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Domination and Factorization of Multilinear Operators

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We study multilinear operators from quasi-Banach lattices to quasi-Banach spaces. We prove that certain vector valued norm inequalities for these operators are equivalent to domination theorems. As an application we show that under some mild assumptions these domination theorems can be expressed in terms of factorization through Orlicz spaces. In the case of the multilinear functionals on $C(K)$-spaces we recover a multilinear variant of the Grothendieck factorization theorem.

Keywords: Factorization, Banach function lattice, Banach envelope, Orlicz space

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1. Introduction

Domination theorems have proved to be relevant tools in the theory of operators between Banach spaces. For instance, they play a fundamental role in the theory of summability in Banach spaces thanks to the Pietsch Domination Theorem for absolutely p-summing operators (see [14]). Several well-known theorems that provide dominations involving weighted L^p-spaces allow to prove factorization theorems, which have powerful applications in Banach space theory as well as in harmonic analysis; recall the Maurey-Rosenthal factorizations through L^p-spaces (see, e.g., [2, 4, 5]). Grothendieck's fundamental theorem should be also mentioned here. It states (see [15, Theorem 5.5]) that every bounded bilinear functional on

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$C(K_1) \times C(K_2)$ is canonically extendable to a bounded bilinear functional on $L^2(\mu_1) \times L^2(\mu_2)$, where μ_1 and μ_2 are probability Borel measures on K_1 and K_2, respectively.

This paper was inspired by the mentioned results and our recent joint work [12], where domination of linear operators has been studied. It is natural to wonder if similar results might hold for multilinear operators. As we will see some of the main results on factorization of linear operators between quasi-Banach lattices to quasi-Banach spaces extend to multilinear operators. Actually, we prove some domination theorems for bilinear operators in which Orlicz quasi-norms appear in a natural way. As a consequence, we prove "square" factorizations for linear operators through Orlicz spaces. Our technique has its roots in a fundamental separation argument based on Ky Fan's Lemma.

We remark that our main results (Theorems 3.2 and 3.4) work also for the multilinear case, since their proofs can be adapted in a straightforward manner. It should be noticed here that some of the well-known domination theorems that hold for bilinear operators cannot be extended to multilinear ones. For instance, a multilinear variant of Grothendieck's theorem fails in higher dimensions (see [1, p. 530]), although some results in this direction are available (see [1]).

The paper is organized as follows. We collect some preliminaries in Sections 1 and 2. In Section 3 we show our main domination results for multilinear operators. We present examples and discuss new inequalities involving p-convex norms that can be analyzed by using our tools. In the last section we show how our results can be stated in terms of factorizations when the bilinear operator is given by a particular linear operator.

2. Preliminaries

In this section we present some preliminary material needed later on. A quasi-normed space is a vector space X with a quasi-norm $\|\cdot\|_X$. If it is complete, we say that it is a *quasi-Banach space*. The Mackey semi-norm $\|\cdot\|_X^c$ on such a space X is the Minkowski functional of the convex hull $\operatorname{conv}(B_X)$ of the unit ball $B_X := \{x \in X;\ \|x\|_X \le 1\}$,

$$\|x\|_X^c = \inf\{\lambda > 0;\ x \in \lambda \operatorname{conv}(B_X)\}, \quad x \in X.$$

For a quasi-Banach space, the topological dual X^* of X is always defined, but it can be the trivial space $\{0\}$. However, if it separates the points of X, then it is a Banach space under the norm

$$\|x^*\|_{X^*} = \sup_{x \in B_X} |x^*(x)|.$$

Therefore, it can be seen that $X^* = (X, \|\cdot\|_X^c)^*$ with equality of norms. The completion of $(X, \|\cdot\|_X^c)$ is called the *Banach envelope* of X and is denoted by $\widehat{X}$. Using the formulas above, we obtain that for $\kappa\colon X \to X^{**}$ being the canonical embedding defined by $\kappa x(x^*) = x^*(x)$ for all $x \in X$ and $x^* \in X^*$,

$$\|\kappa x\|_{X^{**}} = \sup_{\|x^*\|_{X^*} \le 1} |x^*(x)| = \|x\|_X^c .$$

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a complete σ-finite nontrivial measure space. A quasi-normed *function lattice* X on a measure space (Ω, μ) is a quasi-normed space of (classes of μ-almost everywhere equal) measurable functions such that there exists a strictly positive $u \in X$ and X is an ideal in $L_0(\mu)$, i.e., if $|x| \leq |y|$, where $x \in L^0(\mu)$ and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$. If in addition $(X, \|\cdot\|_X)$ is complete then it is called a quasi-Banach function lattice (on (Ω, μ)).

A quasi-Banach function lattice X on (Ω, μ) is said to be a *rearrangement invariant* (r.i. for short) space if for every $f \in L^0(\mu)$ and $g \in X$ with $\mu_f = \mu_g$, we have $f \in X$ and $\|f\|_X = \|g\|_X$. Here μ_f denotes the distribution function of f with respect to μ, i.e., $\mu_f(s) = \mu(\{\omega \in \Omega; |f(\omega)| > s\})$, $s \geq 0$. The decreasing rearrangement f^* of f with respect to μ is defined by $f^*(t) = \inf\{s > 0; \mu_f(s) \leq t\}$, $t \geq 0$.

In the theory of r.i. spaces Marcinkiewicz and Lorentz spaces play an important role (see [10]). Given a measure space (Ω, μ) and a positive function ϕ on $I = (0, \mu(\Omega))$, the Marcinkiewicz space M_ϕ (on (Ω, μ)) consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{M_\phi} := \sup_{t \in I} \frac{\int_0^t f^*(s)\, ds}{\phi(t)} < \infty.$$

If in addition ϕ is concave, then the Lorentz space Λ_ϕ (on (Ω, μ)) consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{\Lambda_\phi} := \int_0^{\mu(\Omega)} f^*(s)\, d\phi(s) < \infty.$$

Both Marcinkiewicz and Lorentz spaces equipped with the norms $\|\cdot\|_{M_\phi}$ and $\|\cdot\|_{\Lambda_\phi}$ respectively, are r.i. spaces.

A quasi-normed function lattice X on (Ω, μ) is said to be order continuous provided $0 \leq x_n \downarrow 0$ a.e. implies $\|x_n\|_X \to 0$. The Köthe dual space (or associate space) X' of a normed function lattice X on (Ω, μ) is defined as the space of all $x \in L^0(\mu)$ such that $\int_\Omega |xy|\, d\mu < \infty$ for every $y \in X$. It is a Banach function lattice on (Ω, μ) when equipped with the norm

$$\|x\|_{X'} = \sup_{\|y\|_X \leq 1} \int_\Omega |xy|\, d\mu\,.$$

Notice that a normed function lattice X on (Ω, μ) is order continuous if and only if the map $X' \ni y \mapsto x_y^* \in X^*$ given by the definition of the elements of X' as elements of X^*, i.e.,

$$x_y^*(x) := \int_\Omega xy\, d\mu, \quad x \in X,$$

is an order isometrical isomorphism of X' onto X^* (see, e.g., [9]). Using these facts, it can be proved that if $(X, \|\cdot\|_X)$ is a quasi-normed function lattice on (Ω, μ) such that the topological dual X^* separates the points of X, then $(X, \|\cdot\|_X^c)$ is a normed function lattice on (Ω, μ), which is order continuous provided X is order continuous.

In what follows Φ will denote the set of all increasing, continuous functions $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(0)=0$. A function $\varphi \in \Phi$ is said to satisfy the Δ_2-condition ($\varphi \in (\Delta_2)$ for short) provided there exists $C>0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t>0$.

Throughout the paper we will work with a special class of quasi-Banach lattices. Let us give the definition of these spaces and collect some facts that will be needed in the paper. For any quasi-normed lattice X on (Ω,μ) and $\varphi \in \Phi$, we define an order ideal in $L^0(\mu)$,

$$X_\varphi = \left\{ f \in L^0(\mu);\ \exists\, \lambda > 0,\ \varphi(\lambda|f|) \in X \right\},$$

and the functional $\|\cdot\|_{X_\varphi}\colon X_\varphi \to [0,\infty)$ by

$$\|f\|_{X_\varphi} = \inf\left\{\lambda>0;\ \|\varphi(|f|/\lambda)\|_X \le 1\right\}, \quad f \in X_\varphi.$$

Clearly, $\|f\|_{X_\varphi}=0$ if and only if $f=0$ and $\|\lambda f\|_{X_\varphi} = |\lambda|\,\|f\|_{X_\varphi}$ for all $\lambda \in \mathbb{R}$, $f \in X_\varphi$.

Note that there is a large class of functions $\varphi \in \Phi$ for which $\|\cdot\|_{X_\varphi}$ is a quasi-norm on X_φ (see [12]). In particular, if φ is a convex function and X is a Banach function lattice, X_φ is a Banach function lattice. In the case when $X = L_1(\mu)$ and $\varphi \in \Phi$, X_φ is the Orlicz space denoted as usual by $L_\varphi(\mu)$. This fact motivates to call X_φ the generalized Orlicz space provided X_φ is a quasi-Banach space.

Observe that for $0<p<\infty$ and $\varphi(t)=t^p$ for all $t \ge 0$, the space X_φ is known as the *p-convexification* X_p of X whose quasi-norm is given by

$$\|f\|_{X_p} = \||f|^p\|_X^{1/p}, \quad f \in X_p.$$

A general construction of the p-convexification for abstract Banach lattices is also available (see p. 53 in [11, Section I.d]).

For a given function $\varphi \in \Phi$ and a quasi-Banach function lattice X, X is said to be *φ-admissible* provided that $\|\cdot\|_{X_\varphi}$ is a quasi-norm on X_φ. If in addition the topological dual $(X_\varphi)^*$ separates the points of X_φ, then X is called *strongly φ-admissible.* For the case $\varphi(t)=t^p$, we simply say that the space X is strongly p-admissible. Notice that if X is a φ-admissible r.i. space on (Ω,μ), then X_φ is also an r.i. space on (Ω,μ).

Under the adequate requirements, order continuity of X_φ is often inherited from X; if an order continuous Banach function lattice X is φ-admissible and φ satisfies the Δ_2-condition, then $\|f_n\|_{X_\varphi} \to 0$ if and only if $\|\varphi(|f_n|)\|_X \to 0$. This implies that X_φ is order continuous if and only if X is order continuous.

3. Domination of bilinear operators

We start this section by proving our main domination theorem for bilinear operators by means of a Hahn-Banach separation argument. The multilinear variants can be obtained by extending the same arguments outlined below. For the proof of the theorem below we will need Ky-Fan's Lemma (see, e.g., [6, Lemma 9.10]) that we include for the sake of completeness.

Lemma 3.1. *Let E be a Hausdorff topological vector space, and let K be a compact convex subset of E. Let Ψ be a set of functions on K with values in $(-\infty,\infty]$ having the following properties*:

(a) *each $f \in \Psi$ is convex and lower semicontinuous,*

(b) *Ψ is concave, i.e., if $g \in conv(\Psi)$, there is an $f \in \Psi$ with $g(x) \le f(x)$, for every $x \in K$,*

(c) *there is an $r \in \mathbb{R}$ such that each $f \in \Psi$ has a value not greater than r.*

Then there is an $x_0 \in K$ such that $f(x_0) \le r$ for all $f \in \Psi$.

Before stating and proving the main theorem of this section, let us mention that we will use the most important feature of the weak* topology, the compactness property of the closed ball of the dual space, which was discovered by Banach in 1932 for separable spaces and was extended to the general case by Alaoglu in 1940: if X is a normed linear space then the closed unit ball B_{X^*} is compact for the weak* topology $\sigma(X^*,X)$ on X^* (see, e.g., [16, p. 29]).

Theorem 3.2. *Let $\phi, \varphi_1, \varphi_2 \in \Phi$ and let X, Y be quasi-Banach function lattices on (Ω_1,μ_1) and (Ω_2,μ_2), respectively, such that X is strongly φ_1^{-1}-admissible and Y is strongly φ_2^{-1}-admissible. Suppose that T is a bilinear operator from $X \times Y$ into a quasi-Banach space E. Assume $0 < C_1, C_2 < \infty$ and that $A \subset X$, $B \subset Y$ are non-empty sets. Consider the following statements*:

(i) *For any finite set of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and any finite sets $\{f_k\}_{k=1}^n$ in A and $\{g_k\}_{k=1}^n$ in B, $n \in \mathbb{N}$, the following inequality holds*:

$$\sum_{k=1}^n \alpha_k \phi\left(\|T(f_k,g_k)\|_E\right) \le C_1 \left\|\sum_{k=1}^n \alpha_k \varphi_1(|f_k|)\right\|^c_{X_{\varphi_1^{-1}}} + C_2 \left\|\sum_{k=1}^n \alpha_k \varphi_2(|g_k|)\right\|^c_{Y_{\varphi_2^{-1}}}.$$

(ii) *There exist positive functionals $x^* \in (X_{\varphi_1^{-1}})^*$ and $y^* \in (Y_{\varphi_2^{-1}})^*$ such that*

$$\phi\left(\|T(f,g)\|_E\right) \le C_1\, x^*(\varphi_1(|f|)) + C_2\, y^*(\varphi_2(|g|)), \quad (f,g) \in A \times B.$$

(iii) *There exist $0 \le u \in B_{(X_{\varphi_1^{-1}})'}$ and $0 \le v \in B_{(Y_{\varphi_2^{-1}})'}$ such that*

$$\phi\left(\|T(f,g)\|_E\right) \le C_1 \int_{\Omega_1} \varphi_1(|f|) u\, d\mu_1 + C_2 \int_{\Omega_2} \varphi_2(|g|) v\, d\mu_2, \quad (f,g) \in A \times B.$$

Then (i) is equivalent to (ii). If $X_{\varphi_1^{-1}}$ and $Y_{\varphi_2^{-1}}$ are order continuous, then all three statements are equivalent.

Proof. (i) $\Rightarrow$ (ii). It follows from the Banach-Alaoglu theorem that if the topological dual of a quasi-normed space E separates the points then the closed unit ball of the dual space E^*, B_{E^*}, is compact for the weak* topology $\sigma(E^*,E)$ on E^*. Put

$$K_1 := \left\{x^* \in B_{(X_{\varphi_1^{-1}})^*};\, x^* \ge 0\right\}, \qquad K_2 := \left\{y^* \in B_{(Y_{\varphi_2^{-1}})^*};\, y^* \ge 0\right\}.$$

Then our hypotheses in combination with the basic facts shown in Section 2 imply that the inequality given in the statement (i) is equivalent to

$$\begin{aligned}&\sum_{k=1}^{n} \alpha_k \phi(\|T(f_k, g_k)\|_E) \\ &\le C_1 \sup_{x^* \in K_1} \sum_{k=1}^{n} \alpha_k x^*(\varphi_1(|f_k|)) + C_2 \sup_{y^* \in K_2} \sum_{k=1}^{n} \alpha_k y^*(\varphi_2(|g_k|)).\end{aligned} \tag{$*$}$$

Let $S_{\ell_1^n}$ be the unit sphere of the n-dimensional ℓ_1^n-space. Consider the family Ψ of convex functions $\psi_{\alpha,f,g}\colon K_1 \times K_2 \to \mathbb{R}$ with $0 \le \alpha = \{\alpha_k\} \in S_{\ell_1^n}$, $f = \{f_k\} \in \prod_{k=1}^n A$, $g = \{g_k\} \in \prod_{k=1}^n B$ defined for every $(x^*, y^*) \in K_1 \times K_2$ by

$$\begin{aligned}&\psi_{\alpha,f,g}(x^*, y^*) \\ &= \sum_{k=1}^{n} \alpha_k \phi(\|T(f_k, g_k)\|_E) - C_1 \sum_{k=1}^{n} \alpha_k x^*(\varphi_1(|f_k|)) - C_2 \sum_{k=1}^{n} \alpha_k y^*(\varphi_2(|g_k|)).\end{aligned}$$

It may be easily verified that Ψ is a concave family. Our hypotheses yield that the unit ball $B_{(X_{\varphi_j^{-1}})^*}$ is a compact convex subset for the weak*-topology $w_j^* = \sigma((X_{\varphi_j^{-1}})^*, X_{\varphi_j^{-1}})$ $(j = 0, 1)$. Since the set K_j is w_j^*-closed, it is w_j^*-compact $(j = 0, 1)$. Clearly, every function in Ψ is continuous in $K_1 \times K_2$ for the product topology $w_1^* \times w_2^*$. Thus the Hahn-Banach theorem together with the inequality $(*)$ imply that for every $\psi \in \Psi$ there exists $(x^*, y^*) \in B_{(X_{\varphi_1^{-1}})^*} \times B_{(X_{\varphi_2^{-1}})^*}$ such that $\psi(x^*, y^*) \le 0$. Then Ky-Fan's lemma applies and we deduce that there exists $(x^*, y^*) \in K_1 \times K_2$ such that $\psi(x^*, y^*) \le 0$ for all $\psi \in \Psi$, and so

$$\phi\left(\|T(f, g)\|_E\right) \le C_1 x^*(\varphi_1(|f|)) + C_2 y^*(\varphi_2(|g|)), \quad (f, g) \in A \times B.$$

This completes the proof of $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (i)$ is obvious.

If $X_{\varphi_j^{-1}}$ is order continuous, then the topological dual $(X_{\varphi_j^{-1}})^*$ can be isometrically identified with the Köthe dual $(X_{\varphi_j^{-1}})'$. This proves the equivalence between (ii) and (iii) and completes the proof. □

Notice that in order to apply the theorem above we require to identify the Mackey completion of the generalized Orlicz spaces as well as their topological duals. The problems related to descriptions of the Mackey completion of the generalized Orlicz space X_φ as well as their duals are interesting on their own. It is not our aim here to discuss these problems in general. We only wish to present some examples, which show that under some mild assumptions we are able to identify $\widehat{X_\varphi}$ and $(X_\varphi)^*$. In order to do this we will need some definitions. Let (Ω, Σ, μ) be a nonatomic measure space. An r.i. space X on (Ω, μ) is said to be generated by a quasi-Banach lattice E on $I = (0, \mu(\Omega))$ equipped with the Lebesgue measure provided that the following conditions are satisfied: $f \in X$ if and only if $f^* \in E$ and $\|f\|_X = \|f^*\|_E$.

In [8, Theorem 2.3] it is proved that if X is an r.i. quasi-Banach space on a non-atomic measure space $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ generated by an r.i. quasi-Banach space

E on I which is 1-concave on the cone Q of decreasing simple functions, i.e., there exists $C > 0$ such that for $f_1, \dots, f_n \in Q$,

$$C\,\|f_1 + \dots + f_n\|_E \ge \|f_1\|_E + \dots + \|f_n\|_E\,,$$

and $\psi(t) = \psi_E(t) := \|\chi_{(0,t)}\|_E$ for all $t \in I$ denotes the fundamental function of E, then

(i) *$X' = M_\psi$ and $X' \neq \{0\}$ if and only if $\widehat{\psi}(t) > 0$ for all $t \in I$, where*

$$\widehat{\psi}(t) = t \inf\{\psi(s)/s; 0 < s \le t\}, \quad t \in I.$$

(ii) *If E is order continuous and $\widehat{\psi}(t) > 0$ for all $t \in I$, then the Banach envelope $\widehat{X}$ of X coincides up to equivalence of norms with the Lorentz space $\Lambda_{\widehat{\psi}}$.*

Observe that if X is a φ-admissible r.i. space on (Ω, μ) generated by an r.i. space F on I, then X_φ is an r.i. space generated by F_φ and the fundamental function ψ of $E := X_\varphi$ is given by $\psi(t) = 1/\varphi^{-1}(1/\psi_F(t))$ for all $t \in I$, where ψ_F is the fundamental function of F. Thus if E is 1-concave on a cone Q, then the above result applies. We refer to [8], where examples of 1-concave r.i. spaces are shown.

We provide examples of Orlicz spaces for which any bilinear mapping on the product of these spaces satisfies an inequality (iii) (and so also each of the two equivalent inequalities in (i) and (ii)) of Theorem 3.2. At first we observe that if X is a Banach function lattice on (Ω, Σ, μ) and $\varphi \in \Phi$, then $X \subset (X_\varphi)_{\varphi^{-1}}$ and

$$\|f\|_{(X_\varphi)_{\varphi^{-1}}} \le \|f\|_X, \quad f \in X.$$

If in addition there exists $C > 0$ such that $\varphi(s)\varphi(t) \le C\varphi(st)$ for all $s, t > 0$, then it is easy to see that $(X_\varphi)_{\varphi^{-1}} \subset X$. Moreover, there exists a constant C_1 such that

$$\|f\|_X \le C_1 \|f\|_{(X_\varphi)_{\varphi^{-1}}}, \quad f \in (X_\varphi)_{\varphi^{-1}}.$$

Thus in the case when $(X_\varphi)_{\varphi^{-1}}$ is a quasi-normed space, we have $(X_\varphi)_{\varphi^{-1}} = X$ up to equivalence of quasi-norms. In particular this implies that the Banach envelope of $(X_\varphi)_{\varphi^{-1}}$ coincides with X up to equivalence of norms. For the special case $X = L_1$, $(L_\varphi)_{\varphi^{-1}} = L_1$.

We state and prove a lemma which provides the mentioned examples. In order to do this we define an operation for pairs of functions from Φ as follows: if φ_1, φ_2 in Φ the function $\varphi_1 \oplus \varphi_2 \colon [0, \infty) \to [0, \infty)$ is given by

$$(\varphi_1 \oplus \varphi_2)(t) := \inf_{s>0} \left(\varphi_1(s) + \varphi_2(t/s)\right), \quad t \ge 0.$$

Lemma 3.3. *For $j = 0, 1$ let $L_{\varphi_j} = (L_{\varphi_j}(\mu_j), \|\cdot\|_{\varphi_j})$ be Orlicz spaces on a measure space (Ω_j, μ_j) generated by $\varphi_j \in \Phi$ satisfying $\varphi_j(s)\varphi_j(t) \le C_j\varphi_j(st)$ for some $C_j > 0$ and all $s, t > 0$. If T is a bilinear operator from $L_{\varphi_1} \times L_{\varphi_2}$ into a quasi-Banach space E with $\|T\| \le 1$, then for $\phi = \varphi_1 \oplus \varphi_2$ we have*

$$\phi(\|T(f,g)\|_E) \le C_1 \int_{\Omega_1} \varphi_1(|f|)\, d\mu_1 + C_2 \int_{\Omega_2} \varphi_2(|g|)\, d\mu_2, \quad (f,g) \in L_{\varphi_1} \times L_{\varphi_2}.$$

Proof. First observe that our hypothesis implies that $\varphi_1, \varphi_2 \in (\Delta_2)$. Thus for every $0 \neq f_j \in L_{\varphi_j}$ with $j = 0, 1$ we have

$$\begin{aligned} C_j \int_{\Omega_j} \varphi_j(|f_j|)\, d\mu_j &= C_j \int_{\Omega_j} \varphi_j \left(\|f_j\|_{\varphi_j} \frac{|f|}{\|f_j\|_{\varphi_j}} \right) d\mu_j \\ &\geq \varphi_j(\|f_j\|_{\varphi_j}) \int_{\Omega_j} \varphi_j \left(\frac{|f_j|}{\|f_j\|_{\varphi_j}} \right) d\mu_j = \varphi_j(\|f_j\|_{\varphi_j}). \end{aligned}$$

It follows from the definition of $\varphi_1 \oplus \varphi_2$ that $\phi(st) \leq \varphi_1(s) + \varphi_2(t)$ for all $s, t \geq 0$. We now combine the inequalities shown above with $\|T\| \leq 1$ to get

$$\begin{aligned} \phi\left(\|T(f,g)\|_E\right) &\leq \phi(\|f\|_{\varphi_1}\|g\|_{\varphi_2}) \leq \varphi_1(\|f\|_{\varphi_1}) + \varphi_2(\|g\|_{\varphi_2}) \\ &\leq C_1 \int_{\Omega_1} \varphi_1(|f|)\, d\mu_1 + C_2 \int_{\Omega_2} \varphi_2(|g|)\, d\mu_2, \end{aligned}$$

and this completes the proof. $\square$

We remark that under some mild conditions on φ_1 and φ_2 it is possible to estimate or even calculate $\phi = \varphi_1 \oplus \varphi_2$. To see this suppose that $\varphi \in \Phi$ is such that

$$\varphi^{-1}(t) \geq \varphi_1^{-1}(t)\, \varphi_2^{-1}(t), \quad t \geq 0.$$

We have that $\phi(st) \leq \varphi_1(s) + \varphi_2(t)$ for all $s, t > 0$, which implies $\varphi \leq \varphi_1 \oplus \varphi_2$.

For the next examples assume that $\varphi \in \Phi$ is subadditive (i.e., $\varphi(u+v) \leq \varphi(u)+\varphi(v)$ for all $u, v > 0$). This yields $\varphi(2\sqrt{uv}) \leq \varphi(u+v) \leq \varphi(u) + \varphi(v)$, $u, v \geq 0$. In consequence $\varphi(2\sqrt{t}) \leq (\varphi \oplus \varphi)(t)$ for all $t \geq 0$. If $\varphi \in \Phi$ is convex, then $\varphi(\sqrt{uv}) \leq \varphi((u+v)/2) \leq (\varphi(u) + \varphi(v))/2$ implies that $2\varphi(\sqrt{t}) \leq (\varphi \oplus \varphi)(t)$ and so $(\varphi \oplus \varphi)(t) = 2\varphi(\sqrt{t})$ for all $t \geq 0$.

Notice that if X is a quasi-Banach lattice which contains no copy of c_0 and has a weak unit then a standard representation theorem can be applied to represent X as an order continuous quasi-Banach function lattice on a measure space (Ω, Σ, μ) where Ω is a compact Hausdorff space and Σ is the σ-algebra of Borel sets of Ω (for details see [7]).

We state now a theorem in which the domain spaces are abstract quasi-Banach lattices. The proof is similar to the one given above.

Theorem 3.4. *Let $0 < p, q < \infty$ and let X and Y be quasi-Banach lattices such that both duals $(X_{1/p})^*$ and $(Y_{1/q})^*$ separates the points of $X_{1/p}$ and $Y_{1/q}$, respectively. Assume $\phi \in \Phi$, $0 < C_1, C_2 < \infty$ and that $A \subset X$, $B \subset Y$ are non-empty sets. The following are equivalent statements about a bilinear operator T from $X \times Y$ to a quasi-Banach space E.*

(*i*) *For any set of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and any sets $\{f_k\}_{k=1}^n$ in A and $\{g_k\}_{k=1}^n$ in B, $n \in \mathbb{N}$,*

$$\sum_{k=1}^n \alpha_k \phi(\|T(f_k, g_k)\|_E) \leq C_1 \left\| \sum_{k=1}^n \alpha_k |f_k|^p \right\|_{X_{1/p}}^c + C_2 \left\| \sum_{k=1}^n \alpha_k |g_k|^q \right\|_{Y_{1/q}}^c.$$

(*ii*) *There exist positive functionals* $x^* \in B_{(X_{1/p})^*}$ *and* $y^* \in B_{(Y_{1/q})^*}$ *such that*

$$\phi\left(\|T(f,g)\|_E\right) \le C_1\, x^*(|f|^p) + C_2\, y^*(|g|^q), \quad (f,g) \in A \times B.$$

Recall that Grothendieck's Theorem states that if K_1, K_2 are compact Hausdorff spaces and U is a bilinear functional on $C(K_1) \times C(K_2)$, then there are probability Borel measures μ_1, μ_2, on K_1, K_2, respectively, such that for every $(f_1, f_2) \in C(K_1) \times C(K_2)$,

$$|U(f_1,f_2)| \le K\|U\| \left(\int_{K_1} |f_1|^2 d\mu_1\right)^{1/2} \left(\int_{K_2} |f_2|^2 d\mu_2\right)^{1/2},$$

where K is an absolute constant (the smallest value of the constant K is usually called the Grothendieck's constant and is denoted by K_G). Notice that the above estimate is equivalent to

$$|U(f_1,f_2)| \le \frac{1}{2} K\|U\| \left(\int_{K_1} |f_1|^2 d\mu_1 + \int_{K_2} |f_2|^2 d\mu_2\right).$$

Therefore the Grothendieck's theorem gives in fact an example of the inequalities that appear in Theorem 3.4.

We show now some applications of our results. Recall that a quasi-Banach lattice X is p-convex, where $0 < p < \infty$, if there is a constant $C > 0$ so that $x_1, ..., x_n \in X$,

$$\left\| \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \right\|_X \le C \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}, \quad x_1, ..., x_n \in X.$$

The smallest value of the constant C is denoted by $M^{(p)}(X)$. We notice the well-known easily verified fact that X is p-convex if and only if $X_{1/p}$ is normable.

By using techniques similar to those of Theorem 3.4 we prove the following more general result which extend Defant's theorem [2, Theorem 1] for bilinear functionals on Banach function lattices defined on measure spaces.

We point out that the proof of the multilinear variant of the result stated can be obtained by using a minor modification of the arguments outlined below.

Theorem 3.5. *Let* $0 < p, q < \infty$ *and let* $1/r = 1/p + 1/q$. *Assume that* X *and* Y *are quasi-Banach lattices such that* X *is strongly* $1/p$*-admissible and* Y *is strongly* $1/q$*-admissible. Assume* $0 < C < \infty$. *The following are equivalent statements about a bilinear operator* T *from* $X \times Y$ *to a quasi-Banach space* E.

(*i*) *For each couple of finite sets* $\{f_k\}_{k=1}^n$ *and* $\{g_k\}_{k=1}^n$ *of elements of* X *and* Y, *respectively,*

$$\left(\sum_{k=1}^n \|T(x_k,y_k)\|_E^r\right)^{1/r} \le C \left(\left\| \left(\sum_{k=1}^n |x_k|^p\right) \right\|_{X_{1/p}}^c \right)^{1/p} \left(\left\| \left(\sum_{k=1}^n |y_k|^q\right) \right\|_{Y_{1/q}}^c \right)^{1/q}.$$

(*ii*) *There are positive functionals* $x^* \in B_{(X_{1/p})^*}$ *and* $y^* \in B_{(Y_{1/q})^*}$ *such that*

$$\|T(x,y)\|_E \le C\ (x^*(|x|^p))^{1/p}\,(x^*(|y|^q))^{1/q}\,,\quad (x,y)\in X\times Y.$$

Proof. $(i) \Rightarrow (ii)$. We apply Theorem 3.4. Consider the spaces $(X_{1/p})^*$ and $(Y_{1/q})^*$, i.e., $(X_{1/p}, \|\cdot\|^c)^*$ and $(Y_{1/q}, \|\cdot\|^c)^*$, respectively, and the corresponding unit balls. They are normed spaces by the strong admissibility of both spaces. Using Young's inequality $\frac{(\alpha\beta)^r}{r} \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$, $\alpha, \beta \ge 0$, we conclude that for any finite collection of positive scalars $\alpha_1, ..., \alpha_n$ with $\sum_{k=1}^n \alpha_k = 1$ and any finite collection of elements $x_1, ..., x_n \in X$ and $y_1, ..., y_n \in Y$ the estimate shown in the statement (i) is equivalent to the following one with $C_1 := rC^r/p$ and $C_2 := rC^r/q$

$$\sum_{k=1}^n \|T(x_k,y_k)\|^r \le C_1 \left\|\sum_{k=1}^n |x_k|^p\right\|^c_{X_{1/p}} + C_2 \left\|\sum_{k=1}^n |y_k|^q\right\|^c_{Y_{1/q}},$$

which is equivalent to

$$\sum_{k=1}^n \|T(x_k,y_k)\|_E^r \le C_1 \sup_{x^*\in B_{(X_{1/p})^*}} \sum_{k=1}^n \alpha_k x^*(|x_k|^p) + C_2 \sup_{y^*\in B_{(Y_{1/q})^*}} \sum_{k=1}^n \alpha_k y^*(|y_k|^q).$$

To see this it is enough to observe that for every $x^* \in B_{(X_{1/p})^*}$ and $y^* \in B_{(Y_{1/q})^*}$ the inequality

$$\sum_{k=1}^n \alpha_k \|T(x_k,y_k)\|^r \le C_1 \sum_{k=1}^n \alpha_k x^*(|x_k|^p) + C_2 \sum_{k=1}^n \alpha_k x^*(|y_k|^q)$$

is equivalent to

$$\sum_{k=1}^n \|T(x_k,y_k)\|_E^r \le C_1 \sum_{k=1}^n x^*(|x_k|^p) + C_2 \sum_{k=1}^n y^*(|y_k|^q),$$

by $\alpha_k \|T(x_k,y_k)\|_E^r = \|T(\alpha_k^{1/p}x_k, \alpha_k^{1/q}y_k)\|_E$, $\alpha_k|x_k|^p = |\alpha_k^{1/p}x_k|^p$ and $\alpha_k|y_k|^q = |\alpha_k^{1/q}y_k|^q$.

Combining the above remarks, we conclude that the statement (i) implies the statement (i) in Theorem 3.2 with shown above constants C_1 and C_2. However, it follows by Theorem 3.4 that the last statement is equivalent to the following: there are positive functionals $x^* \in B_{(X_{1/p})^*}$ and $y^* \in B_{(Y_{1/q})^*}$ such that for all $(x,y) \in X \times Y$,

$$\|T(x,y)\|_E^r \le C_1\, x^*(|x|^p) + C_2\, y^*(|y|^q),$$

and so

$$\frac{1}{r}\|T(x,y)\|_E^r \le \frac{C^r}{p}\, x^*(|x|^p) + \frac{C^r}{q}\, y^*(|y|^q). \qquad (*)$$

We now observe that inequality $(*)$ is equivalent to

$$\|T(x,y)\|_E \le C\ (x^*(|x|^p))^{1/p}\,(x^*(|x|^q))^{1/q}\,,\quad (x,y)\in X\times Y.$$

In fact, it is enough to verify the mentioned equivalence for all $(x,y) \in X \times Y$ such that $\alpha := x^*(|x|^p) > 0$ and $\beta := y^*(|y|^q) > 0$. Clearly, $(*)$ yields

$$\begin{aligned}\|T(x,y)\|_E^r &= \alpha^{r/p}\beta^{r/q}r\,\|T(\alpha^{-1/p}x, \beta^{-1/q}y)\|_E^r \\ &\le \alpha^{r/p}\beta^{r/q}r\left(\frac{C^r}{p}\alpha^{-1}x^*(|x|^p) + \frac{C^r}{q}\beta^{-1}y^*(|y|^q)\right) \\ &= C^r\, x^*(|x|^p)^{r/p}y^*(|y|^q)^{r/q}.\end{aligned}$$

The implication $(ii) \Rightarrow (i)$ follows by a direct calculation using duality with the norms $\|\cdot\|_{X_{1/p}}^c$ and $\|\cdot\|_{Y_{1/q}}^c$. □

Notice that if we assume that X is p-convex and Y is q-convex with $M^{(p)}(X) = M^{(q)}(Y) = 1$, then both $X_{1/p}$ and $Y_{1/q}$ are Banach spaces and so $(X_{1/p})^c = X_{1/p}$ and $(Y_{1/q})^c = Y_{1/q}$ with equality of norms. As a consequence we obtain the result due to Defant [2, Theorem 1].

The following variant of Grothendieck's theorem was proved by Blei [1, Theorem 3.2] and is a consequence of the multilinear version of Theorem 3.5 in combination with the Riesz Representation Theorem and the fact that $C(K)$-spaces are p-convex for every $0 < p < \infty$. Note that Grothendieck's theorem in the form presented after 3.4 fails in higher dimensions (see [1, p. 530]).

Corollary 3.6. *Suppose that K_1,...,K_n are compact Hausdorff spaces and let $1 \le p_j < \infty$, for $j = 1, ..., n$. Let $0 < C < \infty$ and $\sum_{j=1}^n 1/p_j = 1$. The following are equivalent statements about an n-linear functional U on $C(K_1) \times ... \times C(K_n)$:*

(i) *For any set $\left\{f_j^{(k)}\right\}_{j=1}^m$ in $C(K_k)$ with $k = 1, ..., n$*

$$\sum_{j=1}^m \left|U(f_j^{(1)}, ..., f_j^{(n)})\right| \le C\left\|\left(\sum_{j=1}^m |f_j^{(1)}|^{p_1}\right)^{1/p_1}\right\|_{C(K_1)} \cdots \left\|\left(\sum_{j=1}^m |f_j^{(n)}|^{p_n}\right)^{1/p_n}\right\|_{C(K_n)}.$$

(ii) *There exist probability Borel measures $\mu_1, ..., \mu_n$ on $K_1, ..., K_n$, respectively, so that*

$$|U(f_1, ..., f_n)| \le C\left(\int_{K_1} |f_1|^{p_1}\, d\mu_1\right)^{1/p_1} \cdots \left(\int_{K_n} |f_n|^{p_n}\, d\mu_n\right)^{1/p_n}$$

for all $f_1 \in C(K_1)$,...,$f_n \in C(K_n)$.

To show the next application we recall that a Banach lattice E if *p-concave* for $1 < p < \infty$ if there is a constant $C > 0$ so that

$$\left(\sum_{k=1}^n \|x_k\|_E^p\right)^{1/p} \le C\left\|\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\right\|_E, \quad x_1, ..., x_n \in E.$$

Corollary 3.7. *Let $1 < p < \infty$, $1 < p_i < \infty$ for $i = 1, ..., n$ be such that $1/p = 1/p_1 + \ldots + 1/p_n$. Assume that X_k is a p_k-convex Banach lattice for each $k = 1, ..., n$, and let E be a p-concave Banach lattice. For any n-linear positive operator $T\colon X_1 \times \ldots \times X_n \to E$ there exist a constant $C > 0$ and positive functionals $x_k^* \in B_{(X_{1/p_k})^*}$ so that for all $x_1 \in X_1, ..., x_n \in X_n$,*

$$\|T(x_1, ..., x_n)\|_E \leq C\,(x_1^*(|x_1|^{p_1}))^{1/p_1} \cdots \left(x_n^*(|x_n|^{1/p_n})\right)^{1/p_n}.$$

Proof. The key fact which we use is the inequality from [3, Theorem 6.2] which says that, for every choice of finitely many sequences $\{x_j^{(k)}\}_{j=1}^r$ in X_k, $1 \leq k \leq n$,

$$\begin{aligned}&\left\|\left(\sum_{j=1}^r |T(x_j^{(1)}, \ldots, x_j^{(n)})|^p\right)^{1/p}\right\|_E \\ &\leq \|T\| \left\|\left(\sum_{j=1}^r |x_j^{(1)}|^{p_1}\right)^{1/p_1}\right\|_{X_1} \cdots \left\|\left(\sum_{j=1}^r |x_j^{(n)}|^{p_n}\right)^{1/p_n}\right\|_{X_n},\end{aligned}$$

and so our hypothesis on p-concavity yields that for an absolute constant C,

$$\begin{aligned}&\left(\sum_{j=1}^r \left\|T(x_j^{(1)}, \ldots, x_j^{(n)})\right\|_E^p\right)^{1/p} \\ &\leq C\|T\| \left\|\left(\sum_{j=1}^r |x_j^{(1)}|^{p_1}\right)^{1/p_1}\right\|_{X_1} \cdots \left\|\left(\sum_{j=1}^r |x_j^{(n)}|^{p_n}\right)^{1/p_n}\right\|_{X_n}.\end{aligned}$$

Using the multilinear version of Theorem 3.5 we obtain the required statement. □

4. Factorization of bilinear operators

In this section we present our results when they are applied to bilinear forms in terms of factorization theorems for linear operators. As usual for a given σ-algebra of subsets of a set Ω and $A \in \Sigma$, $\Sigma_A = \{F \cap A;\, F \in \Sigma\}$ denotes a σ-algebra of subsets of A.

Theorem 4.1. *Let $\varphi_1, \varphi_2 \in \Phi$ with $\varphi_1, \varphi_2 \in (\Delta_2)$, and let X, Y be order continuous Banach function lattices on (Ω_1, μ_1) and (Ω_2, μ_2), respectively, such that X is strongly φ_1^{-1}-admissible and Y is strongly φ_2^{-1}-admissible. Suppose $0 < C_1, C_2 < \infty$ and consider an operator $S\colon X \to Y'$, where Y' is the Köthe dual of Y. Consider the following statements about the bilinear form $T\colon X \times Y \to \mathbb{R}$ defined by $T(f, g) = \int_{\Omega_2} gS(f)\, d\mu_2$ for all $(f, g) \in X \times Y$:*

(*i*) *For every finite sequence of positive scalars $\{\alpha_k\}_{k=1}^n$ with $\sum_{k=1}^n \alpha_k = 1$ and every finite sequences $\{f_k\}_{k=1}^n$ in X and $\{g_k\}_{k=1}^n$ in Y,*

$$\sum_{k=1}^n \alpha_k |T(f_k, g_k)| \leq C_1 \left\|\sum_{k=1}^n \alpha_k \varphi_1(|f_k|)\right\|^c_{X_{\varphi_1^{-1}}} + C_2 \left\|\sum_{k=1}^n \alpha_k \varphi_2(|g_k|)\right\|^c_{Y_{\varphi_2^{-1}}}.$$

(*ii*) *There exist functions* $0 \leq u \in B_{(X_{\varphi_1^{-1}})'}$ *and* $0 \leq v \in B_{(X_{\varphi_2^{-1}})'}$ *such that*

$$|T(f,g)| \leq C_1 \left(\int_{\Omega_1} \varphi_1(|f|) u d\mu_1 \right) + C_2 \left(\int_{\Omega_2} \varphi_2(|g|) v d\mu_2 \right), \quad (f,g) \in X \times Y.$$

(*iii*) *There exist* $0 \leq u \in (X_{\varphi_1^{-1}})'$ *and* $0 \leq v \in (X_{\varphi_2^{-1}})'$ *such that* S *admits the following factorization*:

$$\begin{tikzcd} X \arrow[rr, "S"] \arrow[d, dotted, "i_1"'] & & Y' \\ L_{\varphi_1}(\nu_1) \arrow[rr, dotted, "\widehat{S}"] & & (L_{\varphi_2}(\nu_2))' \arrow[u, dotted, "i_2"'] \end{tikzcd}$$

where $L_{\varphi_1}(\nu_1)$ *and* $L_{\varphi_1}(\nu_2)$ *are Orlicz spaces on* $(A_1, \Sigma_{A_1}, \nu_1)$ *with* $A_1 = \operatorname{supp} u$, $d\nu_1 = u d\mu_1$ *and on* $(A_2, \Sigma_{A_2}, \nu_2)$ *with* $A_2 = \operatorname{supp} v$, $d\nu_2 = v d\mu_2$ *and* $i_1 \colon X \to L_{\varphi_1}(\nu_1)$, $i_2 \colon L_{\varphi_2}(\nu_2)$ *are operators given by* $i_1(f) = f\chi_{A_1}$ *for all* $f \in X$ *and* $i_2(g) = g\chi_{A_2}$ *for all* $g \in Y$.

Then (*i*) *is equivalent to* (*ii*) *and both of them imply* (*iii*).

Proof. The equivalence between (*i*) and (*ii*) is a consequence of Theorem 3.2.

For (*ii*) $\Rightarrow$ (*iii*), first notice that the last computations in the proof of Theorem 3.5 can also be done for the general case of a couple of Orlicz functions φ_1 and φ_2: for a bilinear functional and positive constants a, b and $\phi(t) = t$ for all $t \geq 0$, condition (*iii*) in Theorem 3.2 can be written as

$$|T(f,g)| = ab|T(f/a, g/b)| \leq ab \left(\int_{\Omega_1} \varphi_1(|f|/a) u d\mu_1 + \int_{\Omega_2} \varphi_2(|g|/b) v d\mu_2 \right).$$

Let us define the measures $d\nu_1 = u d\mu_1$ and $d\nu_2 = v d\mu_2$. Consider now the Orlicz spaces $L_{\varphi_1}(\nu_1)$ and $L_{\varphi_2}(\nu_2)$ for adequate functions u and v. Then for the case $a = \|f\|_{L_{\varphi_1}(\nu_1)}$ and $b = \|g\|_{L_{\varphi_2}(\nu_2)}$ we obtain that the Orlicz space version of the computations in the proof of Theorem 3.5 gives

$$|T(f,g)| \leq 2\|f\|_{L_{\varphi_1}(\nu_1)} \|g\|_{L_{\varphi_2}(\nu_2)}.$$

This means that the equivalent conditions in Theorem 3.2 imply continuity of the bilinear map when defined on the corresponding Orlicz spaces. Clearly, for $f \in X$ we have that $\varphi_1(|f|) \in X_{\varphi_1^{-1}}$ and then $\int \varphi_1(|f|) d\nu_1 < \infty$, and the same holds for g, φ_2 and Y. This implies that $f\chi_{A_1} \in L_{\varphi_1}(\nu_1)$ and $g\chi_{A_2} \in L_{\varphi_2}(\nu_2)$ and finishes the proof. $\square$

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