where ω is a weight function on $[0,\gamma)$. Let us denote by $u(\Phi,\omega)$ the positive number satisfying the equality $\Phi(u(\Phi,\omega)) \int_0^{\gamma_0} \omega(t)dt = 1$. Moreover, in the case when $\gamma_0 < \gamma$, let us define the positive number $v(\Phi,\omega)$ by the formula $\Phi(v(\Phi,\omega)) \int_0^{\gamma-\gamma_0} \omega(t)dt = 1$. Then the following assertions are true:*

(i) *If* $\gamma_0 = \gamma$, *then*

$$\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \frac{a(\Phi)}{u(\Phi,\omega)}.$$

(ii) *If* $\gamma_0 \in (\frac{1}{2}\gamma, \gamma)$, *then*

$$\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \max\left(\frac{a(\Phi)}{u(\Phi,\omega)}, \frac{u(\Phi,\omega)}{v(\Phi,\omega)}\right).$$

(iii) *If* $\gamma_0 \in (0, \frac{1}{2}\gamma]$, *then* $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 0$.

Proof. (i). At the beginning, note that if $a(\Phi) = 0$, then the space $\Lambda_{\Phi,\omega}$ is strictly monotone (see [19, 15]), whence $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1$.

Now assume that $a(\Phi) > 0$. Since Φ satisfies condition $\Delta_2(\infty)$, Φ takes only finite values, so one can find $u(\Phi,\omega) > 0$ such that

$$\int_0^{\gamma_0} \Phi(u(\Phi,\omega))\omega(t)dt = 1. \tag{5}$$

Let x and y be arbitrary elements from the positive cone $(\Lambda_{\Phi,\omega})_+$ such that $y \le x$ and $\|y\|_{\Lambda_{\Phi,\omega}} = \|x\|_{\Lambda_{\Phi,\omega}} = 1$. Then $0 \le y^*(t) \le x^*(t)$ for any $t \in [0,\gamma)$ (see Proposition 1.7, page 41 in [1]) and, by condition $\Delta_2(\infty)$, $\mathrm{I}_{\Phi,\omega}(y) = \mathrm{I}_{\Phi,\omega}(x) = 1$. Hence we obtain that $y^*(t) = x^*(t)$ for any $t \in [0, t_0)$, where $t_0 = \sup\{t \in [0,\gamma) : x^*(t) > a(\Phi)\}$. Defining $A = \{t \in [0,\gamma) : x(t) > a(\Phi)\}$ and $B = \{t \in [0,\gamma) : y(t) > a(\Phi)\}$, by $0 \le y \le x$, we have $B \subset A$, whence by $y^*\chi_{[0,t_0)} = x^*\chi_{[0,t_0)}$ we get $m(A \backslash B) = 0$. We also have $\mu_{x\chi_A}(\lambda) = \mu_x(\lambda) = \mu_y(\lambda) = \mu_{y\chi_A}(\lambda)$ for any $\lambda \ge x^*(t_0)$, whence $(x\chi_A)^* = x^*\chi_{[0,t_0)} = y^*\chi_{[0,t_0)} = (y\chi_A)^*$. Consequently, by Lemma 1.1, we obtain that $y(t) = x(t)$ for m-a.e. $t \in A$. Moreover, $x(t) - y(t) \le x(t) \le a(\Phi)$ for m-a.e. $t \in [0,\gamma)\backslash A$. Therefore, by (5),

$$\|x - y\|_{\Lambda_{\Phi,\omega}} \le \|a(\Phi)\|_{\Lambda_{\Phi,\omega}} = \frac{a(\Phi)}{u(\Phi,\omega)}. \tag{6}$$

Let now (t_n) be a decreasing sequence of numbers from the interval $(0,\gamma)$ such that $\lim_{n\to\infty} t_n = 0$. Since by condition $\Delta_2(\infty)$, the Orlicz function Φ takes only finite values, we can find an increasing sequence (a_n) in $(0,\infty)$ such that $\int_0^{t_n} \Phi(a_n)\omega(t)dt = 1$. Defining $y_n := a_n\chi_{[0,t_n)}$ and $x_n := a_n\chi_{[0,t_n)} + a(\Phi)\chi_{[t_n,\gamma)}$, we get $0 \le y_n \le x_n$, $y_n^* = y_n$, $x_n^* = x_n$ and $\mathrm{I}_{\Phi,\omega}(x_n) = \|x_n\|_{\Lambda_{\Phi,\omega}} = \mathrm{I}_{\Phi,\omega}(y_n) = \|y_n\|_{\Lambda_{\Phi,\omega}} = 1$ for all $n \in \mathbb{N}$. We also have the equalities $x_n - y_n = a(\Phi)\chi_{[t_n,\gamma)}$ and $(x_n - y_n)^* = a(\Phi)\chi_{[0,\gamma - t_n)}$ for any $n \in \mathbb{N}$. Since

$$\begin{aligned}\lim_{n\to\infty} I_{\Phi,\omega}\left(\frac{x_n - y_n}{\frac{a(\Phi)}{u(\Phi,\omega)}}\right) &= \lim_{n\to\infty} \Phi(u(\Phi,\omega)) \int_0^{\gamma - t_n} \omega(t)dt \\ &= \Phi(u(\Phi,\omega)) \int_0^{\gamma} \omega(t)dt = 1,\end{aligned}$$

we obtain that $\lim_{n\to\infty} \|x_n - y_n\|_{\Lambda_{\Phi,\omega}} = \frac{a(\Phi)}{u(\Phi,\omega)}$, whence

$$\begin{aligned}&\sup\{\|x-y\|_{\Lambda_{\Phi,\omega}} : 0 \le y \le x, \|x\|_{\Lambda_{\Phi,\omega}} = \|y\|_{\Lambda_{\Phi,\omega}} = 1\}\\ \ge\ &\lim_{n\to\infty} \|x_n - y_n\|_{\Lambda_{\Phi,\omega}} = \frac{a(\Phi)}{u(\Phi,\omega)}.\end{aligned} \tag{7}$$

Inequalities (6) and (7) give the equality $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \frac{a(\Phi)}{u(\Phi,\omega)}$.

(ii). First suppose $a(\Phi) = 0$. For $x := u(\Phi,\omega)\chi_{[0,\gamma)}$ and $y := u(\Phi,\omega)\chi_{[0,\gamma_0)}$, we have the equalities $x-y = u(\Phi,\omega)\chi_{[\gamma_0,\gamma)}$ and $(x-y)^* = u(\Phi,\omega)\chi_{[0.\gamma-\gamma_0)}$. Obviously, $0 \le y \le x$, $\|x\|_{\Lambda_{\Phi,\omega}} = \mathrm{I}_{\Phi,\omega}(x) = 1$ and $\|y\|_{\Lambda_{\Phi,\omega}} = \mathrm{I}_{\Phi,\omega}(y) = 1$. By $\Phi \in \Delta_2(\infty)$, we can find $v(\Phi,\omega) > u(\Phi,\omega)$ such that

$$\int_0^{\gamma-\gamma_0} \Phi(v(\Phi,\omega))\omega(t)dt = 1.$$

We have $\|x-y\|_{\Lambda_{\Phi,\omega}} = \frac{u(\Phi,\omega)}{v(\Phi,\omega)}$, whence

$$\delta_{m,\Lambda_{\Phi,\omega}}(1) \le 1 - \frac{u(\Phi,\omega)}{v(\Phi,\omega)}. \tag{8}$$

Simultaneously, for any x and y such that $0 \le y \le x$ and $\|y\|_{\Lambda_{\Phi,\omega}} = \|x\|_{\Lambda_{\Phi,\omega}} = 1$, analogously as in (i), we get $0 \le y^* \le x^*$ and $\mathrm{I}_{\Phi,\omega}(x) = \mathrm{I}_{\Phi,\omega}(y) = 1$. Hence we get the equality $y^*(t) = x^*(t)$ for every $t \in [0,\gamma_0)$. Defining $C = \{t \in [0,\gamma) : x(t) > x^*(\gamma_0)\}$ and $D = \{t \in [0,\gamma) : y(t) > x^*(\gamma_0)\}$, analogously as in (i), we obtain that $m(C\backslash D) = 0$ and $y(t) = x(t)$ for m-a.e. $t \in C$. We have $c := m(C) \in [0,\gamma_0]$. In the case when $c < \gamma_0$, by $0 \le y \le x$ and $y^*\chi_{[0,\gamma_0)} = x^*\chi_{[0,\gamma_0)}$, we obtain $F \subset E$ and $m(F) \ge \gamma_0 - c$, where $E = \{t \in [0,\gamma) : x(t) = x^*(\gamma_0)\}$ and $F = \{t \in [0,\gamma) : y(t) = x^*(\gamma_0)\}$. So, defining $G = \{t \in [0,\gamma) : x(t) - y(t) > 0\}$, we get $G \subset [0,\gamma)\backslash(C \cup F)$ and consequently

$$m(G) \le \gamma - c - (\gamma_0 - c) = \gamma - \gamma_0.$$

Moreover, $x(t) - y(t) \le x(t) \le x^*(\gamma_0) \le u(\Phi,\omega)$ for m-a.e. $t \in G$. Indeed, in the case when $x^*(\gamma_0) > u(\Phi,\omega)$, we obtain

$$\int_0^{\gamma_0} \Phi(x^*(t))\omega(t)dt > \int_0^{\gamma_0} \Phi(u(\Phi,\omega))\omega(t)dt = 1, \tag{9}$$

which gives a contradiction with $\|x\|_{\Lambda_{\Phi,\omega}} = 1$. Therefore

$$\|x-y\|_{\Lambda_{\Phi,\omega}} \le \|u(\Phi,\omega)\chi_{[0,\gamma-\gamma_0)}\|_{\Lambda_{\Phi,\omega}} = \frac{u(\Phi,\omega)}{v(\Phi,\omega)},$$

and finally (see (8)) $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \frac{u(\Phi,\omega)}{v(\Phi,\omega)}$.

In the case when $a(\Phi) > 0$, the set A of all pairs (x,y) in $\Lambda_{\Phi,\omega}$ such that $0 \le y \le x$ and $\|y\|_{\Lambda_{\Phi,\omega}} = \|x\|_{\Lambda_{\Phi,\omega}} = 1$ can be divided into two disjoint subsets A_1 and A_2

in the following way: a pair $(x,y) \in A_1$ if $x^*(t) > a(\Phi)$ for all $t \in [0,\gamma_0)$ and $(x,y) \in A_2$ in the opposite case. Proceeding for the set A_1 analogously as in the proof of (ii) when $a(\Phi) = 0$ and for the set A_2 analogously as in the proof of (i), we obtain the equality $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \max\left(\frac{a(\Phi)}{u(\Phi,\omega)}, \frac{u(\Phi,\omega)}{v(\Phi,\omega)}\right)$.

(iii). As in (ii) we define $x := u(\Phi,\omega)\chi_{[0,\gamma)}$ and $y := u(\Phi,\omega)\chi_{[0,\gamma_0)}$. We have $\|x\|_{\Lambda_{\Phi,\omega}} = \mathrm{I}_{\Phi,\omega}(x) = 1$ and $\|y\|_{\Lambda_{\Phi,\omega}} = \mathrm{I}_{\Phi,\omega}(y) = 1$. Simultaneously, by $\gamma_0 \le \frac{1}{2}\gamma$, we obtain that $\|x-y\|_{\Lambda_{\Phi,\omega}} = \mathrm{I}_{\Phi,\omega}(x-y) = 1$, whence $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 0$ which finishes the proof of (iii). $\square$

The example presented below shows how the value $\delta_{m,\Lambda_{\Phi,\omega}}(1)$ depends on γ_0, γ and $a(\Phi)$.

Example 2.4. Let $\gamma > 0$, $\Phi(u) = \max\{0, u-1\}$ for any $u \ge 0$, $\omega(t) = 1$ for any $t \in [0, \min(1,\gamma))$ and $\omega(t) = 0$ for any $t \in [1,\gamma)$ whenever $\gamma > 1$. Then $a(\Phi) = 1$ and $u(\Phi,\omega) = \max((\frac{1}{\gamma}+1), 2)$. By Theorem 2.3 ($(iii)$ and (i)), we get the equalities $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 0$ for $\gamma \ge 2$ and $\delta_{m,\Lambda_{\Phi,\omega}}(1) = \frac{1}{\gamma+1}$ when $\gamma \le 1$. Assume now that $\gamma \in (1,2)$. Then $v(\Phi,\omega) = \frac{\gamma}{\gamma-1}$ and, by Theorem 2.3 (ii), we have $\delta_{m,\Lambda_{\Phi,\omega}}(1) = \frac{1}{2}$ for $\gamma \in (1,\frac{4}{3})$ and $\delta_{m,\Lambda_{\Phi,\omega}}(1) = \frac{2-\gamma}{\gamma}$ for $\gamma \in (\frac{4}{3},2)$.

Theorem 2.5. *Let $0 < \gamma < \infty$ and the numbers γ_0, $u(\Phi,\omega)$ and $v(\Phi,\omega)$ be defined as in Theorem 2.3. Then the following statements hold true:*

(i) *If $\Phi \in \Delta_2(\infty)$, $\gamma_0 = \gamma$ and the weight function ω is regular, then*

$$\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = \frac{a(\Phi)}{u(\Phi,\omega)}.$$

(ii) *If $\Phi \in \Delta_2(\infty)$, $\gamma_0 \in (\frac{1}{2}\gamma,\gamma)$ and the weight function ω is regular, then*

$$\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = \max\left(\frac{a(\Phi)}{u(\Phi,\omega)}, \frac{u(\Phi,\omega)}{v(\Phi,\omega)}\right).$$

(iii) *If $\Phi \notin \Delta_2(\infty)$ or the weight function ω is not regular, then $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$.*

Proof. At the beginning let us recall that, by (2) and (3), we have

$$\begin{aligned}\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) &= \lim_{\varepsilon\to 1^-} \sup\left\{\|x\chi_{A'}\|_{\Lambda_{\Phi,\omega}} : x \in S_+(\Lambda_{\Phi,\omega}), A \in \Sigma, \|x\chi_A\|_{\Lambda_{\Phi,\omega}} \ge \varepsilon\right\} \\ &\ge 1 - \delta_{m,\Lambda_{\Phi,\omega}}(1).\end{aligned}$$

We will show below that the assumption $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) > 1 - \delta_{m,\Lambda_{\Phi,\omega}}(1)$ leads to a contradiction.

(i). Note that if $a(\Phi) = 0$, then $\Lambda_{\Phi,\omega}$ is uniformly monotone (see [19]), whence $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 0$.

Let us assume that $a(\Phi) > 0$ and there exist a sequence (x_n) of elements from $S_+(\Lambda_{\Phi,\omega})$ and a sequence (A_n) of measurable subsets of the interval $[0,\gamma)$ such that $\|x_n\chi_{A_n}\|_{\Lambda_{\Phi,\omega}} \to 1$ and

$$\left\|x_n\chi_{A'_n}\right\|_{\Lambda_{\Phi,\omega}} \geq \frac{a(\Phi)}{u(\Phi,\omega)} + \varepsilon \tag{10}$$

for every $n \in \mathbb{N}$, where $\varepsilon > 0$. We can find $\theta = \theta(\varepsilon) > 0$ such that

$$\left\|(a(\Phi)+\theta)\chi_{[0,\gamma)}\right\|_{\Lambda_{\Phi,\omega}} \leq \frac{a(\Phi)}{u(\Phi,\omega)} + \frac{\varepsilon}{2}. \tag{11}$$

Let us define for all $n \in \mathbb{N}$ the sets

$$B_n = \{t \in [0,\gamma) \colon x(t) > a(\Phi) + \theta\}.$$

Since

$$x_n\chi_{A'_n} = x_n\chi_{A'_n\cap B_n} + x_n\chi_{A'_n\setminus B_n} \leq x_n\chi_{A'_n\cap B_n} + (a(\Phi)+\theta)\chi_{[0,\gamma)},$$

by inequalities (10) and (11) and the triangle inequality for the norm, we get that $\|x_n\chi_{A'_n\cap B_n}\|_{\Lambda_{\Phi,\omega}} \geq \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. By Lemma 1.2, there is $\delta = \delta(\varepsilon) > 0$ such that

$$I_{\Phi,\omega}\left(x_n\chi_{A'_n\cap B_n}\right) \geq \delta(\varepsilon)$$

for all $n \in \mathbb{N}$. Simultaneously, by Lemma 1.3, we obtain

$$I_{\Phi,\omega}\left(x_n\chi_{A_n}\right) \to 1 \quad \text{as } n \to \infty.$$

Let us define the new Orlicz function

$$\Psi(u) = \Phi(|u| + a(\Phi)).$$

Then $\Psi \in \Delta_2(\infty)$ and Ψ vanishes only at zero. Since the weight function ω is regular and $\gamma_0 = \gamma$, the Orlicz-Lorentz space $L_{\Psi,\omega}$ is uniformly monotone. Defining a sequence (y_n) by the formula

$$y_n(t) = \max\{x_n(t) - a(\Phi), 0\}$$

for any $t \in [0,\gamma)$ and all $n \in \mathbb{N}$, it is easy to show that $y_n^*(t) = x_n^*(t) - a(\Phi)$ for any $t \in [0,c_n)$ and all $n \in \mathbb{N}$, where $c_n = m(\{t \in [0,\gamma) \colon x_n(t) > a(\Phi)\})$. Therefore

$$\begin{aligned} I_{\Psi,\omega}(y_n) &= \int_0^{\gamma} \Psi(y_n^*(t))\omega(t)dt = \int_0^{c_n} \Psi(x_n^*(t) - a(\Phi))\omega(t)dt \\ &= \int_0^{c_n} \Phi((x_n^*(t) - a(\Phi)) + a(\Phi))\omega(t)dt = 1, \end{aligned}$$

whence $\|y_n\|_{\Lambda_{\Psi,\omega}} = 1$ for all $n \in \mathbb{N}$. Obviously $y_n \geq 0$ for all $n \in \mathbb{N}$. In a similar way as above, we get that $\lim_{n\to\infty} I_{\Psi,\omega}(y_n\chi_{A_n}) = 1$ and $I_{\Psi,\omega}(y_n\chi_{A'_n\cap B_n}) \geq$

$\delta(\varepsilon)$. Therefore, $\|y_n\chi_{A_n}\|_{\Lambda_{\Psi,\omega}} \to 1$ and $\|y_n\chi_{A_n'}\|_{\Lambda_{\Psi,\omega}} \geq \delta(\varepsilon)$, which contradicts the uniform monotonicity of the Orlicz-Lorentz space $\Lambda_{\Psi,\omega}$.

(ii). First suppose that $a(\Phi) = 0$. Analogously as in (i) we will assume that there exist an $\varepsilon > 0$, a sequence (x_n) of elements from the positive cone X_+ and a sequence (A_n) of measurable subsets of the interval $[0,\gamma)$ such that $\|x_n\|_{\Lambda_{\Phi,\omega}} = 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \|x_n\chi_{A_n}\|_{\Lambda_{\Phi,\omega}} = 1$ and

$$\|x_n\chi_{A_n'}\|_{\Lambda_{\Phi,\omega}} \geq \frac{u(\Phi,\omega)}{v(\Phi,\omega)} + \varepsilon \tag{12}$$

for all $n \in \mathbb{N}$. Let us define for any $n \in \mathbb{N}$ the set $B_n \subset [0,\gamma)$, according to the case, in the following way. If $x_n^*(t) > x_n^*(\gamma_0)$ for every $t < \gamma_0$, then we define

$$B_n = \{t \in [0,\gamma) \colon x_n(t) > x_n^*(\gamma_0)\}.$$

In case when $x_n^*(t) = x_n^*(\gamma_0)$ for some $t < \gamma_0$ and $x_n^*(s) < x_n^*(\gamma_0)$ for any $s > \gamma_0$, we put

$$B_n = \{t \in [0,\gamma) \colon x_n(t) \geq x_n^*(\gamma_0)\}.$$

Finally, if there exist $t < \gamma_0 < s$ such that $x_n^*(t) = x_n^*(s)$, then denoting

$$A_n^1 = \{t \in A_n \colon x_n(t) = x_n^*(\gamma_0)\},$$

we decompose

$$B_n = B_n^1 \cup B_n^2,$$

where $B_n^1 = \{t \colon |x_n(t)| > x_n^*(\gamma_0)\}$, $B_n^2 = \{t \colon |x_n(t)| = x_n^*(\gamma_0)\}$, $m(B_n^2) = \gamma_0 - m(B_n^1)$ and if $m(A_n^1) \leq \gamma_0 - m(B_n^1)$ then $A_n^1 \subset B_n^2$ and $B_n^2 \subset A_n^1$ in the opposite case. It is easy to see that $m(B_n) = \gamma_0$ and $x_n^*(t) = (x_n\chi_{B_n})^*(t)$ for $t \in [0,\gamma_0)$. Let us assume, passing to a subsequence if necessary, that $\|x_n\chi_{A_n\cap B_n}\|_{\Lambda_{\Phi,\omega}} \to 1$ as $n \to \infty$. Since

$$x_n\chi_{A_n'} \leq x_n\chi_{B_n\setminus A_n} + x_n\chi_{A_n'\setminus B_n},$$

from inequality (12) and the triangle inequality for the norm, we get that $\|x_n\chi_{B_n\setminus A_n}\|_{\Lambda_{\Phi,\omega}} \geq \varepsilon$ for any $n \in \mathbb{N}$. Let σ_n $(n \in \mathbb{N})$ be a measure preserving transformation from $[0,\gamma_0)$ to B_n (see Theorem 17, page 410 in [39]). Defining $y_n = x_n \circ \sigma_n$ and $C_n = \sigma_n^{-1}(A_n \cap B_n)$ $(n \in \mathbb{N})$ we have that $y_n \in \Lambda_{\Phi,\omega}([0,\gamma_0))$ and $m(C_n) = m(A_n \cap B_n)$. Since $y_n^*(t) = x_n^*(t)$, $(y_n\chi_{C_n})^*(t) = (x_n\chi_{A_n\cap B_n})^*(t)$ and $(y_n\chi_{[0,\gamma_0)\setminus C_n})^*(t) = (x_n\chi_{B_n\setminus A_n})^*(t)$ for all $t \in [0,\gamma_0)$ and $n \in \mathbb{N}$, we have that $\|y_n\| = 1$, $\|y_n\chi_{[0,\gamma_0)\setminus C_n}\| \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\|y_n\chi_{C_n}\| \to 1$ as $n \to \infty$, where $\|\cdot\|$ denotes the norm in the Orlicz-Lorentz space $\Lambda_{\Phi,\omega}([0,\gamma_0))$, which contradicts uniform monotonicity of the space $\Lambda_{\Phi,\omega}([0,\gamma_0))$.

Now let us assume that there exists $\delta \in (0,1)$ such that $\|x_n\chi_{A_n\cap B_n}\|_{\Lambda_{\Phi,\omega}} \leq 1-\delta$ for all $n \in \mathbb{N}$. Then $\mathrm{I}_{\Phi,\omega}(x_n\chi_{A_n\cap B_n}) \leq 1-\delta$ and, by Lemma 1.3, $\mathrm{I}_{\Phi,\omega}(x_n\chi_{A_n}) \to 1$ as $n \to \infty$. Since, by the definition of the sets B_n, for any $n \in \mathbb{N}$ we obtain

$$\begin{aligned} x(t) \geq x^*(\gamma_0) \ \text{ for } m\text{-a.e. } t \in B_n \\ \text{and } \ x(t) \leq x^*(\gamma_0) \ \text{ for } m\text{-a.e. } t \in A_n\setminus B_n, \end{aligned} \tag{13}$$

denoting $b_n = m(A_n \cap B_n)$, we have

$$\begin{aligned} 1 &= \int_0^{\gamma} \Phi(x_n^*(t))\omega(t)dt = \int_0^{\gamma_0} \Phi\left((x_n\chi_{B_n})^*(t)\right)\omega(t)dt \\ &\geq \int_0^{b_n} \Phi\left((x_n\chi_{B_n\cap A_n})^*(t)\right)\omega(t)dt + \int_{b_n}^{\gamma_0} \Phi\left(\left(x_n\chi_{B_n\setminus A_n}\right)^*(t-b_n)\right)\omega(t)dt \\ &\geq \int_0^{b_n} \Phi\left((x_n\chi_{B_n\cap A_n})^*(t)\right)\omega(t)dt + \int_{b_n}^{\gamma_0} \Phi\left(\left(x_n\chi_{A_n\setminus B_n}\right)^*(t-b_n)\right)\omega(t)dt \\ &= \int_0^{\gamma_0} \Phi\left((x_n\chi_{A_n})^*(t)\right)\omega(t)dt \to 1 \end{aligned} \tag{14}$$

as $n \to \infty$. Therefore

$$\int_{b_n}^{\gamma_0} \Phi\left(\left(x_n\chi_{A_n\setminus B_n}\right)^*(t-b_n)\right)\omega(t)dt \geq \frac{\delta}{2} \tag{15}$$

for all $n \in \mathbb{N}$ starting from some $n_1 \in \mathbb{N}$. Hence, by the inequality $x_n\chi_{A_n\setminus B_n}(t) \leq x^*(\gamma_0) \leq u(\Phi,\omega)$ for m-a.e. $t \in A_n\setminus B_n$ and any $n \in \mathbb{N}$ (see (13) and (9)), we have $b_n \leq b$ for $n \geq n_1$, where

$$\int_b^{\gamma_0} \Phi(u(\Phi,\omega))\omega(t)dt = \frac{\delta}{2}.$$

We have $\|u(\Phi,\omega)\chi_{[0,(\gamma-\gamma_0))}\|_{\Lambda_{\Phi,\omega}} = \frac{u(\Phi,\omega)}{v(\Phi,\omega)}$. Simultaneously, by the order continuity of $\Lambda_{\Phi,\omega}$ (see [19]) one can find $a_0 \in (0, \min(\gamma_0 - \frac{\gamma}{2}, \frac{b}{2}))$ such that $\|u(\Phi,\omega)\chi_{[0,a_0)}\|_{\Lambda_{\Phi,\omega}} \leq \frac{\varepsilon}{4}$. Finally, by the triangle inequality and homogeneity of the norm $\|\cdot\|_{\Lambda_{\Phi,\omega}}$, we obtain that there exists $a \in (0, a_0]$ such that the inequality

$$\|(1+a)u(\Phi,\omega)\chi_{[0,(\gamma-\gamma_0)+a)}\|_{\Lambda_{\Phi,\omega}} \leq \frac{u(\Phi,\omega)}{v(\Phi,\omega)} + \frac{\varepsilon}{2} \tag{16}$$

is satisfied. Let $u_0 \in (0, u(\Phi,\omega))$ be such that

$$\int_0^{\gamma_0} \Phi(u_0)\omega(t)dt \leq \frac{\delta}{2}.$$

We will show that $x_n^*(\gamma_0) \geq u_0$ for any $n \geq n_1$. Indeed, in the opposite case (passing to a subsequence if necessary) we would have (see (13))

$$\int_{b_n}^{\gamma_0} \Phi\left(\left(x_n\chi_{A_n\setminus B_n}\right)^*(t-b_n)\right)\omega(t)dt < \int_0^{\gamma_0} \Phi(u_0)\omega(t)dt \leq \frac{\delta}{2}$$

for any $n \in \mathbb{N}$, which is a contradiction to (15). Now, we claim that

$$m(\mathrm{supp}(x_n \chi_{A_n \backslash B_n})) > m(B_n \backslash A_n) - a = (\gamma_0 - b_n) - a \tag{17}$$

for all $n \in \mathbb{N}$, beginning from some $n_2 \geq n_1$. Indeed, otherwise passing to a subsequence if necessary, we would get

$$\begin{aligned}\int_{\gamma_0 - a}^{\gamma_0} \Phi\left(\left(x_n \chi_{B_n \backslash A_n}\right)^* (t - b_n)\right) \omega(t) dt &\geq \int_{\gamma_0 - a}^{\gamma_0} \Phi(u_0) \omega(t) dt \\ &> \int_{\gamma_0 - a}^{\gamma_0} \Phi\left(\left(x_n \chi_{A_n \backslash B_n}\right)^* (t - b_n)\right) \omega(t) dt = 0,\end{aligned}$$

which is a contradiction to (14). Since $m(\mathrm{supp}(x_n \chi_{A_n \backslash B_n})) \leq \gamma - \gamma_0$, by inequality (17), we have $m(B_n \backslash A_n) < \gamma - \gamma_0 + a$ and, in consequence,

$$b_n > 2\gamma_0 - \gamma - a > \gamma_0 - \frac{\gamma}{2} > 0 \tag{18}$$

for all $n \geq n_2$. Simultaneously,

$$\begin{aligned} m(\mathrm{supp}(x_n \chi_{A'_n})) &= m(\mathrm{supp}(x_n \chi_{B_n \backslash A_n})) + m(\mathrm{supp}(x_n \chi_{A'_n \backslash B_n})) \\ &\leq m(B_n \backslash A_n) + (\gamma - \gamma_0) - m(\mathrm{supp}(x_n \chi_{A_n \backslash B_n})) \\ &< m(\mathrm{supp}(x_n \chi_{A_n \backslash B_n})) + a + (\gamma - \gamma_0) - m(\mathrm{supp}(x_n \chi_{A_n \backslash B_n})) \\ &= (\gamma - \gamma_0) + a, \end{aligned} \tag{19}$$

for the same n.

Observe that if for some $n \geq n_2$ we have $|x_n(t)| \leq (1 + a) u(\Phi, \omega)$ for m-a.e. $t \in B_n \backslash A_n$, then by inequality (16), we get

$$\|x_n \chi_{A'_n}\|_{\Lambda_{\Phi,\omega}} \leq \frac{u(\Phi, \omega)}{v(\Phi, \omega)} + \frac{\varepsilon}{2}$$

for the same n, which gives a contradiction to (12). Therefore, for all $n \geq n_2$, we have

$$m\left\{t \in B_n \backslash A_n \colon\ |x_n(t)| > (1 + a) u(\Phi, \omega)\right\} > 0. \tag{20}$$

Let us denote

$$c_n = \sup\left\{t \geq b_n \colon (x_n \chi_{B_n \backslash A_n})^*(t - b_n) > (1 + a) u(\Phi, \omega)\right\}$$

for all $n \geq n_2$. By virtue of (20), we have that $c_n > b_n$ for all $n \geq n_2$. If there exists $\eta > 0$ such that for any $n \geq n_2$ (passing to a subsequence if necessary), we have

$$\int_{b_n}^{c_n} \left(\Phi\left(\left(x_n \chi_{B_n \backslash A_n}\right)^* (t - b_n)\right) - \Phi(u(\Phi, \omega))\right) \omega(t) dt > \eta,$$

which would be a contradiction to (14) (see (13) and (9)). Therefore

$$\int_{b_n}^{c_n} \left(\Phi\left(\left(x_n\chi_{B_n\setminus A_n}\right)^*(t-b_n)\right) - \Phi(u(\Phi,\omega))\right)\omega(t)dt \to 0$$

as $n\to\infty$, whence by virtue of the definition of c_n we conclude that $c_n - b_n \to 0$. Since $b_n > \gamma_0 - \frac{\gamma}{2} > 0$ for all $n \geq n_2$ (see (18)), we have

$$\int_{b_n}^{c_n} \Phi\left(\left(x_n\chi_{B_n\setminus A_n}\right)^*(t-b_n)\right)\omega(t)dt \to 0. \tag{21}$$

Now we will show in particular that $\|\{t\in B_n\setminus A_n\colon |x_n(t)| > (1+a)u(\Phi,\omega)\}\|_{\Lambda_{\Phi,\omega}}$ is less then $\frac{\varepsilon}{2}$ for all $n\in\mathbb{N}$ starting from some $n_3 \geq n_2$, which consequently will gives that

$$\|x_n\chi_{A_n'}\|_{\Lambda_{\Phi,\omega}} < \frac{u(\Phi,\omega)}{v(\Phi,\omega)} + \varepsilon$$

for all $n \geq n_3$ (see (19) and (16)) and which finally will lead to a contradiction to (12). Let us assume (passing to a subsequence if necessary) that

$$\|\{t\in B_n\setminus A_n\colon |x_n(t)| > (1+a)u(\Phi,\omega)\}\|_{\Lambda_{\Phi,\omega}} \geq \frac{\varepsilon}{2}$$

for any $n \geq n_2$, and let us define

$$z_n(t) = \begin{cases} \left(x_n\chi_{B_n\cap A_n}\right)^*(t) & \text{for } t\in[0,b_n), \\ \left(x_n\chi_{B_n\setminus A_n}\right)^*(t-b_n) & \text{for } t\in[b_n,\gamma_0), \end{cases}$$

$$v_n = z_n\chi_{[0,b_n)\cup[c_n,\gamma_0)},$$

$$w_n = z_n\chi_{[b_n,c_n)}.$$

Then we have $\|z_n\|_{\Lambda_{\Phi,\omega}} = 1$, $z_n - v_n = w_n$, $\|w_n\|_{\Lambda_{\Phi,\omega}} \geq \frac{\varepsilon}{2}$ for all $n \geq n_2$ and from (14) and (21) we get that $\|v_n\|_{\Lambda_{\Phi,\omega}} \to 1$ as $n\to\infty$. But this contradicts the uniform monotonicity of the space $\Lambda_{\Phi,\omega}([0,\gamma_0))$.

Finally, assume that $a(\Phi) > 0$ and there exist an $\varepsilon > 0$, a sequence (x_n) of elements from the positive cone X_+ and a sequence (A_n) of measurable subsets of the interval $[0,\gamma)$ such that $\|x_n\|_{\Lambda_{\Phi,\omega}} = 1$ for all $n\in\mathbb{N}$, $\lim_{n\to\infty}\|x_n\chi_{A_n}\|_{\Lambda_{\Phi,\omega}} = 1$ and

$$\left\|x_n\chi_{A_n'}\right\|_{\Lambda_{\Phi,\omega}} \geq \max\left(\frac{a(\Phi)}{u(\Phi,\omega)}, \frac{u(\Phi,\omega)}{v(\Phi,\omega)}\right) + \varepsilon \tag{22}$$

for all $n\in\mathbb{N}$. If there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k}^*(t) > a(\Phi)$ for any $t < \gamma_0$ and $k\in\mathbb{N}$, then proceeding analogously as above, we get a contradiction. In the opposite case, for any $n\in\mathbb{N}$ there exists $t_n < \gamma_0$ such that $x_n^*(t_n) \leq a(\Phi)$. Then proceeding analogously as in the case (i), we get again a contradiction.

(iii). If $\Phi \notin \Delta_2(\infty)$, then $\Lambda_{\Phi,\omega}$ contains an order linearly isometric copy of ℓ^∞ (see [24]), so $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$. Now assume that $\Phi \in \Delta_2(\infty)$ and ω is not regular. If $\gamma_0 \le \gamma/2$, then by Theorem 2.3*(iii)*, we have $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$. In the opposite case, proceeding analogously as in Theorem 2.1, we get again $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$. □

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