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A Generalization of the Euclidean Angle

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Based on an idea of Ivan Singer we introduce a new concept of an angle in real normed vector spaces which generalizes the Euclidean angle of Hilbert spaces. With this angle we can describe elements of each finite dimensional real normed space by polar coordinates.

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1. Introduction

In a real inner product space $(X, < . \mid . >)$ it is well-known that the inner product can be expressed by the norm, namely for $\vec{x}, \vec{y} \in X$, $\vec{x}, \vec{y} \neq \vec{0}$, we can write

$$
\begin{aligned}
< \vec{x} \mid \vec{y} > &= \frac{1}{4} \cdot \left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 \right) \\
&= \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right].
\end{aligned}
$$

Furthermore we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the usual Euclidean angle

$$
\begin{aligned}
\angle_{\text{Euclid}}(\vec{x}, \vec{y}) :=& \arccos \left(\frac{< \vec{x} \mid \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|} \right) \\
=& \arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right),
\end{aligned}
$$

which is defined in terms of the norm, too.

In this paper we deal with real normed vector spaces $(X, \|\cdot\|)$. Similarly to the case of an inner product we define for such spaces a product $< . | . > : X^2 \longrightarrow \mathbb{R}$.

Definition 1.1. Let $(X, \|\cdot\|)$ be a real normed vector space. We define for two arbitrary elements $\vec{x}, \vec{y} \in X$ the real number $< \vec{x} \mid \vec{y} >$,

$$
< \vec{x} | \vec{y} > := \begin{cases} 0 & \text{for } \vec{x} = \vec{0} \text{ or } \vec{y} = \vec{0}, \\ \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] & \text{for } \vec{x}, \vec{y} \neq \vec{0}. \end{cases}
$$

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It is easy to show that this product fulfils the symmetry ($< \vec{x} \,|\, \vec{y} > \, = \, < \vec{y} \,|\, \vec{x} >$), the positive definiteness ($< \vec{x} \,|\, \vec{x} > \geq 0$ and $< \vec{x} \,|\, \vec{x} > = 0$ only for $\vec{x} = \vec{0}$), and the homogeneity ($< r \cdot \vec{x} \,|\, \vec{y} > = r\cdot < \vec{x} \,|\, \vec{y} >$), for $\vec{x}, \vec{y} \in X$, $r \in \mathbb{R}$. Furthermore, the triple $(X, \|\cdot\|, \; < . \,|\, . >)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or *CSB* inequality, that means we have $| < \vec{x} \,|\, \vec{y} > | \leq \|\vec{x}\| \cdot \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in X$.

Since the *CSB* inequality holds we are able to define an 'angle' $\angle_{\text{Thy}}(\vec{x}, \vec{y})$, according to the Euclidean angle in inner product spaces.

Definition 1.2. In a real normed vector space $(X, \|\cdot\|)$ we define for elements $\vec{x}, \vec{y} \neq \vec{0}$ the number $\angle_{\text{Thy}}(\vec{x}, \vec{y}) \in [0, \pi]$,

$$\begin{aligned} \angle_{\text{Thy}}(\vec{x}, \vec{y}) :=& \; \arccos\left(\frac{< \vec{x} \,|\, \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|} \right) \\ =& \; \arccos\left(\frac{1}{4} \cdot \left[\; \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \; \right] \right). \end{aligned}$$

This new 'angle' has several comfortable properties which are known from the Euclidean angle in inner product spaces, and it corresponds to this well-known angle in the case that $(X, \|\cdot\|)$ already is an inner product space. This reflects the fact that the cosine of an inner angle in a rhombus with the side length 1 can be expressed as the fourth part of the difference of the squares of the two diagonals.

We consider elements $\vec{x}, \vec{y} \in X \backslash \{\vec{0}\}$. We have the properties that

(An 1): $\angle_{\text{Thy}}$ is a continuous map from $(X \backslash \{\vec{0}\})^2$ into the closed interval $[0, \pi]$,
(An 2): $\angle_{\text{Thy}}(\vec{x}, \vec{x}) = 0$,
(An 3): $\angle_{\text{Thy}}(-\vec{x}, \vec{x}) = \pi$,
(An 4): $\angle_{\text{Thy}}(\vec{x}, \vec{y}) = \angle_{\text{Thy}}(\vec{y}, \vec{x})$,
(An 5): for all $r, s > 0$ it holds $\angle_{\text{Thy}}(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_{\text{Thy}}(\vec{x}, \vec{y})$,
(An 6): $\angle_{\text{Thy}}(-\vec{x}, -\vec{y}) = \angle_{\text{Thy}}(\vec{x}, \vec{y})$,
(An 7): $\angle_{\text{Thy}}(\vec{x}, \vec{y}) + \angle_{\text{Thy}}(-\vec{x}, \vec{y}) = \pi$,

which are easy to prove. Moreover, the 'Thy-angle' has the following important property (An 8). This property is the main contribution of our paper.

Theorem 1.3. (An 8)*: For any two linear independent vectors* $\vec{x}, \vec{y}$ *of the space* $(X, \|\cdot\|)$ *there is a decreasing homeomorphism*

$$\Theta : \mathbb{R} \longrightarrow (0, \pi), \quad t \mapsto \angle_{\text{Thy}}(\vec{x}, \vec{y} + t \cdot \vec{x}).$$

The proof will be given in the section **On the Existence of Polar Coordinates**. We use results of a paper by Charles Diminnie, Edward Andalafte, and Raymond Freese [3].

In the last section we abandon the restriction on normed spaces, and we consider also vector spaces provided with a positive functional $\|\cdot\|$ which have a non-convex unit ball. We formulate the conjecture that there exists such a space which fulfils (An 8). Further we show that in such spaces generally we are not able to define the angle $\angle_{\text{Thy}}$, since the *CSB* inequality fails.

2. Preliminary Notes

Let the pair $(X, \|\cdot\|)$ be an arbitrary real normed vector space, i.e. the map $\|\cdot\|$: $X \longrightarrow \mathbb{R}^+ \cup \{0\}$ has the properties $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ ('absolute homogeneity'), $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ ('triangle inequality'), and $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$ ('positive definiteness'), for $\vec{x}, \vec{y} \in X$ and $r \in \mathbb{R}$.

We write for every vector $\vec{v} \neq \vec{0}$ the unit vector $\text{sign}(\vec{v}) := \frac{1}{\|\vec{v}\|} \cdot \vec{v}$, i.e. $\text{sign}(\vec{v})$ is the projection of $\vec{v}$ onto the unit sphere $\mathsf{S} := \{\vec{x} \in X \mid \|\vec{x}\| = 1\}$ of $(X, \|\cdot\|)$. We denote the unit ball of $(X, \|\cdot\|)$ by B. Further we abbreviate the class of real inner product vector spaces by **IP space**. For two real numbers $a \leq b$ the term $[a, b]$ means the closed interval between a and b, and (a, b) means the open interval.

The first attempt to define a concept of generalized 'angles' on metric spaces was made by Menger [9], p. 749. Since then a few ideas have been developed, see the references [3], [4], [6], [10], [11], [12], [13], [15], [18], [19]. In this paper we focus our intention on real normed spaces as a generalization of real inner product spaces. Let $(X, < . \mid . >)$ be an **IP space**, and let $\|\cdot\|$ be the associated norm, $\|\vec{x}\| := \sqrt{< \vec{x}|\vec{x} >}$. The triple $(X, \|\cdot\|, < . \mid . >)$ fulfils the *CSB* inequality, and we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the well-known Euclidean angle $\angle_{\text{Euclid}}(\vec{x}, \vec{y}) := \arccos\left(\frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|}\right)$, which satisfies all properties (*An* 1)–(*An* 8).

3. The Thy-Angle

In a real normed vector space $(X, \|\cdot\|)$ let $< \vec{x} \mid \vec{y} >$ be the real number defined in Definition 1.1 for all $\vec{x}, \vec{y} \in X$. Note that, in the case that $(X, \|\cdot\|)$ is already an **IP space**, this definition corresponds to the usual definition of the inner product.

Lemma 3.1. *For a normed space $(X, \|\cdot\|)$ the pair $(X, < . \mid . >)$ fulfils both the symmetry, the positive definiteness, and the homogeneity. We have $\|\vec{x}\| = \sqrt{< \vec{x} \mid \vec{x} >}$, for all $\vec{x} \in X$. Furthermore, the triple $(X, \|\cdot\|, < . \mid . >)$ satisfies the CSB inequality,*

Proof. To prove the homogeneity we first show $- < \vec{x} \mid \vec{y} > = < -\vec{x} \mid \vec{y} >$. The rest is trivial. $\square$

In real normed spaces $(X, \|\cdot\|)$ for vectors $\vec{x}, \vec{y} \in X \backslash \{\vec{0}\}$, (i.e. $\|\vec{x}\| \cdot \|\vec{y}\| > 0$) we introduced in Definition 1.2 the 'Thy-angle'. (If we would assume the parallelogram identity our definition is equivalent to that one which is discussed in [3]. See also [15], where the 'Singer orthogonality' was defined, and where the author got the idea to write this paper.)

Lemma 3.2. (*a*) *In a real normed vector space $(X, \|\cdot\|)$ the pair $(X, \angle_{\text{Thy}})$ satisfies all seven demands (An 1)–(An 7).*
(*b*) *If $(X, < . \mid . >)$ even is an* **IP space***, we have that the Thy-angle corresponds to the Euclidean angle, i.e. we get for all $\vec{x}, \vec{y} \neq \vec{0}$ the equality $\angle_{\text{Thy}}(\vec{x}, \vec{y}) = \angle_{\text{Euclid}}(\vec{x}, \vec{y})$.*

Proof. All calculations for the proof of the lemma are straightforward. □

There are other properties of the Thy-angle which differ from the Euclidean angle. For instance, the sum of inner angles of a triangle generally is not π, and usually we do not have the desirable equality $\angle_{\text{Thy}}(\vec{x},\vec{y}) = \angle_{\text{Thy}}(\vec{x},\vec{x}+\vec{y}) + \angle_{\text{Thy}}(\vec{x}+\vec{y},\vec{y})$. To show this we can use the norm $\|\cdot\|_1$ of the vector space $\mathbb{R}^2$, $\|(x|y)\|_1 := |x|+|y|$.

We remark that a former version of this paper was published on 'arXiv' [16]. Further, some continuative thoughts are presented in [17]. Properties of the Thy-angle and a comparison with the 'g-angle' are discussed in a paper by Pavle M. Miličić [13].

4. On the Existence of Polar Coordinates

In this section we prove Theorem 1.3, i.e. we show the property $(An\, 8)$. Before we begin with the lengthy proof we formulate a corollary.

Corollary 4.1. *With the above theorem we can describe elements of a two-dimensional real normed vector space $(X, \|\cdot\|)$ by polar coordinates. If we fix a basis $\{\vec{b_1}, \vec{b_2}\}$, then every $\vec{x} = r_1 \cdot \vec{b_1} + r_2 \cdot \vec{b_2} \in X$ is uniquely defined by its norm $\|\vec{x}\|$ and its angle $\angle_X(\vec{x}) := \angle_{\text{Thy}}(\vec{x}, \vec{b_1})$ for $r_2 > 0$ and $\angle_X(\vec{x}) := -\angle_{\text{Thy}}(\vec{x}, \vec{b_1})$ for $r_2 < 0$. This can be generalized easily on a finite number of dimensions.*

Now we start the proof of the theorem. The central idea of the proof is, assuming that the map Θ is not injective contradicts the convexity of the unit ball B of $(X, \|\cdot\|)$.

Proof of Theorem 1.3. We consider an arbitrary real normed vector space $(X, \|\cdot\|)$. Assume a linear independent subset $\{\vec{x}, \vec{y}\} \subset X$. It generates a two-dimensional subspace of X. We consider the map $\Theta : \mathbb{R} \longrightarrow [0, \pi]$. For convenience, we define some abbreviations.

Let $h_+, h_- : \mathbb{R} \longrightarrow [0,2]$, $h_+(t) := \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}+t\cdot\vec{x}}{\|\vec{y}+t\cdot\vec{x}\|} \right\|$, $h_-(t) := \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}+t\cdot\vec{x}}{\|\vec{y}+t\cdot\vec{x}\|} \right\|$. We have

$$\begin{aligned}
\Theta(t) &:= \angle_{\text{Thy}}(\vec{x}, \vec{y} + t\cdot\vec{x}) && (1)\\
&= \arccos\left(\frac{1}{4}\cdot\left[\,\left\|\frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}+t\cdot\vec{x}}{\|\vec{y}+t\cdot\vec{x}\|}\right\|^2 - \left\|\frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}+t\cdot\vec{x}}{\|\vec{y}+t\cdot\vec{x}\|}\right\|^2\,\right]\right) && (2)\\
&= \arccos\left(\frac{1}{4}\cdot\left[[h_+(t)]^2 - [h_-(t)]^2\,\right]\right), && (3)
\end{aligned}$$

and we extend Θ by $\Theta(-\infty) := \pi$, $\Theta(+\infty) := 0$.

We have two limits $\lim_{t\to-\infty} h_+(t) = \lim_{t\to+\infty} h_-(t) = 0$, and we have $\lim_{t\to+\infty} h_+(t) = \lim_{t\to-\infty} h_-(t) = 2$, too. Hence it follows that the map $\Theta : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0,\pi]$ is continuous and surjective. We still have to prove the injectivity of Θ, the difficult part of the proof.

Lemma 4.2. *The maps $h_+, h_- : \mathbb{R} \to [0,2]$ are monotone increasing and decreasing, respectively. If the unit ball* B *of* $(X, \|\cdot\|)$ *is strictly convex, then h_+, h_- even are strictly monotone increasing and decreasing, respectively.*

Proof. See the tricky proof in [3], p. 201, Theorem 2.4 and p. 202, Theorem 2.5. The authors only dealt with the map h_-. Note that they only prove that h_- is monotone. In Theorem 2.5 the case of a strictly convex unit ball B is not explicitly written down. □

Remark 4.3. If we have a unit ball B of $(X, \|\cdot\|)$ that is not strictly convex, then h_+, h_- are generally not strictly monotone. This is shown by the example of the normed space $(\mathbb{R}^2, \|\cdot\|_\infty)$ with the norm $\|(x_1|x_2)\|_\infty := \max\{|x_1|, |x_2|\}$. Choose $\vec{x} := (1|0)$, and $\vec{y} := (0|1)$, then the interval in which h_+ is constant $(= 1)$ is $[-1, 0]$, while h_- is constant $(= 1)$ in $[0, 1]$. This example shows that the intervals where h_+ and h_-, respectively, are constant may intersect in one point. We are just proving the fact that both intervals do not intersect in an interval with nonempty interior.

Note that in the case of a normed space $(X, \|\cdot\|)$ with a strictly convex unit ball it follows from Lemma 4.2 immediately that Θ is strictly monotone decreasing, i.e. $(An\ 8)$ is satisfied.

Now we want to prove that Θ remains to be strictly monotone decreasing, even if the unit ball is not strictly convex. Because h_-, h_+ are monotone, Θ is always monotone decreasing. We have to prove that the monotony is strict.

We prefer a direct proof. Assume two numbers t_1 and t_2 with

$$-\infty < t_1 \leq t_2 < +\infty \quad \text{and} \quad \Theta(t_1) = \Theta(t_2).$$

The case $t_1 = t_2$ is possible. We shall show that this is the only possible case. So let us assume that we have $-\infty < t_1 < t_2 < +\infty$. Now we hunt for contradictions. Because of $\Theta(t_1) = \Theta(t_2)$ and because h_+ is monotone increasing and h_- is monotone decreasing we have $h_+(t_1) = h_+(t_2)$ and $h_-(t_1) = h_-(t_2)$, and for all t in the interval $[t_1, t_2]$ both h_+ and h_- and Θ remain constant.

As from now we calculate in coordinates of the basis $\left\{\frac{\vec{x}}{\|\vec{x}\|}, \frac{\vec{y}}{\|\vec{y}\|}\right\} = \{\text{sign}(\vec{x}), \text{sign}(\vec{y})\}$. Hence $\text{sign}(\vec{x}) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and $\text{sign}(\vec{y}) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. We define $\vec{v} := \left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right) := \text{sign}(\vec{y} + t_1 \cdot \vec{x})$, $\vec{w} := \left(\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right) := \text{sign}(\vec{y}+t_2\cdot\vec{x})$. We get $\vec{v} = \left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right) = \frac{\vec{y}+t_1\cdot\vec{x}}{\|\vec{y}+t_1\cdot\vec{x}\|} = \frac{t_1\cdot\|\vec{x}\|}{\|\vec{y}+t_1\cdot\vec{x}\|}\cdot\frac{\vec{x}}{\|\vec{x}\|} + \frac{\|\vec{y}\|}{\|\vec{y}+t_1\cdot\vec{x}\|}\cdot\frac{\vec{y}}{\|\vec{y}\|}$ $= v_1 \cdot \frac{\vec{x}}{\|\vec{x}\|} + v_2 \cdot \frac{\vec{y}}{\|\vec{y}\|}$, hence $v_2 = \frac{\|\vec{y}\|}{\|\vec{y}+t_1\cdot\vec{x}\|} > 0$. For the same reason we get $w_2 = \frac{\|\vec{y}\|}{\|\vec{y}+t_2\cdot\vec{x}\|} > 0$. Note that $v_1 = \frac{t_1\cdot\|\vec{x}\|}{\|\vec{y}+t_1\cdot\vec{x}\|}$ and $w_1 = \frac{t_2\cdot\|\vec{x}\|}{\|\vec{y}+t_2\cdot\vec{x}\|}$. We write down some facts.

For the following it is important to keep in mind that $\vec{v} \neq \vec{w}$, because $\vec{x}$ and $\vec{y}$ are linear independent. Since $h_+(t_1) = h_+(t_2)$ and $h_-(t_1) = h_-(t_2)$ we have $\|\left(\begin{smallmatrix}v_1+1\\v_2\end{smallmatrix}\right)\| = h_+(t_1) = h_+(t_2) = \|\left(\begin{smallmatrix}w_1+1\\w_2\end{smallmatrix}\right)\|$ and also $\|\left(\begin{smallmatrix}v_1-1\\v_2\end{smallmatrix}\right)\| = \|\left(\begin{smallmatrix}w_1-1\\w_2\end{smallmatrix}\right)\|$.

Remark 4.4. Note that neither the three points $\{\vec{v}, \vec{w}, \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$, nor the three points $\{\left(\begin{smallmatrix}v_1+1\\v_2\end{smallmatrix}\right), \left(\begin{smallmatrix}w_1+1\\w_2\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$, nor $\{\left(\begin{smallmatrix}v_1-1\\v_2\end{smallmatrix}\right), \left(\begin{smallmatrix}w_1-1\\w_2\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$ are collinear.

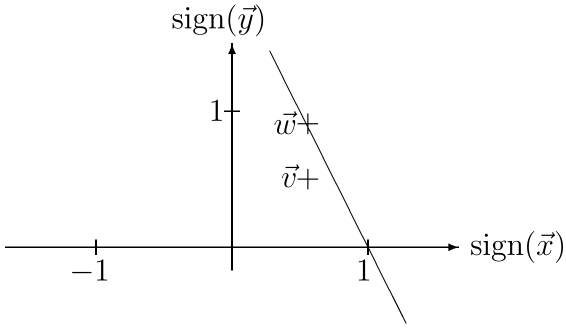

Figure 4.1.

Further note that we can choose the real numbers t_1 and t_2 such that all seven unit vectors

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}v_1\\v_2\end{pmatrix}, \begin{pmatrix}w_1\\w_2\end{pmatrix}, \operatorname{sign}\begin{pmatrix}v_1-1\\v_2\end{pmatrix}, \operatorname{sign}\begin{pmatrix}v_1+1\\v_2\end{pmatrix}, \operatorname{sign}\begin{pmatrix}w_1-1\\w_2\end{pmatrix}, \operatorname{sign}\begin{pmatrix}w_1+1\\w_2\end{pmatrix}$$

are pairwise distinct. See Figure 4.5 (which will also play an important role in the paper later) for a first view on the things to come.

For further investigations we must distinguish three cases with nine subcases.

- Case (**A**): $v_1 < w_1$, with the subcases
  - Case (**A1**): $v_1 < w_1$ and $\{v_1, w_1\} \cap \{-1, +1\} \neq \emptyset$
  - Case (**A2**): $v_1 < w_1$ and $\{v_1, w_1\} \cap (-1, +1) \neq \emptyset$
  - Case (**A3**): $v_1 < w_1 < -1$ or $+1 < v_1 < w_1$.
- Case (**B**): $v_1 > w_1$, with the corresponding subcases (**B1**), (**B2**), (**B3**).
- Case (**C**): $v_1 = w_1$, with the subcases
  - Case (**C1**): $v_1 = w_1 \in \{-1, +1\}$,
  - Case (**C2**): $v_1 = w_1 \in (-1, +1)$,
  - Case (**C3**): $v_1 = w_1 < -1$ or $v_1 = w_1 > +1$.

The easiest cases (**A1**) and (**B1**) can be avoided by replacing $\{t_1, t_2\}$ by another pair of real numbers $\{\widetilde{t_1}, \widetilde{t_2}\}$ with $t_1 < \widetilde{t_1} < \widetilde{t_2} < t_2$.

Let's deal with the easy Case (**C1**): Assume, for instance, $v_1 = w_1 = 1$. Since $h_-(t_1) = h_-(t_2)$ it follows $\|\left(\begin{smallmatrix}0\\v_2\end{smallmatrix}\right)\| = \|\left(\begin{smallmatrix}0\\w_2\end{smallmatrix}\right)\|$, hence $v_2 = w_2$. This contradicts $\vec{v} \neq \vec{w}$.

Now we deal with the Case (**C2**): We have $-1 < v_1 = w_1 < 1$, for instance $0 \leq v_1 = w_1$. Because of $\vec{v} \neq \vec{w}$ we have $v_2 \neq w_2$. Assume for instance $v_2 < w_2$. Since the unit ball B of $(X, \|\cdot\|)$ is convex and $\vec{w}, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathsf{S}$, the straight line between both points is a subset of B. But in this case $\vec{v}$ would be in the interior of B. This is impossible because $\|\vec{v}\| = 1$, (see Figure 4.1).

Let us now turn our attention to 'Grecian Geometry'. With this phrase we mean elementary geometry on the two-dimensional Euclidean plane. We describe and prove two propositions about 'projections', which will be needed for the proof of Theorem 1.3. The first one is a consequence of the theorem of Pappus, but we

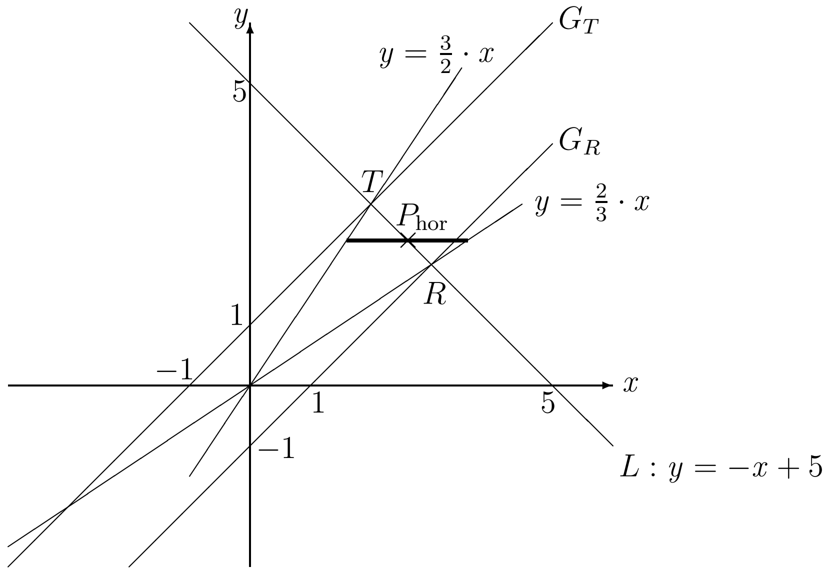

Figure 4.2: Example: $m_L = -1,\ b_L = 5$. We get $R = (3|2)$ and $T = (2|3)$. Hence $P_{\text{hor}} = \left(\frac{13}{5}|\frac{12}{5}\right)$.

give a direct proof. For a better understanding of the following propositions it is reasonable to look at the enclosed figures.

Proposition 4.5. *Let us take $\mathbb{R}^2 = \{(x|y)|\ x, y \in \mathbb{R}\}$, the two-dimensional Euclidean plane, with the horizontal x-axis and the vertical y-axis. Consider the two parallel lines $G_R : y = x - 1$ and $G_T : y = x + 1$. Assume a third straight line L, not parallel to G_R, G_T, respectively, with the additional property that L does not meet the origin $(0|0)$. The intersection of L with G_R is called $R = (x_R|x_R - 1)$, and the intersection of L with G_T is called $T = (x_T|x_T + 1)$.*
Then we state that there is an unique point $P_{\text{hor}} := (x_{\text{hor}}|y_{\text{hor}})$ on L such that the three points $(0|0), (x_{\text{hor}} + 1|y_{\text{hor}})$ and R are collinear, and such that the three points $(0|0), (x_{\text{hor}} - 1|y_{\text{hor}})$, and T are collinear. (See Figure 4.2).

Proof. In the case of $x_R \neq x_T$ we have an equation $L : y = m_L \cdot x + b_L,\ m_L, b_L \in \mathbb{R},\ m_L \neq 1,\ b_L \neq 0$, and then take

$$\begin{aligned} P_{\text{hor}} = (x_{\text{hor}}|y_{\text{hor}}) &= \left(\frac{2 \cdot x_R \cdot x_T + x_R - x_T}{x_R + x_T} \;\Big|\; \frac{2 \cdot (x_R \cdot x_T + x_R - x_T - 1)}{x_R + x_T} \right) \\ &= \left(\frac{m_L - {b_L}^2}{b_L \cdot (m_L - 1)} \;\Big|\; \frac{{m_L}^2 - {b_L}^2}{b_L \cdot (m_L - 1)} \right) \\ &= \left(\frac{x_T \cdot (b_L + 1) + 1}{b_L} \;\Big|\; m_L \cdot \frac{x_T \cdot (b_L + 1) + 1}{b_L} + b_L \right) \\ &= \left(\frac{x_R \cdot (b_L - 1) + 1}{b_L} \;\Big|\; m_L \cdot \frac{x_R \cdot (b_L - 1) + 1}{b_L} + b_L \right). \end{aligned}$$

We have $m_L = \frac{x_R - x_T - 2}{x_R - x_T}$, and $b_L = \frac{x_R + x_T}{x_R - x_T} = x_R \cdot (1 - m_L) - 1 = x_T \cdot (1 - m_L) + 1$.

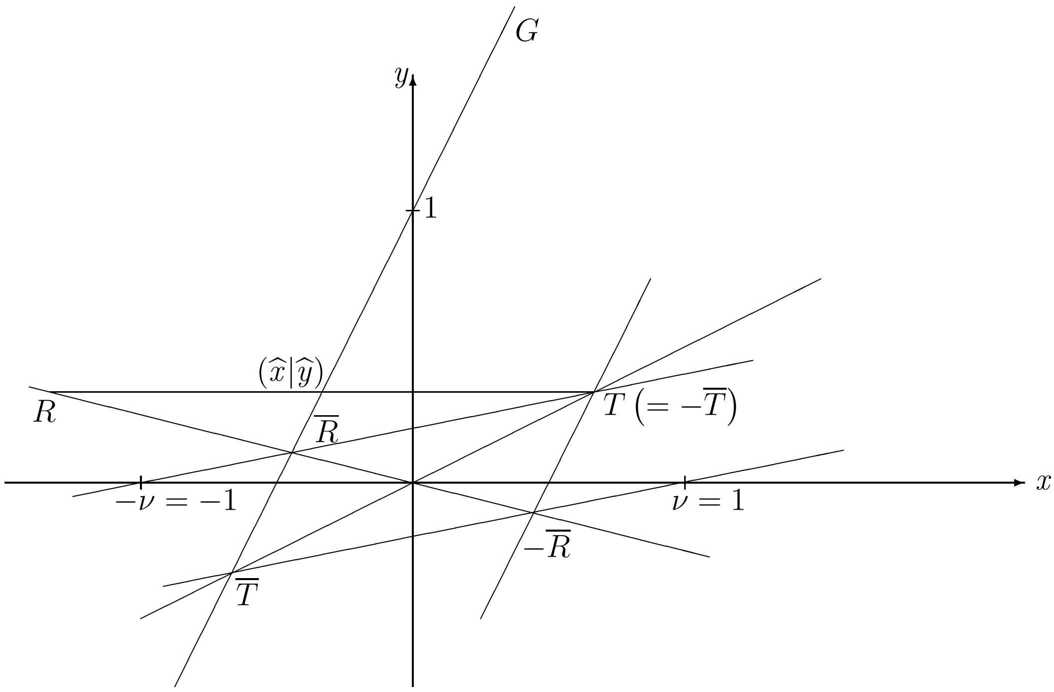

Figure 4.3: Here $G : y = 2 \cdot x + 1$, hence $m = 2$, and we choose $(\widehat{x}|\widehat{y}) := \left(-\frac{1}{3}|\frac{1}{3}\right)$. We get $\overline{R} = \left(-\frac{4}{9}|\frac{1}{9}\right)$, $\overline{T} = \left(-\frac{2}{3}|-\frac{1}{3}\right)$. (In this example is $T = -\overline{T}$). As we claimed, we have $\nu = 1$.

Some elementary calculations confirm that indeed all four formulas of P_{hor} yield the same values for x_{hor} and y_{hor}, and that P_{hor} fulfils the demanded properties.

In the case of $x_R = x_T$ we have an equation $L :\ x = a_L := x_R = x_T$, and

$$P_{\text{hor}} = (\ x_{\text{hor}}|y_{\text{hor}}) = \left(a_L \mid a_L - \frac{1}{a_L}\right), \quad \text{(see Figure 4.6).} \qquad \square$$

Proposition 4.6. *Let us again take* $\mathbb{R}^2$ *with the horizontal x-axis and the vertical y-axis. Consider the straight line G with the equation* $y = m \cdot x + 1$, $m \in \mathbb{R}\backslash\{-1, +1\}$. *Let us choose an arbitrary point* $(\widehat{x}|\widehat{y})$ *on* $G, \widehat{y} \neq 0$. *Take two points* $R := (\widehat{x} - 1|\widehat{y})$ *and* $T := (\widehat{x} + 1|\widehat{y})$. *We call* $\overline{R}$ *the projection of R on the line G, and* $\overline{T}$ *the projection of T on the line G. (That means that the three points* $(0|0), R, \overline{R}$, *and the three points* $(0|0), T, \overline{T}$, *respectively, are collinear,* $\overline{R}, \overline{T} \in G$.*) The four points* $\overline{R}, \overline{T}, -\overline{R}, -\overline{T}$ *are the corners of a parallelogram.*

Then we claim that the (elongated) line that connects $\overline{T}$ *and* $-\overline{R}$ *intersects the x-axis at* $\nu := 1$, *and the (elongated) line that connects* $-\overline{T}$ *and* $\overline{R}$ *intersects the x-axis at* $-\nu = -1$. *Moreover, the two points* $\overline{R}, \overline{T}$ *are on the same side of the x-axis if and only if* $-1 < m < 1$. *(See Figure 4.3).*

Proof. With elementary calculations we obtain

$$\overline{R} = \frac{1}{1+m} \cdot \begin{pmatrix} \widehat{x} - 1 \\ \widehat{y} \end{pmatrix} \quad \text{and} \quad \overline{T} = \frac{1}{1-m} \cdot \begin{pmatrix} \widehat{x} + 1 \\ \widehat{y} \end{pmatrix},$$

and (if $\widehat{x}+m \neq 0$) we have the equation $y = \frac{m\cdot\widehat{x}+1}{\widehat{x}+m}\cdot(x-1)$ of the straight line that intersects $\overline{T}$ and $-\overline{R}$. We get $\nu = 1$.

In the case of $\widehat{x}+m = 0$ the parallelogram of the points $\overline{R}$, $\overline{T}$, $-\overline{R}$, $-\overline{T}$ has a pair of vertical sides. These (elongated) sides intersect the x-axis in 1 and -1, respectively.

The points $\overline{R}, \overline{T}$ are on the same side of the x-axis if and only if their second components have the same signs, hence if and only if $(1-m)\cdot(1+m) > 0$, i.e. $-1 < m < 1$. □

Before we continue the proof of Theorem 1.3 we mention an easy fact about unit balls in $(\mathbb{R}^2, \|\cdot\|)$.

Lemma 4.7. *We assume that* $\mathbb{R}^2$ *is provided with a norm* $\|\cdot\|$*, the unit ball is* B*. Define two closed sets* Set 1 *and* Set 2*,*

$$\text{Set } 1 := \{(x|y) \in \mathbb{R}^2 \mid x+1 \geq y \geq x-1\},$$
$$\text{Set } 2 := \{(x|y) \in \mathbb{R}^2 \mid -x+1 \geq y \geq -x-1\}.$$

Now we assume $\|(1|0)\| = 1 = \|(0|1)\|$*. Then we have the inclusion*

$$\mathsf{B} \subset \text{Set } 1 \cup \text{Set } 2.$$

Proof. The following Figure 4.4 should be sufficient to establish the lemma, noting additionally that B is convex and $\|(-1|0)\| = 1 = \|(0|-1)\|$. □

Let us return to the proof of Theorem 1.3, but we still need some general preparations. Recall that we had defined two unit vectors $\vec{v} = (v_1|v_2) \neq \vec{w} = (w_1|w_2)$. Its coordinates refer to the basis $\{\text{sign}(\vec{x}), \text{sign}(\vec{y})\}$, both $\vec{v}$ and $\vec{w}$ are above the (horizontal) $\text{sign}(\vec{x})$-axis, i.e. $v_2, w_2 > 0$.

Because $\vec{v}$ and $\vec{w}$ are different, they uniquely determine a straight line L that connects them. If we define two lines L_- and L_+ such that L_- connects $\vec{v}-(1|0) = (v_1-1|v_2)$ with $\vec{w}-(1|0) = (w_1-1|w_2)$, and such that L_+ connects $(v_1+1|v_2)$ with $(w_1+1|w_2)$, it is trivial that all three lines L, L_-, and L_+ are parallel, (see Figure 4.5).

Lemma 4.8. *Assume an arbitrary real normed space* $(Y, \|\cdot\|)$*. Let* $\vec{a}, \vec{b} \in Y$ *be linear independent, and let* $\|\vec{a}\| = \|\vec{b}\|$*. Consider any geometric representation of the two-dimensional subspace of* Y *which is generated by the vectors* $\vec{a}$ *and* $\vec{b}$*. Then the line that connects* $\vec{a}$ *and* $\vec{b}$ *is parallel to the line that connects* $\text{sign}(\vec{a})$ *and* $\text{sign}(\vec{b})$*.*

Proof. Trivial by the radiation formula and the fact that $\|\cdot\|$ is absolute homogeneous. □

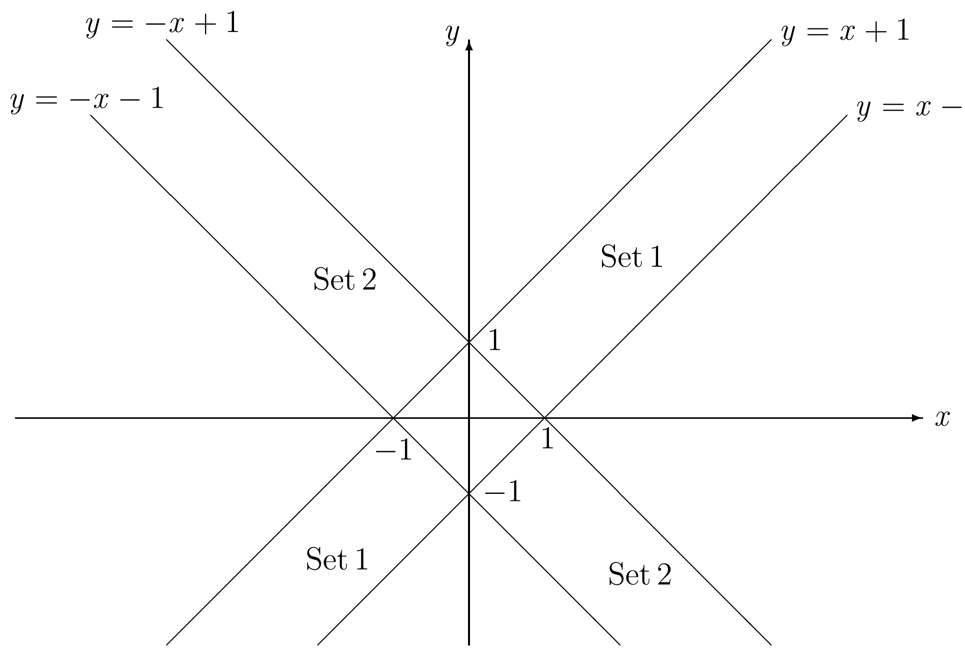

Figure 4.4.

Further we define two straight lines $L_{-,\mathrm{sign}}$ and $L_{+,\mathrm{sign}}$. The line $L_{-,\mathrm{sign}}$ connects the unit vectors $\mathrm{sign}(v_1 - 1|v_2)$ and $\mathrm{sign}(w_1 - 1|w_2)$, and the line $L_{+,\mathrm{sign}}$ connects $\mathrm{sign}(v_1 + 1|v_2)$ with $\mathrm{sign}(w_1 + 1|w_2)$.

Lemma 4.9. *All five lines L, L_-, L_+, $L_{-,\mathrm{sign}}$, and $L_{+,\mathrm{sign}}$ are parallel.*

Proof. Before Remark 4.4 we noted that $h_+(t_1) = \|(v_1 + 1|v_2)\| = h_+(t_2) = \|(w_1 + 1|w_2)\|$, and also $h_-(t_1) = \|(v_1 - 1|v_2)\| = h_-(t_2) = \|(w_1 - 1|w_2)\|$. All norms are positive, the vectors $\vec{v}$ and $\vec{w}$ are linear independent, hence together with the previous Lemma 4.8 the claim is true. $\square$

Recall that we had defined six unit vectors

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \mathrm{sign}\begin{pmatrix} v_1 - 1 \\ v_2 \end{pmatrix}, \mathrm{sign}\begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix}, \mathrm{sign}\begin{pmatrix} w_1 - 1 \\ w_2 \end{pmatrix}, \mathrm{sign}\begin{pmatrix} w_1 + 1 \\ w_2 \end{pmatrix},$$

and we mentioned later that, without loss of generality, all these vectors are pairwise distinct, and they are different from $(0|1)$.

Lemma 4.10. *All six endpoints are collinear. In the case that these six unit vectors are on different sides of the (vertical) $\mathrm{sign}(\vec{y})$-axis, the point $(0|1)$ is located on the same line L, too.*

Proof. All these seven unit vectors have a positive second component, i.e. they are above the $\mathrm{sign}(\vec{x})$-axis. By the previous lemma the three lines L, $L_{-,\mathrm{sign}}$, and $L_{+,\mathrm{sign}}$ are parallel. Let us consider, for instance, L and $L_{-,\mathrm{sign}}$. L intersects $\vec{v}$ and $\vec{w}$, and $L_{-,\mathrm{sign}}$ intersects $\mathrm{sign}(v_1 - 1|v_2)$ and $\mathrm{sign}(w_1 - 1|w_2)$. Because of the convexity of the unit ball B the case $L \neq L_{-,\mathrm{sign}}$ is not possible. (Note the four

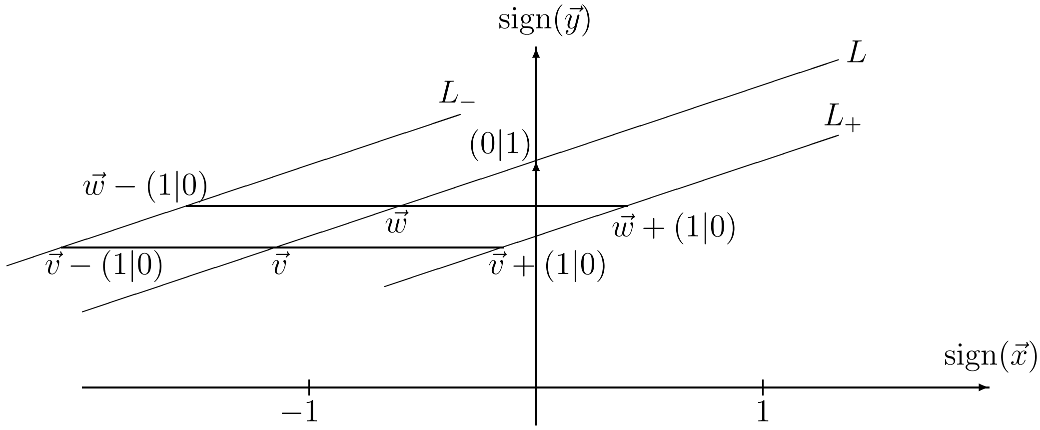

Figure 4.5.

unit vectors $\vec{v}, \vec{w}, \text{sign}(v_1 - 1|v_2)$ and $\text{sign}(w_1 - 1|w_2)$.) Hence all six points that generate these three lines have to be on the same straight line

$$L = L_{-,\text{sign}} = L_{+,\text{sign}}.$$

If in addition these six vectors are on different sides of the sign($\vec{y}$)-axis, by the convexity of the unit ball the point $(0|1)$ has to be on the same line L, too. (See Figure 4.5). $\square$

Now we have collected all the facts we shall need in the following. Recall that we already have shown two trivial cases and (**C1**) and (**C2**) of the proof. The cases (**C3**), (**A2**), (**A3**), (**B2**), and (**B3**) are still left.
Recall that we had defined two unit vectors $\vec{v} = \binom{v_1}{v_2} = \frac{\vec{y}+t_1\cdot\vec{x}}{\|\vec{y}+t_1\cdot\vec{x}\|} \neq \vec{w} = \binom{w_1}{w_2} = \frac{\vec{y}+t_2\cdot\vec{x}}{\|\vec{y}+t_2\cdot\vec{x}\|}$, with $t_1 < t_2$, and $v_2, w_2 > 0$. Because of $\vec{v} \neq \vec{w}$ there is an unique straight line L that connects both endpoints, it has an equation $L : y = m_L \cdot x + b_L$, with $m_L, b_L \in \mathbb{R}$, or a vertical line $L : x = a_L$, (if $v_1 = w_1$).

Case (**C3**): Let $v_1 = w_1 < -1$ or $+1 < v_1 = w_1$, for instance assume $1 < v_1 = w_1$. Thus L is vertical, and it has an equation $L : x = a_L$, with $a_L := v_1 = w_1$.

Thus we deduce with Proposition 4.5 that there is an unique point P_{hor}, (see Figure 4.6),

$$P_{\text{hor}} = (x_{\text{hor}}|y_{\text{hor}}) = \left(a_L \mid a_L - \frac{1}{a_L}\right),$$

such that the three points $(0|0), (a_L - 1|a_L - \frac{1}{a_L})$ and $T := (a_L|a_L + 1)$ are collinear, and such that the three points $(0|0), (a_L + 1|a_L - \frac{1}{a_L})$ and $R := (a_L|a_L - 1)$ are collinear. Because of $a_L > 1$, P_{hor} lies between R and T. By Lemma 4.10 the endpoints of all six unit vectors

$$\vec{v},\ \vec{w},\ \text{sign}(\vec{v} + (1|0)),\ \text{sign}(\vec{w} + (1|0)),\ \text{sign}(\vec{v} - (1|0)),\ \text{sign}(\vec{w} - (1|0))$$

are located on the same line L, and they have to be in the set Set 1 (by Lemma 4.7), and above the sign($\vec{x}$)-axis. Now we use Proposition 4.5. Assume $P_{\text{hor}} = \vec{v}$.

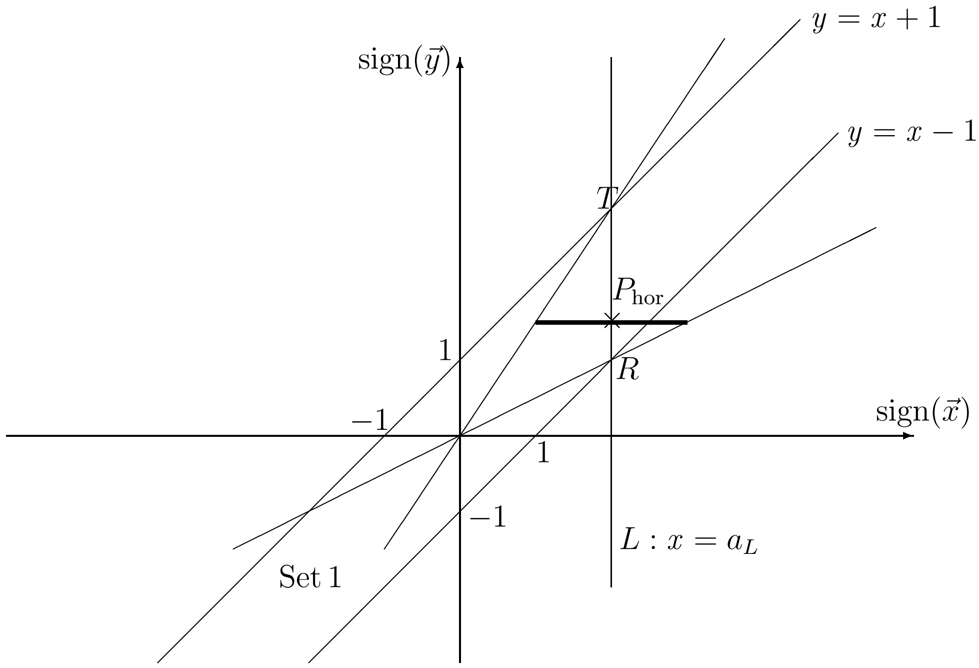

Figure 4.6.

If we have a $\vec{w}$ from the point P_{hor} on L upward, the point $\mathrm{sign}(\vec{w} - (1|0)) \in L$ will be above T, hence not in the set Set 1. If we have a $\vec{w}$ from P_{hor} on L downward, $\mathrm{sign}(\vec{w} + (1|0))$ will be below R, hence not in Set 1. Thus the only possibility is $P_{\mathrm{hor}} = \vec{v} = \vec{w}$, (and $\mathrm{sign}(\vec{v} + (1|0)) = \mathrm{sign}(\vec{w} + (1|0)) = R$ and $\mathrm{sign}(\vec{v} - (1|0)) = \mathrm{sign}(\vec{w} - (1|0)) = T$), and we find a contradiction, and Case (**C3**) is discussed.

The next Case (**A3**) will be proven in a similar way:
We have $v_1 < w_1 < -1$ or $+1 < v_1 < w_1$, assume $1 < v_1 < w_1$. By Lemma 4.7 the two different points $\vec{v}, \vec{w}$ are elements of Set 1, and they define an unique straight line $L : y = m_L \cdot x + b_L,\ m_L, b_L \in \mathbb{R}$. If $m_L = 0$ (i.e. $v_2 = w_2$,) we have $b_L \geq 1$ (otherwise, if $0 < b_L < 1$, it contradicts the convexity of the unit ball B, note the unit vectors $(0|1), \vec{v}, \vec{w}$.) If $m_L \neq 0$, we have a zero $a_L := -b_L/m_L$ of L. It follows both $1 \leq |a_L|$ and $1 \leq |b_L|$. In all other cases, namely $-1 < a_L < 1$ or $-1 < b_L < 1$, we have a contradiction to the convexity of B. (Note the four unit vectors $(1|0), (0|1), \vec{v}, \vec{w}$.) Hence, $1 \leq |b_L|$ and, if $m_L \neq 0$, $1 \leq |a_L|$.

Lemma 4.11. *Furthermore, in the just treated case of* $1 < v_1 < w_1$ *we have* $m_L \neq 1$.

Proof. Assume $m_L = 1$. If $\vec{v}, \vec{w} \in \mathrm{interior}(\mathrm{Set}\,1)$, again we get a contradiction to the convexity of B, (note $(1|0), (0|1), \vec{v}, \vec{w}$.) Or, if $\vec{v}, \vec{w}$ are on the boundary of Set 1, i.e. either $\vec{v}, \vec{w} \in L : y = x + 1$ or $\vec{v}, \vec{w} \in L : y = x - 1$, since $\|\vec{v} + (1|0)\| = \|\vec{w} + (1|0)\|$ or $\|\vec{v} - (1|0)\| = \|\vec{w} - (1|0)\|$ always follows $\vec{v} = \vec{w}$. $\square$

We call $R := (x_R|x_R - 1)$ the intersection of L and $y = x - 1$, and $T := (x_T|x_T + 1)$ the intersection of L and $y = x + 1$, $S, T \in$ Set 1. Then, by Proposition 4.5, we have an unique point $P_{\mathrm{hor}} = (x_{\mathrm{hor}}|y_{\mathrm{hor}})$ on L, such that the three points $(0|0), (x_{\mathrm{hor}}+1|y_{\mathrm{hor}})$, and R are collinear, and such that the three points $(0|0), (x_{\mathrm{hor}}-1|y_{\mathrm{hor}})$, and T are collinear, (see again Figure 4.2 for an example). The vector P_{hor} has the representation

$$P_{\mathrm{hor}} = (\,x_{\mathrm{hor}}|y_{\mathrm{hor}}) = \left(\frac{m_L - {b_L}^2}{b_L \cdot (m_L - 1)} \;\Big|\; \frac{{m_L}^2 - {b_L}^2}{b_L \cdot (m_L - 1)} \right). \tag{4}$$

By Lemma 4.7 and Lemma 4.10 all six points

$$\vec{v},\ \vec{w},\ \mathrm{sign}(\vec{v} + (1|0)),\ \mathrm{sign}(\vec{w} + (1|0)),\ \mathrm{sign}(\vec{v} - (1|0)),\ \mathrm{sign}(\vec{w} - (1|0))$$

are located on the same line L, and they have to be elements of Set 1. Assume $\vec{v} = P_{\mathrm{hor}}$. Then it follows $R = \mathrm{sign}(\vec{v} + (1|0))$ and $T = \mathrm{sign}(\vec{v} - (1|0))$. Since $\vec{w}$ is located away from $\vec{v}$ in one direction on L or the other, either $\mathrm{sign}(\vec{w}+(1|0))$ is not in Set 1 or $\mathrm{sign}(\vec{w}-(1|0))$ is not in Set 1. Hence the only possibility is $\vec{v} = \vec{w} = P_{\mathrm{hor}}$, which contradicts our assumption. Thus we have proven Case (**A3**).

Finally we handle the last Case (**A2**), because the other cases (**B2**) and (**B3**) can be treated in the same manner, and no other ideas will be needed.

Case (**A2**): Let $v_1 < w_1$ and $\{v_1, w_1\} \cap (-1, +1) \neq \emptyset$, for instance let $-1 < w_1 < 1$. The two different points $\vec{v}, \vec{w}$ define an unique straight line $L : y = m_L \cdot x + b_L,\ m_L, b_L \in \mathbb{R}$. The two points $\vec{w} - (1|0)$ and $\vec{w} + (1|0)$ are on different sides of the (vertical) $\mathrm{sign}(\vec{y})$-axis. By Lemma 4.10, all seven points

$$(0|1), \vec{v},\ \vec{w},\ \mathrm{sign}(\vec{v} + (1|0)),\ \mathrm{sign}(\vec{w} + (1|0)),\ \mathrm{sign}(\vec{v} - (1|0)),\ \mathrm{sign}(\vec{w} - (1|0))$$

are located on L, hence $b_L = 1$.

Lemma 4.12. *In the case $-1 < w_1 < 1$ which we are considering we have $-1 < m_L < 1$.*

Proof. If $m_L < -1$ or $m_L > +1$, it would contradict the convexity of the unit ball B. (Note the four unit vectors $(-1|0), \vec{v}, \vec{w}, (1|0)$). Now we assume $m_L \in \{-1, 1\}$, for instance $m_L = 1$. Then $\vec{v}, \vec{w}$ are on $L : y = x + 1$, hence the three points $(0|0), \vec{v} + (1|0), \vec{w} + (1|0)$ would be collinear on the line $y = x$. Because of $\|\vec{v} + (1|0)\| = h_+(t_1) = h_+(t_2) = \|\vec{w} + (1|0)\|$ it follows that $\vec{v} + (1|0) = \vec{w} + (1|0)$, hence $\vec{v} = \vec{w}$, and we get a contradiction. $\square$

Now we use Proposition 4.6. For convenience, we abbreviate up to now four unit vectors by $\overline{T}_w := \mathrm{sign}(\vec{w} + (1|0))$, $\overline{R}_w := \mathrm{sign}(\vec{w} - (1|0))$, $\overline{T}_v := \mathrm{sign}(\vec{v} + (1|0))$, and $\overline{R}_v := \mathrm{sign}(\vec{v} - (1|0))$.

We deal with these four unit vectors $\overline{T}_w, \overline{R}_w, \overline{T}_v, \overline{R}_v$, and their negatives $-\overline{T}_w$, $-\overline{R}_w$, $-\overline{T}_v$, $-\overline{R}_v$. If we take its endpoints, four at a time form a parallelogram, namely

$$\overline{T}_w, \overline{R}_w, -\overline{T}_w, -\overline{R}_w, \quad \text{and} \quad \overline{T}_v, \overline{R}_v, -\overline{T}_v, -\overline{R}_v, \quad \text{respectively.}$$

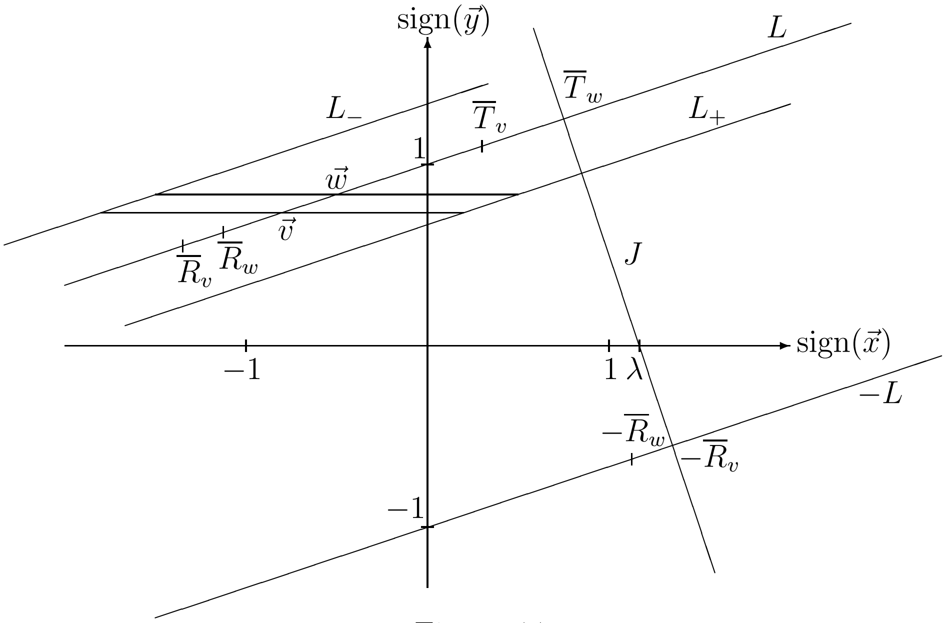

Figure 4.7.

By Proposition 4.6 the line that connects $\overline{T}_v$ and $-\overline{R}_v$ and also the line that connects $\overline{T}_w$ and $-\overline{R}_w$ intersects the horizontal axis in the point $(1|0)$. Further the two points $\overline{T}_v, \overline{R}_v$ and $\overline{T}_w, \overline{R}_w$, respectively, are located above this sign($\vec{x}$)-axis, it follows that the intersection point $(1|0)$ lies between $\overline{T}_v, -\overline{R}_v$ and $\overline{T}_w, -\overline{R}_w$, respectively.

Now let us consider the line J that connects the unit vectors $\overline{T}_w$ and $-\overline{R}_v$. Because of the convexity of the unit ball B, J must be a subset of B.

Because of our assumption $v_1 < w_1$, the leftmost one of the six different points $\overline{R}_v, \overline{R}_w, \vec{v}, \vec{w}, \overline{T}_v, \overline{T}_w$ on the line L is $\overline{R}_v$, and the rightmost one of the six is $\overline{T}_w$. Correspondingly, we consider the endpoints of the six unit vectors $-\overline{R}_v, -\overline{R}_w, -\vec{v}, -\vec{w}$, $-\overline{T}_v, -\overline{T}_w$ on the line $-L$. The leftmost one of these six points is $-\overline{T}_w$, and the rightmost one is $-\overline{R}_v$. We take on the lines L and $-L$, respectively, the point that is the rightmost one, namely $\overline{T}_w$ and $-\overline{R}_v$. The line J that connects both points $\overline{T}_w$ and $-\overline{R}_v$ intersects the sign($\vec{x}$)-axis in $\lambda > 1$. Because $J \subset \mathsf{B}$, we get $\|(\lambda|0)\| \leq 1$. Since $\lambda > 1$, the point $(1|0)$ would be in the interior of B. That contradicts $\|(1|0)\| = 1$, and finally we have found a contradiction also for the last Case (**A2**) . (See Figure 4.7).
In the example of Figure 4.7 we fix $L : y := \frac{1}{3} \cdot x + 1$, and $\vec{v} := \left(-\frac{4}{5} | \frac{11}{15}\right)$, $\vec{w} := \left(-\frac{1}{2} | \frac{5}{6}\right)$. We get $\overline{R}_v = \left(-\frac{27}{20} | \frac{11}{20}\right)$, $\overline{R}_w = \left(-\frac{9}{8} | \frac{5}{8}\right)$, $\overline{T}_v = \left(\frac{3}{10} | \frac{11}{10}\right)$, and $\overline{T}_w = \left(\frac{3}{4} | \frac{5}{4}\right)$. Hence we compute the line $J : y = -3 \cdot x + \frac{7}{2}$, and for the point $(\lambda|0)$ we get $\lambda = \frac{7}{6} > 1$.

This means that all cases (**A1**), (**A2**), (**A3**), (**B1**), (**B2**), (**B3**), (**C1**), (**C2**), and (**C3**) have been discussed, and with the assumption '$t_1 < t_2$', hence '$\vec{v} \neq \vec{w}$', we always found a contradiction, thus $t_1 = t_2$ and $\vec{v} = \vec{w}$ remain as the only possibility. Hence the map Θ is injective, hence bijective, and finally Theorem 1.3 has been proven. $\square$

5. Open Problems and Concave Corners

In this section we extend our considerations on real vector spaces X provided with a positive functional, we demand only the absolute homogeneity, i.e. $\|r\cdot\vec{x}\| = |r|\cdot\|\vec{x}\|$. We call $Z := \{\vec{x} \in X \,|\, \|\vec{x}\| = 0\}$ the 'zero-set' of X. We adapt Definition 1.1 to create a continuous product $< . \,|\, . >: X^2 \longrightarrow \mathbb{R}$, we set $< \vec{x} \,|\, \vec{y} > := 0$ for $\vec{x} \in Z$ or $\vec{y} \in Z$. For elements $\vec{x}, \vec{y} \notin Z$ with the additional property $|< \vec{x} \,|\, \vec{y} >| \leq \|\vec{x}\|\cdot\|\vec{y}\|$ we can define the 'Thy-angle' $\angle_{\text{Thy}}$ as in Definition 1.2,

$$\begin{aligned}\angle_{\text{Thy}}(\vec{x}, \vec{y}) &:= \arccos\left(\frac{< \vec{x} \,|\, \vec{y} >}{\|\vec{x}\|\cdot\|\vec{y}\|}\right) \\ &= \arccos\left(\frac{1}{4}\cdot\left[\left\|\frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|}\right\|^2 - \left\|\frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|}\right\|^2\right]\right).\end{aligned}$$

An interesting example is the following. Consider on $\mathbb{R}^2$ a homogeneous positive functional $\|\cdot\|$ which is not a seminorm. We take $\mathsf{S} := \{(x|y) \in \mathbb{R}^2 \,|\, |x|\cdot|y| = 1\}$ as the unit sphere of $(\mathbb{R}^2, \|\cdot\|)$, and we extend the functional $\|\cdot\|$ by homogeneity. Hence the two axes $Z := \{(x|0) \,|\, x \in \mathbb{R}\} \cup \{(0|y) \,|\, y \in \mathbb{R}\}$ are the zero-set. As it is just described above we define a product $< .|. >$, and the triple $(\mathbb{R}^2, \|\cdot\|, < . \,|\, . >)$ fulfils the *CSB* inequality. We define the Thy-angle for all vectors $\vec{x}, \vec{y} \notin Z$. Instead of ($An$ 1) we have here that the Thy-angle is a continuous map with the domain $(\mathbb{R}^2 \backslash Z)^2$. This angle $\angle_{\text{Thy}}$ satisfies all the other six demands (An 2)–(An 7), but not (An 8). In this example the Thy-angle has only three different values, $\angle_{\text{Thy}}(\vec{x}, \vec{y}) \in \{0, \pi/2, \pi\}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2 \backslash Z$.

In the introduction we said that in a real normed vector space the 'Thy-angle' $\angle_{\text{Thy}}(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y} \neq \vec{0}$, since the *CSB* inequality holds. Further we proved in the fourth section that the property (An 8) is fulfilled. To show this we needed extensively the convexity of the unit ball. The convexity of the unit ball is equivalent to the triangle inequality, which was not used in both definitions 1.1 and 1.2. Hence we can define the product $< . \,|\, . >$ and the Thy-angle in more general spaces, as it was just described. We come to the natural question whether there exists a real vector space X provided with a positive functional $\|\cdot\|$ such that $\|\cdot\|$ fulfils the absolute homogeneity and the positive definiteness but not the triangle inequality, and $(X, \|\cdot\|, < . \,|\, . >)$ satisfies the *CSB* inequality, or even the stronger property (An 8). To seek for examples we consider vector spaces with a positive functional $\|\cdot\|$ which have a non-convex unit ball.

Conjecture 5.1. *There exists a real vector space X provided with a positive functional $\|\cdot\|$ such that $\|\cdot\|$ is absolute homogeneous and positive definite, but its unit ball is not convex, and the triple $(X, \|\cdot\|, < . \,|\, . >)$ satisfies the CSB inequality. (Hence we can define the Thy-angle for all $\vec{x}, \vec{y} \neq \vec{0}$.)*

Conjecture 5.2. *We believe that there exists even a pair $(X, \|\cdot\|)$ which fulfils Conjecture* 5.1 *and it has the additional property* (An 8).

Good candidates for both conjectures are the *Hölder functionals* $\|\cdot\|_p$ on $\mathbb{R}^2$. For

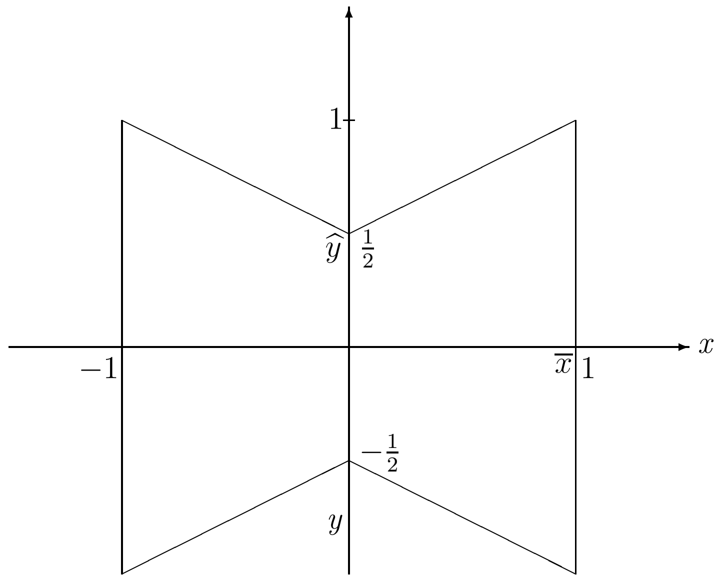

Figure 5.1: The unit sphere of $(\mathbb{R}^2, \|\cdot\|_{hexagon,\frac{1}{2}})$, with the concave corner $\widehat{y} = \big(0|\frac{1}{2}\big)$. We have $\overline{x} = (1|0),\ m_- = -1 < m_+ = +1$.

a positive real number p we set $\|(x|y)\|_p := \sqrt[p]{|x|^p + |y|^p}$. The pairs $(\mathbb{R}^2, \|\cdot\|_p)$ are normed spaces if and only if $p \geq 1$. To look for a positive answer of both questions we can take $0 < p < 1$, but p almost 1. (Hence the unit ball of $(\mathbb{R}^2, \|\cdot\|_p)$ is 'nearly' convex.)

Now we show that both conjectures are not trivial ones. The following examples prove that generally in a triple $(X, \|\cdot\|, <.|.>)$ the *CSB* inequality is not fulfilled. That means we construct counterexamples to Conjecture 5.1.

As a first example we take a Hölder functional on $\mathbb{R}^2$ with $p = \frac{1}{4}$. The Thy-angle of $(1|0)$ and $(1|1)$ does not exist in $\big(\mathbb{R}^2, \|\cdot\|_{\frac{1}{4}}\big)$.

In the second example we define functionals on $\mathbb{R}^2$ such that the unit spheres have something that we now call a 'concave corner'. They will destroy the *CSB* inequality.

Definition 5.3. Let X be a real vector space provided with a positive functional $\|\cdot\|$, let $\widehat{y} \in X$. The vector $\widehat{y}$ is called a *concave corner* if and only if there is an $\overline{x} \in X$, and there are two real numbers $m_-,\ m_+$ with $m_- < m_+$, such that for all $\delta \in [0, 1]$ the equation

$$\|\delta \cdot \overline{x} + (1 + \delta \cdot m_+) \cdot \widehat{y}\| = 1 = \| - \delta \cdot \overline{x} + (1 - \delta \cdot m_-) \cdot \widehat{y}\| \text{ holds.} \qquad \square$$

Remark 5.4. From the definition it follows that $\|\widehat{y}\| = 1$ and $\{\widehat{y}, \overline{x}\}$ is linear independent. Note that we have defined two unit vectors for $0 < \delta \leq 1$.

We get a set of functionals on $\mathbb{R}^2$ if we define for every $r > 0$ a map $\|\cdot\|_{hexagon,r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$, if we fix the unit sphere S of $(\mathbb{R}^2, \|\cdot\|_{hexagon,r})$ with the hexagon through the six points

$$\{(0|r), (1|1), (1|-1), (0|-r), (-1|-1), (-1|1)\}$$

and returning to $(0|r)$, and then extending $\|\cdot\|_{hexagon,r}$ by homogeneity. (See Figure 5.1).

Lemma 5.5. *For all $0 < r < 1$, the space $(\mathbb{R}^2, \|\cdot\|_{hexagon,r})$ has a concave corner at $\widehat{y} := (0|r)$, with $\overline{x} := (1|0)$, $m_- := 1 - \frac{1}{r} < 0 < m_+ := \frac{1}{r} - 1$.*

Proposition 5.6. *Let X be a real vector space, provided with a positive homogeneous functional $\|\cdot\|$, let $\widehat{y} \in X$ be a concave corner. Then the triple $(X, \|\cdot\|, <.|.>)$ does not fulfil the CSB inequality.*

Proof. We use the vectors $\widehat{y}$, $\overline{x}$ from the above definition of a concave corner, and then for all $\delta \in [0,1]$ we take both the unit vector $\delta \cdot \overline{x} + (1 + \delta \cdot m_+) \cdot \widehat{y}$ and $-\delta \cdot \overline{x} + (1 - \delta \cdot m_-) \cdot \widehat{y}$, and we compute their product $P(\delta)$,

$$
\begin{aligned}
& P(\delta) && (5)\\
:= \; & < \delta \cdot \overline{x} + (1 + \delta \cdot m_+) \cdot \widehat{y} \;|\; -\delta \cdot \overline{x} + (1 - \delta \cdot m_-) \cdot \widehat{y} > && (6)\\
= \; & \frac{1}{4} \cdot 1 \cdot 1 \cdot \Big[\, \| [2 + \delta \cdot (m_+ - m_-)] \cdot \widehat{y} \, \|^2 - \| \, 2 \cdot \delta \cdot \overline{x} + \delta \cdot (m_+ + m_-) \cdot \widehat{y} \, \|^2 \, \Big] && (7)\\
= \; & \frac{1}{4} \cdot \Big[\, [2 + \delta \cdot (m_+ - m_-)]^2 \cdot \| \, \widehat{y} \, \|^2 - \delta^2 \cdot \| \, 2 \cdot \overline{x} + (m_+ + m_-) \cdot \widehat{y} \, \|^2 \, \Big] && (8)\\
= \; & 1 + \delta \cdot (m_+ - m_-) + \frac{1}{4} \cdot \delta^2 \cdot \Big[\, (m_+ - m_-)^2 - \| \, 2 \cdot \overline{x} + (m_+ + m_-) \cdot \widehat{y} \, \|^2 \, \Big] && (9)\\
= \; & 1 + \delta \cdot (m_+ - m_-) + \frac{1}{4} \cdot \delta^2 \cdot K, \quad \text{if we define the real constant } K \text{ by} && (10)
\end{aligned}
$$

$$K := (m_+ - m_-)^2 - \| \, 2 \cdot \overline{x} + (m_+ + m_-) \cdot \widehat{y} \|^2 .$$

This calculation holds for all $\delta \in [0,1]$. Hence, because of $m_+ - m_- > 0$, there is a positive $\breve{\delta}$ that is nearly 0 with $P(\breve{\delta}) > 1$, thus, for the two unit vectors

$$\breve{\delta} \cdot \overline{x} + (1 + \breve{\delta} \cdot m_+) \cdot \widehat{y} \quad \text{and} \quad -\breve{\delta} \cdot \overline{x} + (1 - \breve{\delta} \cdot m_-) \cdot \widehat{y}$$

the *CSB* inequality is not satisfied. □

Corollary 5.7. *For all $0 < r < 1$, the triple $(\mathbb{R}^2, \|\cdot\|_{hexagon,r}, <.|.>)$ does not fulfil the CSB inequality. Hence, there are vectors $\vec{x} \neq \vec{0} \neq \vec{y}$ such that the Thy-angle $\angle_{\mathrm{Thy}}(\vec{x}, \vec{y})$ is not defined. Hence, for $0 < r < 1$ the space $(\mathbb{R}^2, \|\cdot\|_{hexagon,r})$ does not fulfil the property $(An\ 1)$.*

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