Journal of Convex Analysis
Volume 20 (2013), No. 4, 971–998

SLLN for Double Array of Mixing Dependence Random Sets and Fuzzy Random Sets in a Separable Banach Space*

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Received: May 8, 2012
Revised manuscript received: December 5, 2012

In this paper, we state several convergence results on Painlevé-Kuratowski-Mosco convergence of strong laws of large numbers for double array of $\varphi(\varphi^*)$-mixing dependence random sets or fuzzy random sets in a separable Banach space. We also obtain the SLLN for double array of $\varphi(\varphi^*)$-mixing random sets in Wijsman sense. Some related results in the literature are extended. Illustrative examples are provided.

Keywords: Double array, random variable, random set, fuzzy random set, strong law of large numbers, Painlevé-Kuratowski-Mosco topology, Wijsman topology, stable type p Banach space

2010 Mathematics Subject Classification: 60F15, 60B12, 28B20

1. Introduction

In recent decades, the strong laws of large numbers (SLLN) for sequence of random sets and fuzzy random sets, gave rise to applications in several fields, such as optimization and control, stochastic and integral geometry, mathematical economics, statistics and related fields. Z. Artstein and S. Hart [2] obtained a SLLN for independent identically distributed (i.i.d.) random sets having values in the closed subsets of $\mathbf{R}^d$ and applied it to a problem of optimal allocations. M. L. Puri and D. A. Ralescu [28] were the first to obtain the SLLN for i.i.d. Banach space-valued compact convex random sets. Later, F. Hiai [20] and C. Hess [15, 16] independently proved similar results for random sets in an infinite dimensional Banach space, with respect to the Painlevé-Kuratowski-Mosco convergence. C. Hess [17] also obtained a convergence result on the Wijsman topology of SLLN for sequence of pairwise independent identically distributed (p.i.i.d.) random sets in a separable Banach space. The multivalued SLLN for random sets in Fréchet spaces was

*This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2011.09.

ISSN 0944-6532 / $ 2.50

proved by P. Raynaud de Fitte [30]. Let us mention also the works of H. Attouch - R. Wets [3] and C. Castaing - F. Ezzaki [8] dealing with the laws of large numbers for normal integrands. Further variants of SLLN have been established under various conditions, see R. L. Taylor and H. Inoue [31, 32], R. L. Taylor, A. N. Vidyashankar and Y. Chen [33], H. Inoue [22], K. A. Fu and L. X. Zhang [14]. For more information on the theory of fuzzy sets we refer to M. L. Puri and D. A. Ralescu [28], H. Inoue and R. L. Taylor [23], S. Li and Y. Ogura [24].

Recently, in [23], H. Inoue and R. L. Taylor derive laws of large numbers for exchangeable random sets in the Painlevé-Kuratowski-Mosco topology. Moreover, K. A. Fu [13] obtained some SLLN for sequence of identically distributed (i.d.) random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence in a separable Banach space. The SLLN for these two sequences are derived for the Painlevé-Kuratowski-Mosco topology. In [9, 10], C. Castaing, N. V. Quang and D. X. Giap obtained several convergence results with respect to the Painlevé-Kuratowski-Mosco, Wijsman and slice topologies of strong laws of large numbers for double array of independent (or pairwise independent) random sets in a separable Banach space with or without geometric property. In [29], N. V. Quang and N. T. Thuan proved the SLLN for double arrays of independent set-valued random variables with respect to the Hausdorff distance.

In this work we present various convergence results in the SLLN for *double array* of random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence in a separable Banach space with respect to the Painlevé-Kuratowski-Mosco topology. It is worth mentioning that, apart from the use of Zhang's result (see Theorem 3.1 in [35]), our techniques are significantly different from those developed earlier (see, e.g., [2, 13, 14, 15, 16, 20, 22, 28, 31, 32, 33] and the references therein). We also obtain the SLLN with respect to the Wijsman convergence for double array of $\varphi(\varphi^*)$-mixing random sets. Some related results in the literature are extended. Illustrative examples are provided.

This paper is organized as follows. In Section 2, we introduce some basic notions of random sets, fuzzy random sets, $\varphi(\varphi^*)$-mixing dependence, Painlevé-Kuratowski-Mosco convergence and Wijsman convergence. Section 3 is concerned with Painlevé-Kuratowski-Mosco convergence of SLLN for double array of $\varphi(\varphi^*)$-mixing and i.d. random sets in a separable Banach space. An application of the above result to the SLLN with respect to the Painlevé-Kuratowski-Mosco convergence for double array of $\varphi(\varphi^*)$-mixing fuzzy random sets is also provided (Corollary 3.7). At this point, we stress that the usual techniques developed in the related literature are no longer applicable here because we deal with *double array* of $\varphi(\varphi^*)$-mixing random sets by contrast to H. Inoue and R. L. Taylor [23] and Ke-Ang Fu [13] dealing the Painlevé-Kuratowski-Mosco convergence in SLLN for sequence of exchangeable (resp. $\varphi(\varphi^*)$-mixing) random sets in a separable Banach space via an adaptation of Theorem 3.2 in [20]. Further we establish in Theorem 3.8 the s-lim inf part of Painlevé-Kuratowski-Mosco convergence of the Marcinkiewicz-Zygmund type SLLN for double array of $\varphi(\varphi^*)$-mixing random sets. We also give a counter-example (Example 3.9) showing that the Painlevé-Kuratowski-Mosco convergence is not valid for this class of random sets. In Section 4 we state the Wijsman convergence of SLLN for double array of $\varphi(\varphi^*)$-mixing and i.d. random sets in a separable Banach space (Theorem 4.4). We also obtain lim sup part of Wijsman convergence for the Marcinkiewicz-Zygmund type SLLN for double array of $\varphi(\varphi^*)$-mixing random sets (Theorem 4.5). We also give an example (Example 4.8) showing that the lim inf part for the Wijsman convergence of Marcinkiewicz-Zygmund type SLLN for double array of $\varphi(\varphi^*)$-mixing and i.d. random sets in a separable Banach space is not valid. Our results extend some related results in the literature and shed a new light in the convergence of SLLN for $\varphi(\varphi^*)$-mixing random sets. In particular, Theorems 3.5 and 4.4 contain Theorem 4.3 in Ke-ang Fu [13] and Theorem 3.1 in Zhang [35], respectively.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $(\mathfrak{X}, \|.\|)$ be a real separable Banach space and $\mathfrak{X}^*$ be its topological dual. The closed unit ball of $\mathfrak{X}^*$ is denoted by B^* and the σ-field of all Borel sets of $\mathfrak{X}$ is denoted by $\mathcal{B}(\mathfrak{X})$. In the present paper, $\mathbf{R}$ (resp. $\mathbf{R}^+$) will be denoted the set of real numbers (resp. non-negative real numbers), $\mathbf{N}$ (resp. $\mathbf{Q}$) will be denoted the set of positive integers (resp. rational numbers).

Let $c(\mathfrak{X})$ (resp. $\mathrm{cc}(\mathfrak{X})$) be the family of all nonempty closed (resp. closed convex) subsets of $\mathfrak{X}$ and $\mathcal{E}$ be the Effros σ-field on $c(\mathfrak{X})$. This σ-field is generated by the subsets $U^- = \{F \in c(\mathfrak{X}) : F \cap U \neq \emptyset\}$, where U ranges over the open subsets of $\mathfrak{X}$. On the other hand, for each $A, C \subset \mathfrak{X}$, $\mathrm{cl}\, C$, $w - \mathrm{cl}\, C$, $\mathrm{co}\, C$ and $\overline{\mathrm{co}}\, C$ denote the *norm-closure*, the *weak-closure*, the *convex hull* and the *closed convex hull* of C, respectively; the *distance function* $d(., C)$ of C, the *Hausdorff distance* $d_H(A, C)$ of A and C, the *norm* $|C|$ of C and the *support function* $\delta^*(., C)$ of C are defined by

$$
\begin{gathered}
d(x, C) = \inf\{\|x - y\| \, : y \in C\}, \quad (x \in \mathfrak{X}), \\
d_H(A, C) = \max\{\sup_{x \in A} d(x, C), \sup_{y \in C} d(y, A)\}, \\
|C| = d_H(C, \{0\}) = \sup\{||x|| \, : x \in C\}, \\
\delta^*(x^*, C) = \sup\{\langle x^*, y\rangle \, : y \in C\}, \quad (x^* \in \mathfrak{X}^*).
\end{gathered}
$$

We note that

$$\delta^*(x^*, C) = \delta^*(x^*, \overline{\mathrm{co}}\, C). \tag{1}$$

In the present paper, we shall use a notion of convergence for sequences of subsets which has been introduced by Mosco [26, 27] and which is related to that of Painlevé-Kuratowski. Let t be a topology on $\mathfrak{X}$ and $(C_n)_{n \geq 1}$ be a sequence in $c(\mathfrak{X})$. We put

$$
\begin{gathered}
t\text{-}\operatorname{li} C_n = \{x \in \mathfrak{X} : x = t\text{-}\lim x_n, \; x_n \in C_n, \forall n \geq 1\}, \\
t\text{-}\operatorname{ls} C_n = \{x \in \mathfrak{X} : x = t\text{-}\lim x_k, \; x_k \in C_{n(k)}, \forall k \geq 1\}
\end{gathered}
$$

where $(C_{n(k)})_{k\geq 1}$ is a subsequence of $(C_n)_{n\geq 1}$. The subsets $t\text{-}\operatorname{li} C_n$ and $t\text{-}\operatorname{ls} C_n$ are the *lower limit* and the *upper limit* of $(C_n)_{n\geq 1}$, relative to topology t. We obviously have $t\text{-}\operatorname{li} C_n \subset t\text{-}\operatorname{ls} C_n$.

A sequence $(C_n)_{n\geq 1}$ converges to C_∞, in the *sense of Painlevé-Kuratowski*, relatively to the topology t, if the two following equalities are satisfied:

$$t\text{-}\operatorname{ls} C_n = t\text{-}\operatorname{li} C_n = C_\infty.$$

In this case, we shall write $C_\infty = t\text{-}\lim_n C_n$; this is true if and only if the next two inclusions hold

$$t\text{-}\operatorname{ls} C_n \subset C_\infty \subset t\text{-}\operatorname{li} C_n.$$

Let us denote by s (resp. w) the strong (resp. weak) topology of $\mathfrak{X}$. It is easily seen that $s\text{-}\operatorname{li} C_n \subset w\text{-}\operatorname{ls} C_n$, and $s\text{-}\operatorname{li} C_n \in c(\mathfrak{X})$ ($s\text{-}\operatorname{li} C_n \in \operatorname{cc}(\mathfrak{X})$ if $C_n \in \operatorname{cc}(\mathfrak{X})$ for all $n \in \mathbf{N}$). A subset C_∞ is said to be the *Painlevé-Kuratowski-Mosco limit* of the sequence $(C_n)_{n\geq 1}$ denoted by $PKM\text{-}\lim_n C_n$ if

$$w\text{-}\operatorname{ls} C_n = s\text{-}\operatorname{li} C_n = C_\infty$$

which is true if and only if

$$w\text{-}\operatorname{ls} C_n \subset C_\infty \subset s\text{-}\operatorname{li} C_n.$$

The corresponding definitions of pointwise convergence and almost sure convergence for a sequence $(X_n, n \geq 1)$ of multifunctions defined on Ω are clear. In fact, in the above definitions, it suffices to replace C_n by $X_n(\omega)$ and C_∞ by $X_\infty(\omega)$ for almost surely $\omega \in \Omega$.

Concerning Painlevé-Kuratowski-Mosco convergence we refer to U. Mosco [26, 27], G. Beer [4, 5, 6, 7], H. Attouch - R. Wets [3], F. Hiai [18, 19, 20].

A closed valued *multifunction* X, i.e., a mapping from Ω to $c(\mathfrak{X})$, is said to be $\mathcal{A}$-*measurable* (or *measurable*) if for every open set U of $\mathfrak{X}$, the subset

$$X^{-1}(U) := \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

belongs to $\mathcal{A}$. A measurable multifunction is also called a *random set* (or *closed valued random variable*). The sub-σ-field $X^{-1}(\mathcal{E})$ generated by X is denoted by $\mathcal{A}_X$. The *domain* and the *graph* of X are respectively defined by

$$\operatorname{dom}(X) = \{\omega \in \Omega : X(\omega) \neq \emptyset\} \quad \text{and} \quad \operatorname{Gr}(X) = \{(\omega, x) \in \Omega \times \mathfrak{X} : x \in X(\omega)\}.$$

The *distribution* $\mathbf{P}_X$ of the measurable multifunction $X : \Omega \to c(\mathfrak{X})$ on the measurable space $(c(\mathfrak{X}), \mathcal{E})$ is defined by

$$\mathbf{P}_X(B) = \mathbf{P}\{X^{-1}(B)\}, \quad \text{for all } B \in \mathcal{E}.$$

A collection of multivalued random variables $\{X_i, i \in I\}$ is said to be *identically distributed* if the $\mathbf{P}_{X_i}$ $(i \in I)$ are identical.

For every sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, consider the space $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$ of all measurable functions from $(\Omega, \mathcal{F})$ into $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$. Further, define the two following subsets associated with the multifunction X involving its measurable selections:

$$S(X, \mathcal{F}) = \{f \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in X(\omega), \text{ for almost every } \omega \in \text{dom}(X)\}.$$

So, $S(X, \mathcal{F})$ is the set of $\mathcal{F}$-measurable selections of X. By Kuratowski-Ryll-Nardzewski Theorem every Effros measurable multifunction, X admits at least one measurable selection. Moreover, X admits a *Castaing representation*, that is, a sequence (f_n) of measurable selections, such that for every $\omega \in \text{dom}(X)$, $X(\omega)$ is equal to the closure of the countable subset $\{f_n(\omega) : n \geq 1\}$ (see [11]).

It is known that $\mathcal{L}^0(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$ $(= \mathcal{L}^0(\mathfrak{X}))$ endowed with the topology of convergence in probability is a metrizable topological vector space. Since a sequence converging in probability admits an almost sure converging subsequence, it is clear that, for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, the set $S(X, \mathcal{F})$ is closed in $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$.

For $1 \leq p < \infty$, $\mathcal{L}^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$ $(= \mathcal{L}^p(\mathfrak{X}))$ denotes the subspace of $\mathcal{L}^0(\mathfrak{X})$ whose members $f : \Omega \to \mathfrak{X}$ satisfy

$$\|f\|_p = (\mathbf{E}\|f\|^p)^{\frac{1}{p}} = \left(\int_\Omega \|f(\omega)\|^p \mathbf{P}(d\omega)\right)^{\frac{1}{p}} < +\infty.$$

$\mathcal{L}^p(\mathbf{R})$ is denoted by $\mathcal{L}^p$. Given a sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ and a random set X, define the following subsets of $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$

$$S^p(X, \mathcal{F}) = \{f \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in X(\omega), \text{ for almost every } \omega \in \text{dom}(X)\}.$$

Using standard measurable selection arguments, it is not hard to see that, when $\mathcal{A}_X \subseteq \mathcal{F} \subseteq \mathcal{A}$, the set $S^1(X, \mathcal{F})$ is non empty if and only if the positive function $d(0, X)$ is integrable. In such a situation, we shall say that the multifunction X is *integrable*. Observe that when X is integrable, $\mathbf{P}(\text{dom}(X)) = 1$. On the other hand, X is said to be *integrably bounded* if the function $|X|(\omega) = \sup\{\|x\| : x \in X(\omega)\}$ is integrable. In this case, we have $S^1(X, \mathcal{A}) = S(X, \mathcal{A})$. An integrably bounded multifunction is also integrable, but the converse implication is false. For any measurable multifunction X and any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, the *multivalued integral* of X over Ω, with respect to $\mathcal{F}$, is defined by

$$\mathbf{E}(X, \mathcal{F}) = \{\mathbf{E}(f) : f \in S^1(X, \mathcal{F})\},$$

where $\mathbf{E}(f) = \int_\Omega f d\mathbf{P}$ is the usual Bochner integral of f. We note that $\mathbf{E}(X, \mathcal{A})$ is not always closed. $\mathbf{E}(X, \mathcal{A})$ is non empty if and only if X is integrable.

The Wijsman topology $\mathcal{T}_W$ on $c(\mathfrak{X})$ is the topology of pointwise convergence of distance functions. This means that a net (C_α) of closed sets converges to C in the Wijsman topology if, for every $x \in \mathfrak{X}$, one has

$$d(x, C) = \lim_\alpha d(x, C_\alpha).$$

It is known that for any closed convex subset C of $\mathfrak{X}$ and for any $x \in \mathfrak{X}$, we have

$$d(x, C) = \sup_{z \in B^*} \{\langle z, x\rangle - \delta^*(z, C)\}. \tag{2}$$

C. Hess (see [17], Lemma 3.1) showed that, for any closed convex subset C of $\mathfrak{X}$, there exists a countable subset D^* of B^* verifying, for any $x \in \mathfrak{X}$,

$$d(x, C) = \sup_{z \in D^*} \{\langle z, x\rangle - \delta^*(z, C)\}. \tag{3}$$

Now we introduce some notions of fuzzy random sets. A fuzzy set in $\mathfrak{X}$ is a function $u : \mathfrak{X} \to [0, 1]$. Let $F(\mathfrak{X})$ denote the family of fuzzy subsets u satisfying the following conditions:

(a) u is upper semicontinuous, that is, the α-upper level set of u, that is, $u_\alpha = \{x \in \mathfrak{X} : u(x) \geq \alpha\}$ is a closed subset of $\mathfrak{X}$ for each $\alpha \in (0, 1]$,
(b) $\{x \in \mathfrak{X} : u(x) > 0\}$ has compact closure,
(c) $\{x \in \mathfrak{X} : u(x) = 1\} \neq \emptyset$.

A linear structure in $F(\mathfrak{X})$ is defined by the following operations:

$$(u + v)(x) = \sup_{y+z=x} \min[u(y), v(z)],$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0, \\ I_0(x) & \text{if } \lambda = 0, \end{cases}$$

where $u, v \in F(\mathfrak{X}), \lambda \in \mathbb{R}$ and I_0 is the indicator function of $\{0\}$. This of course implies $(u + v)_\alpha = u_\alpha + v_\alpha$ and $(\lambda u)_\alpha = \lambda u_\alpha$. Then we adopt the metric $d_r(d_\infty)$ as a generalization of the Hausdorff metric from $c(\mathfrak{X})$ to $F(\mathfrak{X})$, where

$$d_r(u, v) = \left[\int_0^1 d_H^r(u_\alpha, v_\alpha) d\alpha\right]^{1/r} \quad \text{if } 1 \leq r < \infty,$$
$$d_\infty(u, v) = \sup_{\alpha \in (0,1]} d_H(u_\alpha, v_\alpha) \quad \text{if } r = \infty,$$

where $u, v \in F(\mathfrak{X})$.

For each $u \in F(\mathfrak{X})$, denote $|u| := d_\infty(u, I_0) = \sup_{\alpha>0} |u_\alpha|$. The concept of a fuzzy random set as a generation for a random set was extensively studied by Puri and Ralescu [28]. A *fuzzy random set* is a function $X : \Omega \to F(\mathfrak{X})$ such that for each $\alpha \in (0, 1]$, the α-upper level set of X (that is, $X_\alpha(\omega) = \{x \in \mathfrak{X} : X(\omega)(x) \geq \alpha\}$) is a random set. The expectation of a fuzzy random set X, denoted by $\mathbf{E}(X)$, is an element in $F(\mathfrak{X})$ such that for each $\alpha \in (0, 1]$,

$$(\mathbf{E}(X))_\alpha = \text{cl}\{\mathbf{E}(f) : f \in S^1(X_\alpha, \mathcal{A})\},$$

where the closure is taken in $\mathfrak{X}$. By virtue of the existence theorem (see [25]), we have an equivalent definition as follows:

$$\mathbf{E}(X)(x) = \sup\{\alpha \in (0, 1] : x \in \mathbf{E}(X_\alpha)\}.$$

A fuzzy random set X_∞ is said to be the *Painlevé-Kuratowski-Mosco limit* of the sequence of fuzzy random sets $\{X_n : n \geq 1\}$ denoted by $PKM\text{-}\lim X_n$ if $L_\alpha X_\infty = PKM\text{-}\lim L_\alpha X_n$ for every $\alpha \in (0,1]$ a.s.

Given two σ-fields $\mathcal{U}, \mathcal{V}$ in $\mathcal{A}$, write

$$\varphi(\mathcal{U}, \mathcal{V}) := \sup\{|\mathbf{P}(B \mid A) - \mathbf{P}(B)| : A \in \mathcal{U}, B \in \mathcal{V}, P(A) \neq 0\}.$$

Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of random sets on $(\Omega, \mathcal{A}, \mathbf{P})$. For two nonempty disjoint sets $S, T \subset \mathbf{N}^2$, we define $\operatorname{dist}(S,T)$ as

$$\min\left\{\sqrt{(j_1-k_1)^2+(j_2-k_2)^2} : (j_1,j_2) \in S, (k_1,k_2) \in T\right\}.$$

Let $\sigma(S)$ and $\sigma(T)$ be the σ-fields generated by $\{X_{mn} : (m,n) \in S\}$ and $\{X_{mn} : (m,n) \in T\}$, respectively. Now we define φ^*-mixing coefficients for the double array of $\{X_{mn} : m \geq 1, n \geq 1\}$. For any real number $s \geq 1$, set

$$\varphi^*(s) = \sup\{\varphi(\sigma(S), \sigma(T)) : S, T \subset \mathbf{N}^2, \operatorname{dist}(S,T) \geq s\},$$
$$\varphi(s) = \sup\{\varphi(\sigma(S), \sigma(T)) : S, T \subset \mathbf{N}^2, \operatorname{dist}(\operatorname{co} S, \operatorname{co} T) \geq s\},$$

where $\operatorname{co} S$ denote the *convex hull* of S.

If $\varphi(s)$ (resp. $\varphi^*(s)$) tends to zero as $s \to \infty$, then we say that the double array is φ-mixing (resp. φ^*-mixing). Obviously, a double array of φ^*-mixing random sets is φ-mixing. In fact, a double array of independent random sets is $\varphi(\varphi^*)$-mixing.

Let $0 < p \leq 2$ and let $\{\theta_n, n \geq 1\}$ be a sequence of i.i.d. stable random variables each with characteristic function $\phi(t) = \exp\{-|t^p|\}$. The real separable Banach space $\mathfrak{X}$ is said to be of *stable type* p if $\sum_{n=1}^{\infty} \theta_n v_n$ converges a.s. whenever $v_n \in \mathfrak{X}$, $n \geq 1$ with $\sum_{n=1}^{\infty} \|v_n\|^p < \infty$.

A double array $\{f_{mn} : m \geq 1, n \geq 1\}$ of random elements is said to be *stochastically dominated* by a random element f if for some positive constant $C < \infty$

$$\mathbf{P}\{\|f_{mn}\| \geq t\} \leq C\mathbf{P}\{\|f\| \geq t\}, \quad t \geq 0, \ m \geq 1, \ n \geq 1.$$

A double array $\{X_{mn} : m \geq 1, n \geq 1\}$ of closed valued random variables is said to be *stochastically dominated* by a random element X if for some positive constant $C < \infty$

$$\mathbf{P}\{|X_{mn}| \geq t\} \leq C\mathbf{P}\{\|X\| \geq t\}, \quad t \geq 0, \ m \geq 1, \ n \geq 1.$$

This condition is satisfied when the $\{X_{mn} : m \geq 1, n \geq 1\}$ is identically distributed.

For notational convenience, for $a, b \in \mathbf{R}$, $\max\{a,b\}$ is denoted by $a \vee b$, $\min\{a,b\}$ is denoted by $a \wedge b$ and the symbol C denotes a generic positive constant which is not necessarily the same one in each circumstance. The logarithms are to the base e, for $a \in \mathbf{R}$, $\ln(a \vee e)$ will be denoted by $\log^+ a$.

For basic notions of probability theory and set-valued analysis, we refer to F. Akhiat, C. Castaing and F. Ezzaki [1], C. Castaing and M. Valadier [11], F. Hiai and H. Umegaki [21], Y. S. Chow and H. Teicher [12].

Now we proceed to state our main results.

3. SLLN in Painlevé-Kuratowski-Mosco convergence for random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence in a separable Banach space

In this section, the strong laws of large numbers in Painlevé-Kuratowski-Mosco convergence will be obtained for double array of random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence in a separable Banach space. We need some lemmas which will be used later.

Lemma 3.1. *Let X be a random set. Then one has the following:*

(a) *For each X with $S^1(X, \mathcal{A}) \neq \emptyset$,*

$$\overline{\mathrm{co}}\, \mathbf{E}(X, \mathcal{A}) = \overline{\mathrm{co}}\, \mathbf{E}(X, \mathcal{A}_X).$$

(b) *Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of $\varphi(\varphi^*)$-mixing and identically distributed random sets. For each $f_{11} \in S^1(X_{11}, \mathcal{A}_{X_{11}})$, there exists $\{f_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}}) : m \geq 1, n \geq 1, (m, n) \neq (1, 1)\}$ such that $\{f_{mn} : m \geq 1, n \geq 1\}$ is a double array of $\varphi(\varphi^*)$-mixing and identically distributed random elements.*

(c) *For each identically distributed random sets X, Y with $S^1(X, \mathcal{A}) \neq \emptyset$, one has*

$$\mathbf{E}(X, \mathcal{A}_X) = \mathbf{E}(Y, \mathcal{A}_Y).$$

Proof. (a) follows from [20].

(b) Since $\mathfrak{X}$ is separable and f_{11} is $\mathcal{A}_{X_{11}}$-measurable, there exists a $(\mathcal{E}, \mathcal{B}(\mathfrak{X}))$-measurable function $\Psi : c(\mathfrak{X}) \to \mathfrak{X}$ satisfying that $f_{11}(\omega) = \Psi(X_{11}(\omega))$ for every $\omega \in \Omega$. Now define $f_{mn}(\omega) = \Psi(X_{mn}(\omega)), \omega \in \Omega$. As $\{X_{mn} : m \geq 1, n \geq 1\}$ is $\varphi(\varphi^*)$-mixing and identically distributed, so is $\{f_{mn} : m \geq 1, n \geq 1\}$, since the definitions of $\varphi(\varphi^*)$-mixing dependence rely on σ-fields. In fact, the $\varphi(\varphi^*)$-mixing coefficients of $\{f_{mn} : m \geq 1, n \geq 1\}$ are less than those of $\{X_{mn} : m \geq 1, n \geq 1\}$. Thus

$$\int_{\Omega} \|f_{11}\| d\mathbf{P} = \int_{c(\mathfrak{X})} \|\Psi(X_{11})\| d\mathbf{P}_{X_{11}} = \int_{c(\mathfrak{X})} \|\Psi(X_{mn})\| d\mathbf{P}_{X_{mn}} = \int_{\Omega} \|f_{mn}\| d\mathbf{P} < \infty.$$

Since the function $d(x, X)$ of $\mathfrak{X} \times c(\mathfrak{X})$ into $\mathbf{R}$ is $(\mathcal{B}(\mathfrak{X}), \mathcal{E})$-measurable for every integrable random sets X, thus $\{d(f_{mn}(\cdot), X_{mn}(\cdot))\}$ is $\varphi(\varphi^*)$-mixing and identically distributed.

Hence, $\{d(f_{11}(\omega), X_{11}(\omega))\} = 0$ a.s. implies $\{d(f_{mn}(\omega), X_{mn}(\omega))\} = 0$ a.s., which leads to $f_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$.

(c) follows from (b). □

Lemma 3.2. *If $\{X_i, i \in I\}$ is a collection of $\varphi(\varphi^*)$-mixing integrable random sets and $f_i \in S^1(X_i, \mathcal{A}_{X_i})$ for every $i \in I$, then the collection $\{f_i, i \in I\}$ is $\varphi(\varphi^*)$-mixing.*

Proof. Since the definitions of $\varphi(\varphi^*)$-mixing dependence rely on σ-fields and f_i is $\mathcal{A}_{X_i}$-measurable then the $\varphi(\varphi^*)$-mixing coefficients of $\{f_i, i \in I\}$ are less than those of $\{X_i, i \in I\}$. It follows that the collection $\{f_i, i \in I\}$ is $\varphi(\varphi^*)$-mixing. $\square$

The two following propositions constitute a key ingredient for proving the main convergence result in this section.

Proposition 3.3 (see [10], Proposition 3.3). *Let $\{x_{mn} : m \geq 1, n \geq 1\}$ be a double array of elements in a Banach space $\mathfrak{X}$ such that*

(*i*) *For each $m \geq 1$, $\frac{1}{n}\sum_{j=1}^{n} x_{mj} \to x$ as $n \to \infty$,*

(*ii*) *For each $n \geq 1$, $\frac{1}{m}\sum_{i=1}^{m} x_{in} \to x$ as $m \to \infty$,*

(*iii*) $\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x$ *as* $m \wedge n \to \infty$,

where x is a member of $\mathfrak{X}$. Then,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x \quad \text{as } m \vee n \to \infty.$$

Proposition 3.4 (see [35]). *Let $\{f_{mn} : m \geq 1, n \geq 1\}$ be a double array of mixing and identically distributed random elements in a real separable Banach space $\mathfrak{X}$ with $\mathbf{E}(\|f_{11}\| \log^+ \|f_{11}\|) < \infty$. Suppose that one of the following conditions is satisfied:*

$$(a) \quad \lim_{s\to\infty} \varphi^*(s) < \frac{1}{2}, \qquad (b) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty.$$

Then one has

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(f_{ij} - \mathbf{E}f_{ij}) \to 0 \quad \text{a.s. as } m \vee n \to \infty.$$

For the sake of completeness, we state and summarize a useful lemma.

Lemma 3.5. *Let $\mathfrak{X}$ be a Banach space and $C \subset \mathfrak{X}$. Then for each $x \in \overline{\mathrm{co}}\, C$ and $\epsilon > 0$, we can choose $x_1, \ldots, x_k \in C$ such that*

$$\left\| \frac{1}{k}\sum_{i=1}^{k} x_i - x \right\| < \epsilon.$$

Proof. For any $x \in \overline{\mathrm{co}}\, C$ and $\epsilon > 0$, there exists $y \in \mathrm{co}\, C$ such that $\|x - y\| < \frac{\epsilon}{2}$. Obviously, it is possible to write

$$y = \sum_{j=1}^{m} \lambda_j y_j, \quad \text{where } \lambda_j \in \mathbb{R},\ 0 \leq \lambda_j \leq 1,\ \sum_{j=1}^{m} \lambda_j = 1,\ y_j \in C.$$

Put $M = \max\{\|y_j\|/1 \leq j \leq m\}$. We can only consider $m \geq 2$:

By the density of the set of all rational numbers and $0 \leq \lambda_j \leq 1$ then for each $1 \leq j \leq m-1$, there exists $\beta_j \in \mathbf{Q}$ such that

$$0 \leq \beta_j \leq \lambda_j, \tag{I}$$

$$|\lambda_j - \beta_j| < \frac{\epsilon}{2m(m-1)M}. \tag{II}$$

Next, put $\beta_m = 1 - \sum_{j=1}^{m-1} \beta_j$. By (I) we have

$$\beta_m = 1 - \sum_{j=1}^{m-1} \beta_j \geq 1 - \sum_{j=1}^{m-1} \lambda_j = \lambda_m \geq 0.$$

Moreover, (II) yields

$$|\lambda_m - \beta_m| = \left|\sum_{j=1}^{m-1} (\lambda_j - \beta_j)\right| \leq \sum_{j=1}^{m-1} |\lambda_j - \beta_j| < \sum_{j=1}^{m-1} \frac{\epsilon}{2m(m-1)M} = \frac{\epsilon}{2mM}.$$

Thus, for each $1 \leq j \leq m$, we obtain

$$\beta_j \in \mathbf{Q},\ 0 \leq \beta_j \leq 1,\ |\lambda_j - \beta_j| < \frac{\epsilon}{2mM} \text{ and } \sum_{i=1}^{m} \beta_i = 1.$$

Putting $y' = \sum_{j=1}^{m} \beta_j y_j$, we have

$$\|y - y'\| = \left\|\sum_{j=1}^{m} (\lambda_j - \beta_j) y_j\right\| \leq \sum_{j=1}^{m} (|\lambda_j - \beta_j|.\|y_j\|) < \sum_{j=1}^{m} \frac{\epsilon}{2mM}.M = \frac{\epsilon}{2}$$

and $\|x - y'\| \leq \|x - y\| + \|y - y'\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

On the other hand

$$\begin{aligned} y' &= \sum_{j=1}^{m} \beta_j y_j = \sum_{j=1}^{m} \frac{p_j}{q_j} y_j = \sum_{j=1}^{m} \frac{k_j}{k} y_j \ \text{ where } p_j, q_j, k_j \in \mathbf{N}, k = \sum_{i=1}^{m} k_i \\ &= \frac{1}{k}[\underbrace{(y_1 + \cdots + y_1)}_{k_1} + \cdots + \underbrace{(y_m + \cdots + y_m)}_{k_m}] \\ &= \frac{1}{k} \sum_{j=1}^{k} x_j, \quad \text{where } x_j \in C. \end{aligned}$$

This completes the proof. □

H. Inoue and R. L. Taylor [23] proved the strong law of large numbers for sequence of exchangeable random sets in a separable Banach space in Painlevé-Kuratowski-Mosco convergence. Recently, Ke-Ang Fu [13] replaced exchangeable dependence by $\varphi(\varphi^*)$-mixing dependence and obtain a strong law of large numbers. Here we extend K. A. Fu's results for the case of double array of identically distributed random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence.

Theorem 3.6. *Let $\{X_{mn} : m \geq 1,\ n \geq 1\}$ be a double array of mixing and identically distributed random sets in a real separable Banach space $\mathfrak{X}$ with $\mathbf{E}(|X_{11}| \log^+ |X_{11}|) < \infty$. Suppose that one of the following conditions is satisfied:*

$$(a) \quad \lim_{s\to\infty} \varphi^*(s) < \frac{1}{2}, \qquad (b) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty.$$

Then one has

$$PKM\text{-}\lim_{m\vee n\to\infty} \frac{1}{mn} \mathrm{cl} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} = \overline{\mathrm{co}}\, \mathbf{E}(X_{11}, \mathcal{A}) \quad a.s.$$

Proof. Here we only consider the φ-mixing case, since the φ^*-mixing case can be proved similarly. Let $D = \overline{\mathrm{co}}\, \mathbf{E}(X_{11}, \mathcal{A})$ and $G_{mn}(\omega) = \frac{1}{mn} \mathrm{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)$ for each $m \geq 1, n \geq 1$. We will show that $D \subset s\text{-}\mathrm{li}_{m\vee n\to\infty} G_{mn}(\omega)$ a.s. To do this, we will use Proposition 3.3. Let $d \in D$ and $\epsilon > 0$, by Lemmas 3.1(a) and 3.5, there exists $f_j \in S^1(X_{11}, \mathcal{A}_{X_{11}}), 1 \leq j \leq k$ such that $\|\frac{1}{k}\sum_{j=1}^{k} \mathbf{E}(f_j) - d\| < \epsilon$. By Lemma 3.1$(b)$ and (c) we can choose $f_{ij} \in S^1(X_{ij}, \mathcal{A}_{X_{ij}}), 1 \leq i \leq k, 1 \leq j \leq k$ such that

$$\mathbf{E}f_{ij} = \begin{cases} \mathbf{E}f_{i+j-1} & \text{if } i+j \leq k+1, \\ \mathbf{E}f_{i+j-1-k} & \text{if } i+j > k+1. \end{cases}$$

Let $d_j = \mathbf{E}f_j, 1 \leq j \leq k$ and let $d_{ij} = \mathbf{E}f_{ij}, 1 \leq i \leq k, 1 \leq j \leq k$. It is easy to check that

$$\frac{1}{k}\sum_{i=1}^{k} d_i = \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} d_{ij}, \tag{4}$$

$$\frac{1}{k}\sum_{i=1}^{k} d_i = \frac{1}{k}\sum_{i=1}^{k} d_{ij} \text{ for each } j = 1, 2, \ldots, k, \tag{5}$$

$$\frac{1}{k}\sum_{i=1}^{k} d_i = \frac{1}{k}\sum_{j=1}^{k} d_{ij} \text{ for each } i = 1, 2, \ldots, k. \tag{6}$$

By Lemma 3.1 (b) and (c), there exists a double array $\{f_{ij} : i \geq 1, j \geq 1\}$ of $f_{ij} \in S^1(X_{ij}, \mathcal{A}_{X_{ij}})$ such that $\{f_{(s-1)k+i,(t-1)k+j} : s \geq 1, t \geq 1\}$ is φ-mixing and identically distributed for each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k$.

If $m = (s-1)k+p, n = (t-1)k+q$ where $1 \leq p \leq k, 1 \leq q \leq k$, then the following estimation holds:

$$\begin{aligned}
&\left\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega)-\frac{1}{k}\sum_{i=1}^{k} d_i\right\| \\
=&\left\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega)-\frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} d_{ij}\right\| \qquad \text{(by (4))} \\
\le&\frac{st}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\left\|\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t} f_{(l-1)k+i,(r-1)k+j}(\omega)-d_{ij}\right\| \\
&\quad+\frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\|f_{(s-1)k+i,(r-1)k+j}(\omega)\| \\
&\quad+\frac{s}{mn}\sum_{i=1}^{k}\sum_{j=q+1}^{k}\frac{1}{s}\sum_{l=1}^{s}\|f_{(l-1)k+i,(t-1)k+j}(\omega)\| \\
&\quad+\left(\frac{st}{mn}-\frac{1}{k^2}\right)\left\|\sum_{i=1}^{k}\sum_{j=1}^{k} d_{ij}\right\|.
\end{aligned}$$

Now we prove the desired estimations that constitute the main technical part of the proof.
Since $\{X_{mn} : m \ge 1, n \ge 1\}$ is φ-mixing and identically distributed random sets, it follows that $\{f_{(s-1)k+i,(t-1)k+j} : s \ge 1, t \ge 1\}$ is a double array of φ-mixing and identically distributed random elements in $\mathcal{L}^1(\mathfrak{X})$ with

$$\mathbf{E}(\|f_{ij}\|\log^+\|f_{ij}\|) \le \mathbf{E}(|X_{ij}|\log^+|X_{ij}|) < \infty.$$

Hence, applying Lemma 3.4 for random elements to each double array $\{f_{(s-1)k+i,(t-1)k+j} : s \ge 1, t \ge 1\}$ for any $1 \le i \le k, 1 \le j \le k$, we get

$$\left\|\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t} f_{(l-1)k+i,(r-1)k+j}(\omega)-d_{ij}\right\| \to 0 \ \text{ a.s. as } s\vee t\to\infty.$$

Thus

$$\frac{st}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\left\|\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t} f_{(l-1)k+i,(r-1)k+j}(\omega)-d_{ij}\right\| \to 0 \ \text{ a.s. as } m\vee n\to\infty.$$

Since $\{f_{(s-1)k+i,(r-1)k+j} : r \ge 1\}$ is a sequence of φ-mixing and identically distributed random elements in $\mathcal{L}^1(\mathfrak{X})$ for each $s \ge 1$, it follows from Theorem 3.1 of Zhang [35] that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{r=1}^{t}\|f_{(s-1)k+i,(r-1)k+j}(\omega)-d_{ij}\| = 0 \ \text{ a.s.}$$

Hence,

$$\begin{aligned}
&\frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\|f_{(s-1)k+i,(r-1)k+j}(\omega)\| \\
&\le \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\|f_{(s-1)k+i,(r-1)k+j}(\omega)-d_{ij}\| + \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\|d_{ij}\| \\
&\le \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\|f_{(s-1)k+i,(r-1)k+j}(\omega)-d_{ij}\| \\
&\qquad + \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\|d_{ij}\| \to 0 \ \text{ a.s. as } m\wedge n\to\infty.
\end{aligned}$$

Similarly, we get

$$\frac{s}{mn}\sum_{i=1}^{k}\sum_{j=q+1}^{k}\frac{1}{s}\sum_{l=1}^{s}\|f_{(l-1)k+i,(t-1)k+j}(\omega)\| \to 0 \ \text{ a.s. as } m\wedge n\to\infty.$$

We have

$$\lim_{m\wedge n\to\infty}\frac{st}{mn} = \lim_{m\wedge n\to\infty}\left(\frac{1}{k}+\frac{1}{m}-\frac{p}{mk}\right)\left(\frac{1}{k}+\frac{1}{n}-\frac{q}{nk}\right) = \frac{1}{k^2},$$

it follows that

$$\left(\frac{st}{mn}-\frac{1}{k^2}\right)\left\|\sum_{i=1}^{k}\sum_{j=1}^{k}d_{ij}\right\| \to 0 \ \text{ as } m\wedge n\to\infty.$$

Combining the above limits, we get

$$\left\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}f_{ij}(\omega)-\frac{1}{k}\sum_{i=1}^{k}d_i\right\| \to 0 \ \text{ a.s. as } m\wedge n\to\infty. \tag{7}$$

Next, for each $n=(t-1)k+j,\ 1\le j\le k$. If $m=(s-1)k+p,\ 1\le p\le k$, then the following estimation holds:

$$\begin{aligned}
&\left\|\frac{1}{m}\sum_{i=1}^{m}f_{in}(\omega)-\frac{1}{k}\sum_{i=1}^{k}d_i\right\| = \left\|\frac{1}{m}\sum_{i=1}^{m}f_{in}(\omega)-\frac{1}{k}\sum_{i=1}^{k}d_{ij}\right\| \qquad \text{(by (5))} \\
&\le \frac{s}{m}\sum_{i=1}^{k}\left\|\frac{1}{s}\sum_{h=1}^{s}f_{(h-1)k+i,n}(\omega)-d_{ij}\right\| + \frac{s}{m}\sum_{i=p+1}^{k}\frac{1}{s}\|f_{(s-1)k+i,n}(\omega)\| \\
&\qquad + \left(\frac{s}{m}-\frac{1}{k}\right)\left\|\sum_{i=1}^{k}d_{ij}\right\|.
\end{aligned}$$

For $1 \le i \le k$, since $\{f_{(s-1)k+i,n} : s \ge 1\}$ is a sequence of φ-mixing and identically distributed random elements in $\mathcal{L}^1(\mathfrak{X})$, then again by Theorem 3.1 of Zhang [35],

$$\left\| \frac{1}{s} \sum_{h=1}^{s} f_{(h-1)k+i,n}(\omega) - d_{ij} \right\| \to 0 \quad \text{a.s. as } s \to \infty,$$

and hence $\frac{1}{s}\|f_{(s-1)k+i,n}(\omega)\| \to 0$ a.s. as $s \to \infty$. Therefore,

$$\left\| \frac{1}{m} \sum_{i=1}^{m} f_{in}(\omega) - \frac{1}{k} \sum_{i=1}^{k} d_i \right\| \to 0 \quad \text{a.s. as } m \to \infty. \tag{8}$$

Similarly, we have, for each $m \ge 1$,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} f_{mj}(\omega) - \frac{1}{k} \sum_{i=1}^{k} d_i \right\| \to 0 \quad \text{a.s. as } n \to \infty. \tag{9}$$

Combining (7), (8), (9) and Proposition 3.3 we get

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(\omega) \to \frac{1}{k} \sum_{i=1}^{k} d_i \quad \text{a.s. as } m \vee n \to \infty.$$

Hence, $\frac{1}{k}\sum_{i=1}^{k} d_i \in s\text{-}\mathrm{li}_{m\vee n\to\infty} G_{mn}(\omega)$ a.s. Thus $D \subset s\text{-}\mathrm{li}_{m\vee n\to\infty} G_{mn}(\omega)$ a.s.

In the following we show $w\text{-}\mathrm{ls}_{m\vee n\to\infty} G_{mn}(\omega) \subset D$ a.s. Let $\{d_k, k \ge 1\}$ be a sequence dense in $\mathfrak{X}\backslash D$. Since $\mathfrak{X}$ is separable, by the separation theorem, there exists a sequence $\{d_k^*, k \ge 1\}$ in $\mathfrak{X}^*$ with $\|d_k^*\| = 1$ such that

$$\langle d_k^*, d_k \rangle \le \delta^*(d_k^*, D), \quad \forall k \ge 1.$$

Thus, it follows that $d \in D$ if and only if $\langle d_k^*, d \rangle \le \delta^*(d_k^*, D)$ for all $k \ge 1$.

Since $\{X_{mn} : m \ge 1, n \ge 1\}$ is φ-mixing and identically distributed random sets, then for each $k \ge 1$, $\{\delta^*(d_k^*, X_{mn}) : m \ge 1, n \ge 1\}$ is a double array of φ-mixing and identically distributed random variables in $\mathcal{L}^1$. On the other hand,

$$\mathbf{E}(|\delta^*(d_k^*, X_{11})| \log^+ |\delta^*(d_k^*, X_{11})|) \le \mathbf{E}(|X_{11}| \log^+ |X_{11}|) < \infty \quad \text{for each } k \ge 1.$$

So, there exists a negligible subset $N \in \mathcal{A}$ such that for every $\omega \in \Omega\backslash N$ and $k \ge 1$,

$$\delta^*(d_k^*, G_{mn}(\omega)) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^*(d_k^*, X_{mn}(\omega)) \to \delta^*(d_k^*, D) < \infty \quad \text{as } m \vee n \to \infty.$$

If $d \in w\text{-}\mathrm{ls}_{m\vee n\to\infty} G_{mn}(\omega)$ for $\omega \in \Omega\backslash N$, then $d_{rs} \xrightarrow{w} d$ as $r \vee s \to \infty$ for some $d_{rs} \in G_{m_r n_s}(\omega)$ and hence

$$\langle d_k^*, d \rangle = \lim_{r\vee s\to\infty} \langle d_k^*, d_{rs} \rangle \le \lim_{r\vee s\to\infty} \delta^*(d_k^*, G_{m_r n_s}(\omega)) = \delta^*(d_k^*, D), \quad k \ge 1,$$

which implies $d \in D$. Thus $w\text{-}\mathrm{ls}_{m\vee n\to\infty} G_{mn}(\omega) \subset D$ a.s. $\square$

Next, from Theorem 3.6 we prove an SLLN with respect to Painlevé-Kuratowski-Mosco convergence for double array of mixing and identically distributed fuzzy random sets.

Corollary 3.7. *Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of mixing fuzzy random sets taking values in $F(\mathfrak{X})$. If $\mathbf{E}(|X_{11}| \log^+ |X_{11}|) < \infty$ and one of the following conditions is satisfied:*

$$(a) \quad \lim_{s\to\infty} \varphi^*(s) < \frac{1}{2}, \qquad (b) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty,$$

then

$$PKM\text{-} \lim_{m\vee n\to\infty} \frac{1}{mn} \operatorname{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} = \overline{\operatorname{co}}\, \mathbf{E}(X_{11}) \quad a.s.$$

Proof. Since $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of mixing and identically distributed fuzzy random sets in $F(\mathfrak{X})$, by the definitions of mixing we have that $\{(X_{mn})_\alpha : m \geq 1, n \geq 1\}$ is mixing random sets with $S^1((X_{mn})_\alpha, \mathcal{A}) \neq \emptyset$ for any $\alpha \in (0, 1]$. Thus the desired result follows from Theorem 3.6 immediately. □

Remark. By the definitions of mixing dependence and σ-fields, it follows that the mixing coefficients in Corollary 3.7 is less than those in Theorem 3.6.

The following is a result on Marcinkiewicz-Zygmund strong laws of large numbers for double array of mixing random sets.

Theorem 3.8. *Let $1 \leq p \leq 2$ and let $\mathfrak{X}$ be a stable type p Banach space with $\mathfrak{X}^*$ having the Radon-Nikodym property (RNP). Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of φ^*-mixing random sets with $0 \in \mathbf{E}(X_{mn}, \mathcal{A}_{X_{mn}})$ for all $m \geq 1, n \geq 1$. Suppose that $\{X_{mn} : m \geq 1, n \geq 1\}$ is stochastically dominated by a random element X. If $\mathbf{E}\big(\|X\|^p \log^+ \|X\|\big) < \infty$ then*

$$0 \in s\text{-}\operatorname{li}_{m\vee n\to\infty} \frac{1}{(mn)^{1/p}} \operatorname{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega) \quad a.s. \tag{10}$$

Furthermore, if $\mathfrak{X}$ is reflexive, then the condition φ^*-mixing (i.e. $\lim_{s\to\infty} \varphi^*(s) = 0$) can be replaced by $\lim_{s\to\infty} \varphi^*(s) < \frac{1}{4}$.

Proof. For all $m \geq 1, n \geq 1$, since $0 \in \mathbf{E}(X_{mn}, \mathcal{A}_{X_{mn}})$, there exists $f_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ such that $\mathbf{E} f_{mn} = 0$.

Moreover, since $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of φ^*-mixing random sets and f_{mn} is a $\mathcal{A}_{X_{mn}}$-measurable selection of X_{mn}, then by Lemma 3.2, $\{f_{mn} : m \geq 1, n \geq 1\}$ is φ^*-mixing.

Since the event of $\{\|f_{mn}\| \geq t\} \subset \{|X_{mn}| \geq t\}$ for all $t \geq 0, m \geq 1, n \geq 1$, then

$$\mathbf{P}\{\|f_{mn}\| \geq t\} \leq \mathbf{P}\{|X_{mn}| \geq t\} \leq C\, \mathbf{P}\{\|X\| \geq t\} \quad \text{for all } t \geq 0, m \geq 1, n \geq 1.$$

Thus $\{f_{mn} : m \geq 1, n \geq 1\}$ is stochastically dominated by a random element X. Using Theorem 6 [34], we get

$$\frac{1}{(mn)^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(\omega) \to 0 \ \text{ a.s. as } m \vee n \to \infty,$$

it follows that

$$0 \in s\text{-}\mathrm{li}_{m\vee n\to\infty} \frac{1}{(mn)^{1/p}} \operatorname{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega) \ \text{ a.s.}$$

The proof is therefore completed. $\square$

The following example shows that in Theorem 3.8, the conclusion (10) cannot be replaced by the stronger one

$$PKM\text{-}\lim_{m\vee n\to\infty} \frac{1}{(mn)^{1/p}} \operatorname{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega) = \{0\} \ \text{ a.s.} \tag{I}$$

Example 3.9. Let $\mathfrak{X} = \mathbf{R}$, then $\mathfrak{X}$ is stable type p $(1 \leq p \leq 2)$. Put $X(\omega) = [\frac{-1}{2}, \frac{1}{2}], \omega \in \Omega$ then X is integrable random sets and $\mathcal{A}_X = \{\emptyset, \Omega\}$.

Consider a random variable f such that $\mathbf{P}(f = \frac{1}{2}) = \mathbf{P}(f = \frac{-1}{2}) = \frac{1}{2}$. Obviously, f is a measurable selection of X and $Ef = \frac{1}{2}.\frac{1}{2} + \frac{-1}{2}.\frac{1}{2} = 0$, $\mathbf{P}(f \neq 0) = 1 > 0$.

Put $X_{mn} = X$, $f_{mn} = f$ for all $m \geq 1, n \geq 1$. We infer that $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent random sets. Since then, $\{X_{mn} : m \geq 1, n \geq 1\}$ is φ^*-mixing. Next, $f_{mn} \in S^1(X_{mn}, \mathcal{A})$ and $\mathbf{E}f_{mn} = 0$. Put $g(\omega) = 0$ for all $\omega \in \Omega$, we get $g \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ and $\mathbf{E}g = 0$. Hence, we have $0 \in \mathbf{E}(X_{mn}, \mathcal{A}_{X_{mn}})$. The random variable h is defined by $h(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$. It is easy to check that $\{X_{mn} : m \geq 1, n \geq 1\}$ is stochastically dominated by the random variable h satisfying $\mathbf{E}\big(\|h\|^p \log^+ \|h\|\big) < \infty$. However,

$$w\text{-}\mathrm{ls}_{m\vee n\to\infty} \frac{1}{(mn)^{1/p}} \operatorname{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega) \nsubseteq \{0\} \ \text{ a.s.}$$

Hence, $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies all the conditions of Theorem 3.8 but the conclusion (I) is not true.

4. SLLN in Wijsman convergence for $\varphi(\varphi^*)$-mixing random sets in a separable Banach space

In this section, we will provide several Wijsman convergence results relative to the strong laws of large numbers for double array of $\varphi(\varphi^*)$-mixing random sets in a separable Banach space.

Proposition 4.1. *Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of $\varphi(\varphi^*)$-mixing random sets having the same distribution as an integrable r.s. X such that $\mathbf{E}(|X| \log^+ |X|) < \infty$ and one of the following conditions is satisfied:*

$$(a) \quad \lim_{s\to\infty} \varphi^*(s) < \frac{1}{2}, \qquad (b) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty.$$

Let C' be the set of all convex combinations of $\mathbf{E}(X, \mathcal{A}_X)$, with rational coefficients. Then, for each $y \in C'$, there exists a negligible subset $N(y)$ of Ω and a double array $\{g_{mn} : m \geq 1, n \geq 1\}$ in $\mathcal{L}^1(\mathfrak{X})$ verifying:

(i) *for each $m \geq 1, n \geq 1$, $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$,*

(ii) *for any $\omega \in \Omega \setminus N(y)$,*

$$y = \lim_{m\vee n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}(\omega).$$

Proof. Here we only consider the φ-mixing case, since the φ^*-mixing case can be proved similarly. Consider $y \in C'$. From the definition of C' we have $y = \sum_{j=1}^{k} \lambda_j y_j$ where k is a positive integer, λ_j $(j = \overline{1,k})$ are positive rational numbers with $\sum_{j=1}^{k} \lambda_j = 1$ and where, for each $j \geq 1$, $y_j \in \mathbf{E}(X, \mathcal{A}_X)$. Obviously, for every $j = 1, \ldots, k$, it is possible to write $\lambda_j = \frac{d_j}{d}$ where d and the d_j are positive integers satisfying $d = \sum_{j=1}^{k} d_j$. Put $z_1 = y_1, \ldots, z_{d_1} = y_1$, $z_{d_1+1} = y_2$, $\ldots$, $z_{d_1+d_2} = y_2$, $\ldots$, $z_{d_1+\cdots+d_{k-1}+1} = y_k$, $\ldots$, $z_d = y_k$. Since then, we have $y = \frac{1}{d}\sum_{i=1}^{d} z_i$. For each $1 \leq j \leq d$, $z_j = \mathbf{E}(f_j)$ of $f_j \in S^1(X, \mathcal{A}_X)$.

The proof will be performed in several steps.

Step 1. By Lemma 3.1, we can choose $f_{ij} \in S^1(X_{ij}, \mathcal{A}_{X_{ij}})$, $1 \leq i \leq d, 1 \leq j \leq d$ such that

$$\mathbf{E}(f_{ij}) = \begin{cases} \mathbf{E}(f_{i+j-1}) & \text{if } i + j \leq d + 1, \\ \mathbf{E}(f_{i+j-1-d}) & \text{if } i + j > d + 1. \end{cases}$$

Let $z_{ij} = \mathbf{E}(f_{ij}), 1 \leq i \leq d, 1 \leq j \leq d$. It is easy to check that (4), (5) and (6) are true.

By Lemma 3.1, for each $m \geq 1, n \geq 1$, there exists a double array $\{g_{mn} : m \geq 1, n \geq 1\}$ of $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ such that $\{g_{(s-1)d+i,(t-1)d+j} : s \geq 1, t \geq 1\}$ is a double array of φ-mixing random elements having the same distribution as f_{ij} for each $i = 1, \ldots, d$ and $j = 1, \ldots, d$.

For any $m \geq 1, n \geq 1$, there exists the integers s_m, p_m, t_n and q_n satisfying

$$m = s_m d + p_m, \quad s_m \geq 0, \ 1 \leq p_m \leq d, \tag{11}$$

$$n = t_n d + q_n, \quad t_n \geq 0, \ 1 \leq q_n \leq d. \tag{12}$$

From the above relationships, we deduce that the sequences (p_m) and (q_n) are bounded, whereas from (11) and (12) we deduce

$$\lim_{m\to\infty} s_m = \infty \quad \text{and} \quad \lim_{n\to\infty} t_n = \infty.$$

Furthermore it is not difficult to show for all $\omega \in \Omega$ that following equality holds

$$\begin{aligned}
(*):\quad & \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \\
&= \frac{s_m t_n}{mn}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} g_{(l-1)d+i,(r-1)d+j}(\omega) \\
&\quad + \frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n} g_{s_m d+i,(r-1)d+j}(\omega) \\
&\quad + \frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m} g_{(l-1)d+i,t_n d+j}(\omega) + \frac{1}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{q_n} g_{s_m d+i,t_n d+j}(\omega).
\end{aligned}$$

The proof will be performed as follows.

Step 2. Claim 1: There is a negligible subset $N_{11}(y)$ such that for every $\omega \in \Omega \setminus N_{11}(y)$,

$$\frac{s_m t_n}{mn}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} g_{(l-1)d+i,(r-1)d+j}(\omega) \to \frac{1}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d} z_{ij} \quad \text{as } m\wedge n\to\infty. \tag{13}$$

Since $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of φ-mixing and identically distributed closed valued random variables, then $\{g_{(s-1)d+i,(t-1)d+j} : s \geq 1, t \geq 1\}$ is a double array of φ-mixing and identically distributed random elements in $\mathcal{L}^1(\mathfrak{X})$ with

$$\mathbf{E}(\|g_{ij}\| \log^+ \|g_{ij}\|) \leq \mathbf{E}(|X_{ij}| \log^+ |X_{ij}|) < \infty.$$

Hence, applying Proposition 3.4 to each double array of random elements $\{g_{(s-1)d+i,(t-1)d+j} : s \geq 1, t \geq 1\}$ in $\mathcal{L}^1(\mathfrak{X})$ yields a negligible subset $N_{11}(y)$ such that for every $\omega \in \Omega \setminus N_{11}(y)$, for any $1 \leq i \leq d,\ 1 \leq j \leq d$,

$$\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} g_{(l-1)d+i,(r-1)d+j}(\omega) \to z_{ij} \quad \text{as } m\vee n\to\infty. \tag{14}$$

We have

$$\begin{aligned}
\lim_{m\to\infty}\frac{s_m}{m} &= \lim_{m\to\infty}\left(\frac{1}{d}+\frac{1}{m}-\frac{p_m}{md}\right) = \frac{1}{d} \\
\text{and}\quad \lim_{n\to\infty}\frac{t_n}{n} &= \lim_{n\to\infty}\left(\frac{1}{d}+\frac{1}{n}-\frac{q_n}{nd}\right) = \frac{1}{d}.
\end{aligned} \tag{15}$$

Since then, (13) holds.

Step 3. Claim 2:

$$\lim_{m\wedge n\to\infty}\left(\frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n}g_{s_md+i,(r-1)d+j}(\omega)\right.\\ \left.+\frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m}g_{(l-1)d+i,t_nd+j}(\omega)\right)=0 \quad \text{a.s.} \tag{16}$$

Since $\{g_{(s-1)d+i,(r-1)d+j} : r \geq 1\}$ is a sequence of φ-mixing and identically distributed random elements in $\mathcal{L}^1(\mathfrak{X})$ for each $s \geq 1,\ 1 \leq i \leq d,\ 1 \leq j \leq d$, applying Theorem 3.1 of Zhang [35] to this sequence yields a negligible subset $N_{12}(y)$ such that for every $\omega \in \Omega \setminus N_{12}(y)$, for any $1 \leq i \leq d,\ 1 \leq j \leq d,\ m \geq 1$, we have

$$\frac{1}{t_n}\sum_{r=1}^{t_n}g_{s_md+i,(r-1)d+j}(\omega)\to z_{ij} \quad \text{as } n\to\infty. \tag{17}$$

Thus, for every $\omega \in \Omega \setminus N_{12}(y)$,

$$\frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n}g_{s_md+i,(r-1)d+j}(\omega)\to 0 \quad \text{as } m\wedge n\to\infty.$$

Similarly, there exists a negligible subset $N_{13}(y)$ such that for every $\omega \in \Omega\setminus N_{13}(y)$,

$$\frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m}g_{(l-1)d+i,t_nd+j}(\omega)\to 0 \quad \text{as } m\wedge n\to\infty.$$

Step 4. Claim 3: There is a negligible $N_1(y)$ such that for every $\omega \in \Omega \setminus N_1(y)$,

$$\frac{1}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{q_n}g_{s_md+i,t_nd+j}(\omega)\to 0 \quad \text{as } m\wedge n\to\infty. \tag{18}$$

We define the negligible subset $N_1(y)$ as the union of the $N_{1j}(y)$ where $j \in \{1, 2, 3\}$, from (14), (15) and (17) we have that for every $\omega \in \Omega \setminus N_1(y)$, for any $1 \leq i \leq d,\ 1 \leq j \leq d$,

$$\begin{aligned}&\frac{1}{mn}g_{s_md+i,t_nd+j}(\omega)\\ &=\frac{s_mt_n}{mn}\left(\frac{1}{s_mt_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n}g_{ld+i,rd+j}(\omega)-\frac{s_m-1}{s_m}.\frac{1}{(s_m-1)t_n}\sum_{l=1}^{s_m-1}\sum_{r=1}^{t_n}g_{ld+i,rd+j}(\omega)\right.\\ &\qquad\left.-\frac{t_n-1}{s_mt_n}.\frac{1}{t_n-1}\sum_{r=1}^{t_n-1}g_{s_md+i,rd+j}(\omega)\right)\\ &\to\frac{1}{d^2}(z_{ij}-1.z_{ij}-0.z_{ij})=0 \quad \text{as } m\wedge n\to\infty,\end{aligned}$$

whence Claim 3 follows by applying this estimate.

Combining the above limits and using (4) and coming back to $(*)$ we have that for every $\omega \in \Omega \setminus N_1(y)$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_i \quad \text{as } m \wedge n \to \infty. \tag{19}$$

Step 5. Next, for each $n = td + j,\ 1 \le j \le d$. If $m = s_m d + p_m,\ 1 \le p_m \le d$ then for all $\omega \in \Omega$, the following equality holds

$$(**): \quad \frac{1}{m}\sum_{i=1}^{m} g_{in}(\omega) = \frac{s_m}{m}\sum_{i=1}^{d}\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) + \frac{1}{m}\sum_{i=1}^{p_m} g_{s_m d+i,n}(\omega).$$

Claim 4:

$$\lim_{m\to\infty}\frac{s_m}{m}\sum_{i=1}^{d}\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) = \frac{1}{d}\sum_{i=1}^{d} z_{ij} \quad \text{a.s.} \tag{20}$$

Since $\{g_{(s-1)d+i,n} : s \ge 1\}$ is a sequence of φ-mixing and identically distributed random elements in $\mathcal{L}^1(\mathfrak{X})$ for each $1 \le i \le d$, $n \ge 1$ then applying again Theorem 3.1 of Zhang [35] to this sequence yields a negligible subset $N_{21}(n, y)$ such that for every $\omega \in \Omega \setminus N_{21}(n, y)$, for any $1 \le i \le d$,

$$\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) \to z_{ij} \quad \text{as } m \to \infty.$$

Thus, for every $\omega \in \Omega \setminus N_{21}(n, y)$,

$$\frac{s_m}{m}\sum_{i=1}^{d}\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_{ij} \quad \text{as } m \to \infty.$$

Claim 5:

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{p_m} g_{s_m d+i,n}(\omega) = 0 \quad \text{a.s.} \tag{21}$$

Applying Theorem 3.1 of Zhang [35] yields a negligible subset $N_{22}(n, y)$ such that for every $\omega \in \Omega \setminus N_{22}(n, y)$, for any $1 \le i \le d$,

$$\begin{aligned}\frac{1}{m} g_{s_m d+i,n}(\omega) &= \frac{s_m}{m}\left(\frac{1}{s_m}\sum_{l=1}^{s_m} g_{ld+i,n}(\omega) - \frac{s_m - 1}{s_m}\frac{1}{s_m - 1}\sum_{l=1}^{s_m - 1} g_{ld+i,n}(\omega)\right)\\ &\to \frac{1}{d}(z_{ij} - 1.z_{ij}) = 0 \quad \text{as } m \to \infty.\end{aligned}$$

Thus, for every $\omega \in \Omega \setminus N_{22}(n, y)$,

$$\frac{1}{m}\sum_{i=1}^{p_m} g_{s_m d+i,n}(\omega) \to 0 \quad \text{as } m \to \infty.$$

We define the negligible subset $N_2(n, y)$ as the union of the N_{2j} where $j \in \{1, 2\}$, combining the above limits and by (5) and coming back to $(**)$ we have that for every $\omega \in \Omega \setminus N_2(n, y)$,

$$\frac{1}{m}\sum_{i=1}^{m} g_{in}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_i \quad \text{as } m \to \infty. \tag{22}$$

Similarly, for each $m \geq 1$, there exists a negligible subset $N_3(m, y)$ such that for every $\omega \in \Omega \setminus N_3(m, y)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g_{mj}(\omega) = \frac{1}{d}\sum_{i=1}^{d} z_i. \tag{23}$$

Final Step and Conclusion:

We define the negligible subset $N(y)$ as the union of the $N_1(y)$, $N_2(n, y)$ and $N_3(m, y)$ where $m \geq 1, n \geq 1$. Combining (19), (22), (23) and Proposition 3.3, we have that for every $\omega \in \Omega \setminus N(y)$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \to y \quad \text{as } m \vee n \to \infty.$$

The proof is therefore completed. □

We also need an useful lemma that is formally derived from (2).

Lemma 4.2. *Let $\{C_{mn} : m \geq 1, n \geq 1\}$ be a double array in $c(\mathfrak{X})$. Also consider $C \in c(\mathfrak{X})$ and a countable dense subset D^* of B^* such that*

$$d(x, \overline{\text{co}}\, C) = \sup_{z \in D^*} \{\langle z, x\rangle - \delta^*(z, \overline{\text{co}}\, C)\}, \quad x \in \mathfrak{X}$$

(*which is possible by* (2)*. If, for every $z \in D^*$, one has*

$$\limsup_{m\vee n\to\infty} \delta^*(z, C_{mn}) \leq \delta^*(z, C) \tag{24}$$

then, for every $x \in \mathfrak{X}$,

$$\liminf_{m\vee n\to\infty} d(x, C_{mn}) \geq d(x, \overline{\text{co}}\, C).$$

Proof. For every $x \in \mathfrak{X}$, we have by (1), (2), (24) and elementary calculations

$$\begin{aligned}
\liminf_{m\vee n\to\infty} d(x, C_{mn}) &\geq \liminf_{m\vee n\to\infty} d(x, \overline{\mathrm{co}}\, C_{mn}) \\
&\geq \liminf_{m\vee n\to\infty} \Big[\sup_{z\in B^*} \{\langle z, x\rangle - \delta^*(z, C_{mn})\}\Big] \\
&\geq \sup_{z\in B^*} \Big\{ \liminf_{m\vee n\to\infty} \big[\langle z, x\rangle - \delta^*(z, C_{mn})\big]\Big\} \\
&\geq \sup_{z\in B^*} \Big\{\langle z, x\rangle - \delta^*(z, C)\Big\} \\
&\geq \sup_{z\in D^*} \Big\{\langle z, x\rangle - \delta^*(z, C)\Big\} = d(x, \overline{\mathrm{co}}\, C). \qquad \square
\end{aligned}$$

Proposition 4.1 allows to prove the "lim sup" part of Wijsman convergence for the SLLN under consideration.

Proposition 4.3. *Under the same hypotheses as in Proposition* 4.1, *there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,*

$$\limsup_{m\vee n\to\infty} d\left(x, \mathrm{cl}\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\right) \leq d(x, \overline{\mathrm{co}}\, \mathbf{E}(X, \mathcal{A})). \tag{25}$$

Proof. The proof is omitted since it is similar to the proof of Proposition 3.4 in C. Hess [17]. $\square$

Now, we can state and prove the main result of this section, namely, the multivalued SLLN for double array of $\varphi(\varphi^*)$-mixing and identically distributed random sets when $c(\mathfrak{X})$ is endowed with the Wijsman topology.

Theorem 4.4. *Let $\mathfrak{X}$ be a separable Banach space and let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of $\varphi(\varphi^*)$-mixing random sets having the same distribution as an integrable r.s. X such that $\mathbf{E}(|X| \log^+ |X|) < \infty$ and one of the following conditions is satisfied:*

$$(a) \quad \lim_{s\to\infty} \varphi^*(s) < \frac{1}{2}, \qquad (b) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty.$$

Then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$,

$$\overline{\mathrm{co}}\, \mathbf{E}(X, \mathcal{A}) = \mathcal{T}_W - \lim_{m\vee n\to\infty} \mathrm{cl}\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)$$

that is, for any $x \in \mathfrak{X}$,

$$d(x, \overline{\mathrm{co}}\, \mathbf{E}(X, \mathcal{A})) = \lim_{m\vee n\to\infty} d\left(x, \mathrm{cl}\, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\right).$$

Proof. Let $Z_{mn}(\omega) = \text{cl}\ \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)$ for every $m \geq 1, n \geq 1, \omega \in \Omega$ and let $C = \overline{\text{co}}\,\mathbf{E}(X, \mathcal{A})$. By (3) there is a countable set D^* in B^* such that

$$d(x, C) = \sup_{z \in D^*} \{\langle z, x\rangle - \delta^*(z, C)\} \quad \forall x \in \mathfrak{X}.$$

Let z be fixed in D^*. Notice that $\{\delta^*(z, X_{mn}) : m \geq 1, n \geq 1\}$ is a double array of $\varphi(\varphi^*)$-mixing random variables having the same distribution as $\delta^*(z, X)$. Further, by the inequality

$$\mathbf{E}(|\delta^*(z, X)| \log^+ |\delta^*(z, X)|) \leq \mathbf{E}(|X| \log^+ |X|) < \infty,$$

we may apply Proposition 3.4 to double array of $\mathbf{R}$-valued random variables $\{\delta^*(z, X_{mn}) : m \geq 1, n \geq 1\}$. This yields the existence of a negligible subset $N(z)$ of Ω verifying, for every $\omega \in \Omega \setminus N(z)$,

$$\delta^*(z, C) = \lim_{m \vee n \to \infty} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \delta^*(z, X_{ij}(\omega)) = \lim_{m \vee n \to \infty} \delta^*(z, Z_{mn}(\omega)).$$

Now, defining the negligible subset N_1 as the union of the $N(z)$, for $z \in D^*$, we deduce that

$$\delta^*(z, C) = \lim_{m \vee n \to \infty} \delta^*(z, Z_{mn}(\omega)) \quad \forall z \in D^*,\ \forall \omega \in \Omega \setminus N_1. \tag{26}$$

The above equality and Lemma 4.2 entail

$$\liminf_{m \vee n \to \infty} d(x, Z_{mn}(\omega)) \geq d(x, C) \quad \forall x \in \mathfrak{X},\ \forall \omega \in \Omega \setminus N_1. \tag{27}$$

On the other hand, Proposition 4.3 yields the existence of a negligible subset N_2 such that

$$\limsup_{m \vee n \to \infty} d(x, Z_{mn}(\omega)) \leq d(x, C) \quad \forall x \in \mathfrak{X},\ \forall \omega \in \Omega \setminus N_2. \tag{28}$$

Finish the proof by defining the negligible subset $N = N_1 \cup N_2$ and by combining inequalities (27) and (28). □

The following is an immediate consequence of Theorem 4.5. It is even new because it is the first result dealing with Wijsman convergence for $\varphi(\varphi^*)$-mixing random sets.

Corollary 4.5. *Let $\mathfrak{X}$ be a separable Banach space and let $(X_n)_{n\geq 1}$ be a $\varphi(\varphi^*)$-mixing random sets having the same distribution as an integrable r.s. X such that $\mathbf{E}(|X| \log^+ |X|) < \infty$ and one of the following conditions is satisfied:*

$$(a) \quad \lim_{s \to \infty} \varphi^*(s) < \frac{1}{2}, \qquad (b) \quad \sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty.$$

Then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$,

$$d(x, \overline{\text{co}}\,\mathbf{E}(X, \mathcal{A})) = \lim_{n \to \infty} d\left(x, \text{cl}\,\frac{1}{n}\sum_{j=1}^{n} X_j(\omega)\right).$$

In the next theorem, we obtain the Marcinkiewicz-Zygmund type law of large numbers for double array of $\varphi(\varphi^*)$-mixing random sets.

Theorem 4.6. *Let $\mathfrak{X}$ be a stable type p $(1 \le p \le 2)$ Banach space with $\mathfrak{X}^*$ having the RNP and let $\{X_{mn} : m \ge 1, n \ge 1\}$ be a double array of φ^*-mixing random sets with $0 \in \mathbf{E}(X_{mn}, \mathcal{A}_{X_{mn}})$ for all $m \ge 1, n \ge 1$. Suppose that $\{X_{mn} : m \ge 1, n \ge 1\}$ is stochastically dominated by a random element X. If $\mathbf{E}(\|X\|^p \log^+ \|X\|) < \infty$ then there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,*

$$\limsup_{m \vee n \to \infty} d\left(x, \mathrm{cl}\frac{1}{(mn)^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\right) \le d(x, \{0\}). \tag{29}$$

Furthermore, if $\mathfrak{X}$ is reflexive, then the condition φ^*-mixing (i.e. $\lim_{s\to\infty} \varphi^*(s) = 0$) can be replaced by $\lim_{s\to\infty} \varphi^*(s) < \frac{1}{4}$.

Proof. Using the same argument as in the proof of Theorem 3.8 provides a selection f_{mn} of X_{mn} such that

$$\frac{1}{(mn)^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(\omega) \to 0 \ \text{ a.s. as } m \vee n \to \infty,$$

so, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\begin{aligned} &\limsup_{m \vee n \to \infty} d\left(x, \mathrm{cl}\,\frac{1}{(mn)^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\right) \\ \le\ &\limsup_{m \vee n \to \infty} \left\| x - \frac{1}{(mn)^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(\omega) \right\| = d(x, \{0\}). \end{aligned}$$

The proof is completed. $\square$

The following lemma will be used to next example.

Lemma 4.7. *For all $p \ge 1$, there exists a double array of constants $\{a_{mn} : m \ge 1, n \ge 1\}$ satisfying the following conditions*

$$\begin{aligned} &i) \quad S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = (mn)^{\frac{1}{p}}, \quad \textit{for all } m \ge 1, n \ge 1, \\ &ii) \quad 0 \le a_{ij} \le 1, \quad \textit{for all } i \ge 1, j \ge 1. \end{aligned}$$

Proof. Since the formula $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$ then

$$a_{mn} = S_{mn} - S_{m-1,n} - S_{m,n-1} + S_{m-1,n-1},$$

in which $S_{m,0} = S_{0,n} = 0$ for all $m \ge 1, n \ge 1$.

It implies that

$$\begin{aligned}a_{mn} &= (mn)^{\frac{1}{p}} - (m-1)^{\frac{1}{p}}(n)^{\frac{1}{p}} - (m)^{\frac{1}{p}}(n-1)^{\frac{1}{p}} + (m-1)^{\frac{1}{p}}(n-1)^{\frac{1}{p}}\\ &= n^{\frac{1}{p}}\left(m^{\frac{1}{p}} - (m-1)^{\frac{1}{p}}\right) - (n-1)^{\frac{1}{p}}\left(m^{\frac{1}{p}} - (m-1)^{\frac{1}{p}}\right)\\ &= \left(m^{\frac{1}{p}} - (m-1)^{\frac{1}{p}}\right)\left(n^{\frac{1}{p}} - (n-1)^{\frac{1}{p}}\right).\end{aligned}$$

Next, we prove that the condition *ii*) holds. Put $f(x) = x^{\frac{1}{p}} - (x-1)^{\frac{1}{p}}$ for all $x \geq 1$. We have

$$\begin{aligned}f'(x) &= \frac{1}{p}.x^{\frac{1}{p}-1} - \frac{1}{p}.(x-1)^{\frac{1}{p}-1}\\ &= \frac{1}{p}\left(\frac{1}{x^{\frac{p-1}{p}}} - \frac{1}{(x-1)^{\frac{p-1}{p}}}\right) \leq 0, \quad \forall x \geq 1.\end{aligned}$$

It follows that $f(x)$ is a decreasing function on $[1, +\infty)$. This leads to $f(x) \leq f(1) = 1$, $\forall x \geq 1$ (1). Moreover, it is easy to see that $f(x) \geq 0, \forall x \geq 1$ (2).

The proof is completed by (1), (2) and the equal $a_{mn} = f(m).f(n)$. □

The following example shows that in Theorem 4.6, for each p $(1 \leq p \leq 2)$, the conclusion (29) cannot be replaced by the stronger one

$$\mathcal{T}_W - \lim_{m\vee n\to\infty} \frac{1}{(mn)^{1/p}} \operatorname{cl} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega) = \{0\} \text{ a.s.} \tag{II}$$

Example 4.8. Let $\mathfrak{X} = \mathbf{R}$, then $\mathfrak{X}$ is stable type p $(1 \leq p \leq 2)$ Banach space. Put $X(\omega) = [-1, 1]$ for all $\omega \in \Omega$, then X is a random set and $\mathcal{A}_X = \{\emptyset, \Omega\}$. Put $f(\omega) = 0$ for all $\omega \in \Omega$, we get $f \in S^1(X, \mathcal{A}_X)$ and $\mathbf{E}(f) = 0$.

Put $X_{mn} = X$, $f_{mn} = f$ for all $m \geq 1, n \geq 1$. We infer that $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent random sets. Since then, $\{X_{mn} : m \geq 1, n \geq 1\}$ is φ^*-mixing. Moreover, $0 \in \mathbf{E}(X_{mn}, \mathcal{A}_{X_{mn}})$.

Next, the double array of constants $\{a_{mn} : m \geq 1, n \geq 1\}$ is defined as in Lemma 4.7. Put $g_{mn}(\omega) = a_{mn}$ and $g(\omega) = 1$ for all $\omega \in \Omega$ and $m \geq 1, n \geq 1$, we get $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$. Then, we have

$$1 = g(\omega) = \frac{1}{(mn)^{\frac{1}{p}}}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \in \operatorname{cl}\frac{1}{mn)^{\frac{1}{p}}}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega) \text{ for all } \omega \in \Omega.$$

Since then, for $x = 1$,

$$\begin{aligned}0 &\leq \liminf_{m\vee n\to\infty} d\left(x, \operatorname{cl}\frac{1}{mn)^{\frac{1}{p}}}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\right)\\ &\leq \liminf_{m\vee n\to\infty} \|x - g(\omega)\| = 0 \text{ for all } \omega \in \Omega,\end{aligned}$$

but $d(x, \{0\}) = 1$.

It is easy to check that $\{X_{mn} : m \geq 1, n \geq 1\}$ is stochastically dominated by the random variable g satisfying $\mathbf{E}\big(\|g\|^p \log^+ \|g\|\big) < \infty$. However, for all $\omega \in \Omega$, for each p $(1 \leq p \leq 2)$

$$\liminf_{m \vee n \to \infty} d\left(x, \mathrm{cl}\, \frac{1}{(mn)^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\right) \not\geq d(x, \{0\}) \quad \text{a.s. for } x = 1.$$

Thus $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies all the conditions of Theorem 4.6 but the conclusion (II) is not true.

Acknowledgements. We would like to thank the referees for very careful reading of the paper and valuable comments. We also wish to thank Professor Charles Castaing (Département de Mathématiques, Université Montpellier II, France) for helpful discussions.

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