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A Generalization of Blaschke's Convergence Theorem in Metric Spaces

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A metric space (X, d) together with a set-valued mapping $G : X \times X \to 2^X$ is said to be a *generalized segment space* (X, d, G) if $G(x, y) \neq \emptyset$ for all $x, y \in X$ and for any sequences $x_n \to x$ and $y_n \to y$ in X, $d_H(G(x_n, y_n), G(x, y)) \to 0$ as $n \to \infty$, where d_H is the Hausdorff distance. Normed linear spaces, nonempty convex sets, and proper uniquely geodesic spaces, etc are generalized segment spaces for suitable G. A subset A of X is called *G-type convex* if $G(x, y) \subset A$ whenever $x, y \in A$. We prove a generalization of Blaschke's convergence theorem for metric spaces: if (X, d, G) is a proper generalized segment space, then every uniformly bounded sequence of nonempty G-type convex subsets of X contains a subsequence which converges to some nonempty compact G-type convex subset in X.

Keywords: Blaschke's convergence theorem; convex sets; generalized convexity, geodesic convex sets; geodesic segments; Hausdorff distance; uniquely geodesic spaces

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1. Introduction

Blaschke's convergence theorem is a classical theorem in convex analysis. It says that a uniformly bounded infinite collection of nonempty closed convex sets in a finite dimensional space contains a sequence which converges to a nonempty compact convex set (see [9]). There are generalizations of this theorem for infinite dimensional normed linear spaces (see [7]) or for sets that are not convex in the

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usual sense such as star-shaped sets (see [5]). Thus it is natural to find versions of the theorem for generalized convex sets. One of the first papers dealing with geodesic convexity in a simple polygon on the plane was presented by Toussaint [8]. In [4], we dealt with analytic properties of geodesic convex sets in a simple polygon. Among the others, Blaschke's convergence theorem for geodesic convex sets was presented. The aim of this paper is to develop some basic ideas in [4] to get a generalization of Blaschke's convergence theorem in metric spaces. The concept of generalized segment spaces is introduced in Section 2. Loosely speaking, it is a triple (X, d, G) where (X, d) is a metric space and $G : X \times X \to 2^X$ a "continuous" set-valued mapping with nonempty values. We present several familiar classes of spaces that are generalized segment spaces such as normed linear spaces, nonempty convex sets, and proper uniquely geodesic spaces, etc. Generally, any metric spaces whose segments vary continuously with their end points are generalized segment spaces. It is obvious that on X, by changing either d or G or both, we may have different generalized segment spaces. So in Section 3, some ways of obtaining generalized segment spaces from a given one are showed. Among the others, we prove that in normed linear spaces, the concept of generalized segment space depends only on the topology of the space but not on equivalent norms (Proposition 3.6). The same is also true for metric spaces if G takes on relatively compact values (Theorem 3.7). This holds, in particular, if X is compact. In Section 4, we introduce the notion of G-type convexity, namely, a subset A of X is said to be *G-type convex* if $G(x, y) \subset A$ whenever $x, y \in A$. The usual convexity in linear spaces and geodesic convexity in uniquely geodesic spaces are special cases of G-type convexity. Some properties of convex sets are still true for G-type convex sets (Propositions 4.4 and 4.6 (*ii*)). We then get a generalization of Blaschke's convergence theorem for metric spaces: if (X, d, G) is a proper generalized segment space, then every uniformly bounded sequence of nonempty G-type convex subsets of X contains a subsequence which converges to some nonempty compact G-type convex subset in X (Theorem 4.10). In particular, in a proper uniquely geodesic space, every uniformly bounded sequence of nonempty geodesically convex sets contains a subsequence which converges to a nonempty compact geodesically convex set (Corollary 4.11). Theorem 4.10 also reduces to the classical Blaschke's convergence theorem when X is a finite dimensional normed linear space and $G(x, y)$ is the line segment joining x and y.

Before starting the analysis, we recall some definitions and properties from [2]. Let (X, d) be a metric space. For $a \in X$ and $r > 0$ we denote

$$B(a, r) = \{x \in X : \ d(x, a) < r\} \quad \text{and} \quad \bar{B}(a, r) = \{x \in X : \ d(x, a) \leq r\},$$

the open and closed balls center a and radius r, respectively. For nonempty subsets A and B of X, and $x \in X$, let $d(x, A) = \inf_{a \in A} d(x, a)$, the distance from x to A, and

$$d_H(A, B) = \max\left\{\sup_{u \in A} d(u, B), \ \sup_{v \in B} d(v, A)\right\}.$$

Then $d_H(\bar{A}, \bar{B}) = d_H(\bar{A}, B) = d_H(A, \bar{B}) = d_H(A, B)$, where $\bar{S}$ denotes the closure of S. Let $\mathcal{K}(X)$ be the family of all nonempty compact subsets of X. Then d_H is a metric on $\mathcal{K}(X)$, called the *Hausdorff metric.* If X is compact, so is $(\mathcal{K}(X), d_H)$ (see [2]).

If X is a linear space, for $x, y \in X$, the set $[x, y] = \{(1-\lambda)x + \lambda y : \ 0 \le \lambda \le 1\}$ is called the *line segment* joining x and y.

2. Definition of Generalized Segment Spaces and Examples

In this paper, let 2^X denote the collection of all subsets of X.

Definition 2.1. A *generalized segment space* is a metric space (X, d) together with a set-valued mapping $G : X \times X \to 2^X$ that satisfy the following conditions:

(A1) $G(x, y) \neq \emptyset$ for all $x, y \in X$;
(A2) If (x_n) and (y_n) are sequences in X, $x_n \to x$ and $y_n \to y$, then

$$d_H\left(G(x_n, y_n), G(x, y)\right) \to 0 \quad \text{as } n \to \infty.$$

We denote this generalized segment space by (X, d, G). It should be noted that in Definition 2.1, it is not required $x, y \in G(x, y)$. Let's consider some examples of generalized segment spaces. In the first one, G takes on convex values (i.e., $G(x, y)$ is convex for all $x, y \in X$) and may be single-valued.

Example 2.2. Suppose that X is a normed linear space, $d(x, y) = \|x - y\|$, and $\lambda \ge 0$ is a fixed number. Let $G_\lambda(x, y) = \bar{B}((x + y)/2, \lambda\|x - y\|)$, the closed ball center $\frac{1}{2}(x + y)$ and radius $\lambda\|x - y\|$. Then (X, d, G_λ) is a generalized segment space. To prove this, we first show that for all $a, b, x, y \in X$,

$$\begin{aligned} & d_H\left(\bar{B}\left(\frac{x+y}{2}, \lambda\|x-y\|\right), \bar{B}\left(\frac{a+b}{2}, \lambda\|a-b\|\right)\right) \\ & \le (2\lambda + 1)\max\{\|a - x\|, \|b - y\|\}. \end{aligned} \tag{1}$$

Let

$$c = \frac{1}{2}(a + b), \qquad z = \frac{1}{2}(x + y), \qquad r = \lambda\|a - b\|, \qquad \sigma = \lambda\|x - y\|,$$

and

$$\alpha = \max\{\|a - x\|, \|b - y\|\}.$$

Since $\big|\|a - b\| - \|x - y\|\big| \le \|a - x\| + \|b - y\| \le 2\alpha$, we have

$$|r - \sigma| = \lambda\big|\|a - b\| - \|x - y\|\big| \le 2\lambda\alpha. \tag{2}$$

Moreover,

$$\|z - c\| = \left\|\frac{x + y}{2} - \frac{a + b}{2}\right\| \le \frac{1}{2}(\|a - x\| + \|b - y\|) \le \alpha. \tag{3}$$

Now take any $v \in \bar{B}(z, \sigma)$. If $\|v - c\| \le r$, then $d(v, \bar{B}(c, r)) = 0 \le (2\lambda + 1)\alpha$. Suppose $\|v - c\| > r$. Let $u = c + \frac{r(v-c)}{\|v-c\|}$, then $u \in \bar{B}(c, r)$ and by (2)–(3),

$$\|v - u\| = \|v - c\| - r \le \|v - z\| + \|z - c\| - r \le \sigma + \alpha - r \le (2\lambda + 1)\alpha.$$

Hence $d(v, \bar{B}(c,r)) \leq \|v-u\| \leq (2\lambda+1)\alpha$ and so $\sup_{v\in\bar{B}(z,\sigma)} d(v, \bar{B}(c,r)) \leq (2\lambda+1)\alpha$. Likewise, $\sup_{u\in\bar{B}(c,r)} d(u, \bar{B}(z,\sigma)) \leq (2\lambda+1)\alpha$ and (1) is proved.

Suppose now that $x_n \to x$ and $y_n \to y$. Using (1) we get

$$d_H\left(G_\lambda(x_n,y_n), G_\lambda(x,y)\right) \leq (2\lambda+1)\max\{\|x_n-x\|, \|y_n-y\|\} \to 0.$$

Therefore, (X,d,G_λ) is a generalized segment space.

From now on, we assume that metric on a normed linear space $(X, \|\cdot\|)$ is the usual metric $d(x,y) = \|x-y\|$. The next theorem gives more interesting examples of generalized segment spaces.

Theorem 2.3.

(*i*) *Any normed linear space is a generalized segment space with* $G(x,y) = [x,y]$.

(*ii*) *Any nonempty convex set in a normed linear space is a generalized segment space.*

Proof. Let $G(x,y) = [x,y]$ then (A1) is satisfied. To prove (A2), we need the following inequality:

$$d_H\left([u,v],[x,y]\right) \leq \max\{\|u-x\|, \|v-y\|\} \quad \text{for all } u,v,x,y \in X. \tag{4}$$

Take any $w \in [u,v]$, $w = (1-t)u + tv$ for some $t \in [0,1]$. Let $z := (1-t)x + ty \in [x,y]$. As $w - z = (1-t)(u-x) + t(v-y)$,

$$\|w-z\| \leq (1-t)\|u-x\| + t\|v-y\| \leq \max\{\|u-x\|, \|v-y\|\}.$$

Thus

$$d\left(w,[x,y]\right) \leq \|w-z\| \leq \max\{\|u-x\|, \|v-y\|\} \quad \text{for all } w \in [u,v].$$

Similarly,

$$d\left(z,[u,v]\right) \leq \max\{\|u-x\|, \|v-y\|\} \quad \text{for all } z \in [x,y]$$

so that (4) holds. That (A2) is satisfied follows directly from (4). □

Example 2.4. Consider a simple polygon $\mathcal{P}$ on the plane equipped with the Euclidian distance d. For any $x, y \in \mathcal{P}$, there is a unique shortest segment $G(x,y)$ that joins x and y and is totally contained in $\mathcal{P}$. This shortest segment is a polyline (see [8]). In [4], Lemma 3.1, we showed that the set-valued mapping G satisfies (A2). Thus $(\mathcal{P}, d, G)$ is a generalized segment space.

Given a metric space (X,d). A *path* in X is a continuous map γ from a compact interval $[a,b] \subset I\!R$ to X. We say that γ *joins* the point $\gamma(a)$ to the point $\gamma(b)$. A *geodesic path* in X is a path $\gamma : [a,b] \to X$ such that $d(\gamma(t), \gamma(t')) = |t-t'|$ for all $t, t' \in [a,b]$; the image $\gamma([a,b])$ is called a *geodesic segment* in X joining $\gamma(a)$ and $\gamma(b)$. X is said to be a *geodesic space* if every two points in X are joined by a geodesic path. We say that X is *uniquely geodesic* if for any x and y in X, there is a unique geodesic segment joining them (see [6]). X is said to be *proper* if every closed bounded subset of X is compact. In this paper we are interested in the convergence of geodesic segments in proper uniquely geodesic spaces: if a proper metric space X is uniquely geodesic, then geodesic segments in X vary continuously with their endpoints (see [3], p. 38). This statement can be rephrased as follows.

Theorem 2.5. *If (X, d) is a proper uniquely geodesic space then (X, d, G) is a generalized segment space where $G(x, y)$ is the geodesic segment joining x and y.*

3. Making New Generalized Segment Spaces from a Given One

In Section 2 we presented several classes of spaces that are generalized segment spaces. In this section, we consider some ways of obtaining generalized segment spaces from a given one.

Proposition 3.1. *Suppose (X, d, G) is a generalized segment space and $\varphi, \psi : X \to X$ are continuous mappings. Then $(X, d, G_{\varphi\psi})$ is also a generalized segment space, where $G_{\varphi\psi}(x, y) = G(\varphi(x), \psi(y))$.*

Proof. Suppose $x_n \to x$ and $y_n \to y$. Since $\varphi, \psi : X \to X$ are continuous, $\varphi(x_n) \to \varphi(x)$ and $\psi(y_n) \to \psi(y)$. Hence

$$d_H(G_{\varphi\psi}(x_n, y_n), G_{\varphi\psi}(x, y)) = d_H(G(\varphi(x_n), \psi(y_n)), G(\varphi(x), \psi(y))) \to 0.$$

Thus $(X, d, G_{\varphi\psi})$ is a generalized segment space. □

For example, Theorem 2.3 and Proposition 3.1 imply that if X is a normed linear space, v a fixed vector, and α a scalar, then (X, d, G') and (X, d, G'') are generalized segment spaces, where $G'(x, y) = [x + v, y + v]$ and $G''(x, y) = [\alpha x, \alpha y]$.

Recall that a set A is *relatively* compact if its closure $\bar{A}$ is compact.

Theorem 3.2. *Suppose (X, d, G) is a generalized segment space and ρ is a metric on X such that $\rho(x_n, x) \to 0$ implies $d(x_n, x) \to 0$. Let $f : (X, d) \to (X, \rho)$ be a mapping and $f \circ G : X \times X \to 2^X$ defined by $f \circ G(x, y) = f(G(x, y))$. Then $(X, \rho, f \circ G)$ will be a generalized segment space if one of the following conditions holds.*

(i) *f is uniformly continuous;*

(ii) *f is continuous and $G(x, y)$ is relatively compact in (X, d) for all $x, y \in X$. In this case, $f \circ G(x, y)$ is also relatively compact.*

Proof. (i) Suppose $\rho(x_n, x) \to 0$, $\rho(y_n, y) \to 0$ and $\epsilon > 0$. Denote $\Gamma = G(x, y)$ and $\Gamma_n = G(x_n, y_n)$. Since f is uniformly continuous, there exists $\delta > 0$ such that $\rho(f(x'), f(y')) < \epsilon$ whenever $d(x', y') < \delta$. From the assumption it follows that $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$. As (X, d, G) is a generalized segment space,

$d_H(\Gamma_n, \Gamma) \to 0$ and so there is an integer N such that $d_H(\Gamma_n, \Gamma) < \delta$ for $n \geq N$. For each $u \in \Gamma$ and $n \geq N$,

$$d(u, \Gamma_n) \leq \sup_{z \in \Gamma} d(z, \Gamma_n) \leq d_H(\Gamma_n, \Gamma) < \delta.$$

Hence there is a $u_n \in \Gamma_n$ satisfying $d(u, u_n) < \delta$ for $n \geq N$. Thus, $\rho(f(u), f(u_n)) < \epsilon$, which implies $\rho(f(u), f(\Gamma_n)) < \epsilon$. Accordingly, $\sup_{u\in\Gamma} \rho(f(u), f(\Gamma_n)) \leq \epsilon$. Likewise, $\sup_{v\in\Gamma_n} \rho(f(v), f(\Gamma)) \leq \epsilon$ for $n \geq N$. Therefore,

$$\rho_H(f(\Gamma_n), f(\Gamma)) \leq \epsilon \quad \text{for } n \geq N.$$

Consequently, $\rho_H(f(\Gamma_n), f(\Gamma)) \to 0$ and $(X, \rho, f \circ G)$ is a generalized segment space.

(ii) Suppose $\rho(x_n, x) \to 0$, $\rho(y_n, y) \to 0$. Let

$$F = \overline{G(x, y)} \quad \text{and} \quad F_n = \overline{G(x_n, y_n)}, \quad n = 1, 2....$$

By assumption (ii), F, F_n are compact, so are $f(F), f(F_n)$. Thus we have

$$f(F) = \overline{f(G(x, y))} \quad \text{and} \quad f(F_n) = \overline{f(G(x_n, y_n))}, \quad n = 1, 2....$$

So $\rho_H(f(G(x_n, y_n)), f(G(x, y))) = \rho_H(f(F_n), f(F))$ and we only need to show that $\rho_H(f(F_n), f(F)) \to 0$. This follows from the fact that $d_H(F_n, F) \to 0$ and the following result. □

Lemma 3.3. *Suppose $f : (X, d) \to (Y, \rho)$ is continuous and F, F_n are subsets of X, F is relatively compact in (X, d). If $d_H(F_n, F) \to 0$, then $\rho_H(f(F_n), f(F)) \to 0$.*

Proof. Assume $d_H(F_n, F) \to 0$. If $\sup_{x\in F} \rho(f(x), f(F_n)) \not\to 0$, then there are an $\epsilon > 0$, a subsequence $(f(F_{n_i}))$ of $(f(F_n))$, and $x_{n_i} \in F$ such that

$$\rho(f(x_{n_i}), f(F_{n_i})) \geq \epsilon \quad \text{for all } i. \tag{5}$$

As $d(x_{n_i}, F_{n_i}) \leq d_H(F_{n_i}, F) \to 0$, there exist $x'_{n_i} \in F_{n_i}$ satisfying $d(x_{n_i}, x'_{n_i}) \to 0$. Since $\bar{F}$ is compact in (X, d), we can assume $d(x_{n_i}, x_0) \to 0$ for some $x_0 \in \bar{F}$. Hence $d(x'_{n_i}, x_0) \to 0$. As f is continuous,

$$\rho(f(x_{n_i}), f(F_{n_i})) \leq \rho\left(f(x_{n_i}), f(x'_{n_i})\right) \to \rho(f(x_0), f(x_0)) = 0,$$

which contradicts (5). Thus, $\sup_{x\in F} \rho(f(x), f(F_n)) \to 0$. Likewise, $\sup_{y\in F_n} \rho(f(y), f(F)) \to 0$ and we have $\rho_H(f(F_n), f(F)) \to 0$. □

Applying Lemma 3.3 for the case when f is the identity mapping of X we get

Corollary 3.4. *Suppose the identity map $(X, d) \to (X, \rho)$ is continuous and F, F_n are subsets of X, F is relatively compact in (X, d). If $d_H(F_n, F) \to 0$ then $\rho_H(F_n, F) \to 0$. Thus, if d and ρ are equivalent metrics on X and F is a relatively compact subset of X, then $d_H(F_n, F) \to 0$ iff $\rho_H(F_n, F) \to 0$.*

Coming back to Theorem 3.2, we note that in general, if (X, d, G) is a generalized segment space and $f : X \to X$ is merely a continuous mapping then $(X, d, f \circ G)$ may not be a generalized segment space. This is illustrated with the following example. Let d be the Euclidian metric on $\mathbb{R}^2$ and $G : \mathbb{R}^2 \times \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ given by

$$G\left((x^1, x^2), (y^1, y^2)\right) = \left\{(u^1, u^2) \in \mathbb{R}^2 : \ u^2 = \max\{x^2, y^2\}\right\}.$$

If $(x_n^1, x_n^2) \to (x_0^1, x_0^2)$ and $(y_n^1, y_n^2) \to (y_0^1, y_0^2)$, then $x_n^2 \to x_0^2$ and $y_n^2 \to y_0^2$. Thus $\max\{x_n^2, y_n^2\} \to \max\{x_0^2, y_0^2\}$ and so

$$d_H\left(G\left((x_n^1, x_n^2), (y_n^1, y_n^2)\right), G\left((x_0^1, x_0^2), (y_0^1, y_0^2)\right)\right) \to 0,$$

i.e., $(\mathbb{R}^2, d, G)$ is a generalized segment space. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x^1, x^2) = (0, x^1x^2)$. It is easily seen that f is continuous. Consider the sequence $\left((0, \frac{1}{n})\right)$. Clearly, $(0, \frac{1}{n}) \to (0, 0)$,

$$G\left(\left(0, \frac{1}{n}\right), (0,0)\right) = \left\{\left(x, \frac{1}{n}\right) : \ x \in \mathbb{R}\right\}$$

and

$$G\left((0,0), (0,0)\right) = \{(x, 0) : \ x \in \mathbb{R}\}.$$

Hence,

$$f \circ G\left(\left(0, \frac{1}{n}\right), (0,0)\right) = \left\{\left(0, \frac{x}{n}\right) : \ x \in \mathbb{R}\right\}$$

and

$$f \circ G\left((0,0), (0,0)\right) = \{(0,0)\}.$$

Thus, $d_H\left(f \circ G\left((0, \frac{1}{n}), (0,0)\right), f \circ G\left((0,0), (0,0)\right)\right) = \infty$, which shows that $(\mathbb{R}^2, d, f \circ G)$ is not a generalized segment space.

A direct consequence of Theorem 3.2 is the following.

Corollary 3.5. *Suppose (X, d, G) is a generalized segment space and $f : X \to X$ is continuous. Then $(X, d, f \circ G)$ is a generalized segment space if G takes on bounded values and X is proper.*

Next we consider the case when the metric on a generalized segment space is replaced with another equivalent one. Recall that two metrics d and ρ on a set X are said to be *equivalent* if a sequence (x_n) of X satisfies $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$. In other words, d and ρ are equivalent if and only if d and ρ generate the same topology on X (see [1]).

Proposition 3.6.

(*i*) *Suppose d and ρ are equivalent metrics on X such that*

$$\rho(x, y) \le \beta d(x, y) \ \text{ for all } x, y \in X,$$

where β is a positive constant. If (X, d, G) is a generalized segment space, then so is (X, ρ, G).

(ii) *If there are positive constants α and β such that*

$$\alpha d(x,y) \le \rho(x,y) \le \beta d(x,y) \quad \text{for all } x,y \in X, \tag{6}$$

then (X,d,G) is a generalized segment space iff so is (X,ρ,G).

Proof. (i) It is easy to see that the identical mapping $i : (X,d) \to (X,\rho)$ is uniformly continuous. Applying Theorem 3.2 part (i) for $f = i$ we get the required result.

(ii) Two metrics d and ρ satisfying (6) are equivalent. So part (ii) follows directly from part (i). □

A particular case of Proposition 3.6 is when $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on Y and X is a nonempty subset of Y. Let $d_i(x,y) = \|x-y\|_i$, $i = 1,2$, and $G : X \times X \to 2^X$ be a set-valued mapping that takes on nonempty values. Then (X,d_1,G) is a generalized segment space iff so is (X,d_2,G). This is because there are positive constants α and β such that $\alpha\|z\|_1 \le \|z\|_2 \le \beta\|z\|_1$ for all $z \in Y$. Accordingly, for normed linear spaces, the concept of generalized segment space depends only on the topology of the space but not on a representative norm.

In the next theorem, some requirements on d and ρ are relaxed but a condition is imposed on G.

Theorem 3.7. *Suppose d and ρ are equivalent metrics on X and $G : X \times X \to 2^X$ takes on relatively compact values. Then (X,d,G) is a generalized segment space iff so is (X,ρ,G).*

Proof. Indeed, by Corollary 3.4, for any sequences (x_n) and (y_n) in X we have $d_H\left(G(x_n,y_n),G(x,y)\right) \to 0$ iff $\rho_H\left(G(x_n,y_n),G(x,y)\right) \to 0$. Thus, (X,d,G) is a generalized segment space iff so is (X,ρ,G). □

Theorem 3.7 shows that in the case G takes on relatively compact values, the concept of generalized segment space depends only on the topology of the space but not on equivalent metrics. This holds, in particular, if X is compact.

For proper uniquely geodesic spaces, we have the following statement whose proof follows directly from Theorems 2.5 and 3.7.

Corollary 3.8. *If (X,d) is a proper uniquely geodesic space and ρ is an equivalent metric on X, then (X,ρ,G) is a generalized segment space where $G(x,y)$ is the geodesic segment joining x and y.*

4. G-type Convex Sets and Blaschke's Convergence Theorem

Definition 4.1. Let X be a set and $G : X \times X \to 2^X$ a set-valued mapping that takes on nonempty values. A set $A \subset X$ is called *G-type convex* if $G(x,y) \subset A$ whenever $x,y \in A$.

If X is a linear space and $G(x,y) = [x,y]$, then G-type convexity coincides with the usual convexity.

Example 4.2. Let $\mathcal{P}$ be a simple polygon on the plane with the Euclidian distance. For any $x, y \in \mathcal{P}$, let $G(x,y)$ be the shortest segment in $\mathcal{P}$ joining x and y. A subset A of $\mathcal{P}$ is said to be *geodesic convex* if $G(x,y) \subset A$ whenever $x, y \in A$ (see [8]). This geodesic convexity is clearly the same as G-type convexity.

Example 4.3. Let X be a uniquely geodesic space. A subset A of X is said to be *geodesically convex* if for every x and y in A, the geodesic segment joining x and y is contained in A (see [6], p. 67). If we denote $G(x,y)$ the geodesic segment joining x and y, then geodesic convexity means G-type convexity.

In general a G-type convex set may be disconnected. For instance, let X be a normed linear space, and $G(x,y) = \frac{1}{4}[x,y]$. Let $A = \{x : \|x\| \le 1\} \cup \{x : 3 \le \|x\| \le 4\}$. This set is disconnected (and hence is not convex in the usual sense). However, for any $x, y \in A$, $\|x\| \le 4$, $\|y\| \le 4$ so $\|z\| \le 1$ for all $z \in G(x,y)$, i.e., $z \in A$. Thus, $G(x,y) \subset A$ and A is G-type convex.

Proposition 4.4. *The intersection of an arbitrary family of G-type convex sets is G-type convex.*

Proof. Suppose $(A_i)_{i\in I}$ is a family of G-type convex sets. If $x, y \in \cap_{i\in I} A_i$ then $x, y \in A_i$ and as A_i is G-type convex, $G(x,y) \subset A_i$ for each $i \in I$. Thus $G(x,y) \subset \cap_{i\in I} A_i$, which shows that $\cap_{i\in I} A_i$ is G-type convex. □

Proposition 4.5. *If A is a nonempty G-type convex set in a generalized segment space (X,d,G) then (A,d,G) is a generalized segment space, too.*

Proof. It follows directly from the definition of G-type convex sets. □

Proposition 4.6. *Let (X,d,G) be a generalized segment space.*

(i) *If $x_n, y_n \in X$, $x_n \to x$, $y_n \to y$ and $z \in G(x,y)$ then there is $z_n \in G(x_n, y_n)$, $n = 1, 2, \dots$ such that $z_n \to z$ as $n \to \infty$.*

(ii) *If A is a G-type convex subset of X, then so is its closure $\bar{A}$.*

Proof. (i) There is $z_n \in G(x_n, y_n)$ such that $d(z, z_n) < d\left(z, G(x_n, y_n)\right) + \frac{1}{n}$. Then, by (A2), $d(z, z_n) < d_H\left(G(x_n, y_n), G(x,y)\right) + \frac{1}{n} \to 0$ so that $z_n \to z$.

(ii) Suppose that $x, y \in \bar{A}$ and $z \in G(x,y)$. There are sequences $(x_n), (y_n)$ in A such that $x_n \to x$ and $y_n \to y$. Part (i) says that there is $z_n \in G(x_n, y_n)$, $n = 1, 2, \dots$ such that $z_n \to z$. As A is G-type convex, $G(x_n, y_n) \subset A$ for all n and hence $z_n \in A$, $n = 1, 2, \dots$. Thus, $z \in \bar{A}$. This shows that $G(x,y) \subset \bar{A}$ and $\bar{A}$ is G-type convex. □

The proof of the following result follows immediately from Theorem 2.5, Example 4.3, and Proposition 4.6 (ii).

Corollary 4.7 (See [6], p. 68). *Let X be a proper uniquely geodesic space and let A be a geodesically convex subset of X. Then its closure $\bar{A}$ is geodesically convex.*

Consider now the collection $\mathcal{G}(X) = \{G(x,y) : x, y \in X\}$.

Theorem 4.8. *Let (X,d,G) be a compact generalized segment space.*

(i) *For any sequence $(G(x_n,y_n))$, there exists a subsequence $\big(G(x_{n_j},y_{n_j})\big)$ such that $d_H\big(G(x_{n_j},y_{n_j}), G(x,y)\big) \to 0$ for some $x, y \in X$. So if $G(u,v)$ is closed for all $u, v \in X$ then $(\mathcal{G}(X), d_H)$ is a compact metric space.*

(ii) *If A is a nonempty compact subset of X and $d_H(G(x_n,y_n), A) \to 0$, then $A = \overline{G(x,y)}$ for some $x, y \in X$.*

Proof. (i) Let $(G(x_n,y_n))$ be a sequence in $\mathcal{G}(X)$. As X is compact, there exist convergent subsequences (x_{n_j}) of (x_n) and (y_{n_j}) of (y_n). Let $x := \lim_{j\to\infty} x_{n_j}$ and $y := \lim_{j\to\infty} y_{n_j}$. By (A2), $d_H\big(G(x_{n_j},y_{n_j}), G(x,y)\big) \to 0$.

(ii) It follows from part (i) that there exists a subsequence $\big(G(x_{n_j},y_{n_j})\big)$ of $(G(x_n,y_n))$ such that $d_H\big(G(x_{n_j},y_{n_j}), G(x,y)\big) \to 0$ for some $x, y \in X$, so $d_H\left(\overline{G(x_{n_j},y_{n_j})}, \overline{G(x,y)}\right) \to 0$. Since $d_H\left(\overline{G(x_{n_j},y_{n_j})}, A\right) \to 0$ and the limit of a sequence in the metric space $(\mathcal{K}(X), d_H)$ is unique, we must have $A = \overline{G(x,y)}$. $\square$

Proposition 4.9. *Let (A_n) be a sequence of nonempty G-type convex subsets of a generalized segment space (X,d,G) and let A be a nonempty closed subset of X. If $d_H(A_n, A) \to 0$, then A is also G-type convex.*

Proof. Suppose that $x, y \in A$ and $z \in G(x,y)$. There is an $x_n \in A_n$ such that $d(x,x_n) < d(x,A_n) + \frac{1}{n}$, for each $n = 1, 2, \ldots$. Because $d(x,A_n) \le d_H(A,A_n) \to 0$, we have $x_n \to x$. Similarly, for each n, there is $y_n \in A_n$ such that $y_n \to y$. By Proposition 4.6 (i), there exists $z_n \in G(x_n,y_n)$ such that $z_n \to z$. Since A_n is G-type convex, $G(x_n,y_n) \subset A_n$ and so $z_n \in A_n$. Take $w_n \in A$ satisfying $d(z_n,w_n) \le d_H(A_n,A) + \frac{1}{n}$. We have

$$d(w_n,z) \le d(w_n,z_n) + d(z_n,z) \le d_H(A_n,A) + \frac{1}{n} + d(z_n,z) \to 0,$$

which shows that $w_n \to z$. As A is closed, $z \in A$. Therefore, $G(x,y) \subset A$ and A is G-type convex. $\square$

Let us denote by $\mathcal{GC}(X)$ the collection of all nonempty G-type convex closed subsets of a generalized segment space (X,d,G). A sequence of sets in X is said to be *uniformly bounded* if there exists some ball in X that contains every member of the sequence (see [10], p. 96). We can now formulate our main result of this section.

Theorem 4.10. *Let (X,d,G) be a generalized segment space.*

(i) *If X is compact, then so is $(\mathcal{GC}(X), d_H)$.*

(*ii*) *If X is proper, then every uniformly bounded sequence of nonempty G-type convex sets in X contains a subsequence which converges to some nonempty compact G-type convex set in X.*

Proof. (*i*) Proposition 4.9 shows that $\mathcal{GC}(X)$ is a closed set in the compact space $(\mathcal{K}(X), d_H)$. Thus $(\mathcal{GC}(X), d_H)$ is also a compact space.

(*ii*) Suppose (A_n) is a uniformly bounded sequence of nonempty G-type convex sets in X. Since X is proper, there is a compact set Y that contains all $\bar{A}_n$. As $(\mathcal{K}(Y), d_H)$ is a compact space, there is a subsequence $(\bar{A}_{n_i})$ that converges to some nonempty compact set A. Hence $d_H(A_{n_i}, A) = d_H(\bar{A}_{n_i}, A) \to 0$ and Proposition 4.9 states that A is G-type convex, too. □

A particular case of Theorem 4.10 (*i*) when X is a simple polygon $\mathcal{P}$ on the plane and $G(x, y)$ is the shortest segment joining x and y (see Examples 2.4 and 4.2) was proved in [4]. We now have a direct consequence of Theorem 4.10. Its proof follows immediately from Theorems 2.5, 4.10 (*ii*), and Example 4.3.

Corollary 4.11. *If X is a proper uniquely geodesic space then every uniformly bounded sequence of nonempty geodesically convex subsets of X contains a subsequence which converges to some nonempty compact geodesically convex subset in X.*

In fact, in Theorem 4.10 and Corollary 4.11, X may be any metric space provided that there is a compact set Y containing all members of the sequence (A_n). The proof of Theorem 4.10 (*ii*) remains valid for this case.

Proposition 4.12. *Let (A_n) be a sequence of nonempty G-type convex subsets of a generalized segment space (X, d, G). If there exists a compact set that contains all members of (A_n), then there exists a subsequence of (A_n) which converges to some nonempty compact G-type convex set in X.*

Corollary 4.13 (See [7]). *If C is a nonempty compact convex set in a normed linear space, then every sequence of nonempty compact convex subsets of C contains a subsequence which converges to some nonempty compact convex set in C.*

Finally we can obtain the classical Blaschke's convergence theorem for convex sets in $\mathbb{R}^n$, which is a special case of Theorem 4.10 (*ii*).

Corollary 4.14 (See [10], p. 97). *Every uniformly bounded sequence of nonempty compact convex sets in $\mathbb{R}^n$ contains a subsequence which converges to some nonempty compact convex set in $\mathbb{R}^n$.*

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