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An Evolutionary Structure of Convex Pentagons on a C^2 Complete Surface and a Creation Principle of some Weighted Dendrites of Order Three

Anastasios N. Zachos
University of Patras, Department of Mathematics, 26500 Rion, Greece
azachos@gmail.com, zaxos@master.math.upatras.gr

In memory of Pierre-Louis Moreau de Maupertuis (1698–1759), the founder of the least action principle.

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We reveal some new evolutionary structures of convex pentagons on a C^2 surface in the three dimensional Euclidean Space which is hidden on the notion of plasticity of weighted convex pentagons and quadrilaterals. We prove a creation principle for a weighted dendrite of order 3 on a C^2 surface which is derived by decreasing the degree of plasticity of weighted convex pentagons from 2 to 1. The creation principle of weighted dendrites of order 3 characterize the process of creation of various leaf tree types on surfaces.

Keywords: Weighted Fermat-Torricelli problem, Steiner minimal tree, convex pentagons, surface, dendrite

1991 Mathematics Subject Classification: 51E12, 52A10, 52A55, 51E10

1. Introduction

The Steiner problem is given in [5], page 360: "Given n points $A_1A_2...A_n$ to find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system."

A fundamental result which was proved in [7] (see also [2], page 328–329) states that:

A solution of the Steiner problem is a Steiner tree with at most $n-2$ Fermat-Torricelli (or Steiner) points which are vertices of the polygonal tree which do not belong in $\{A_1A_2...A_n\}$, where each Fermat-Torricelli point has valency 3, and the angle between any two edges incident with a Fermat-Torricelli point is of 120°.

A study of the definition of topologies of Steiner trees and variants of Steiner problem which are uniquely determined on the two dimensional Euclidean Space

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is given in [7]. Some other classical studies on Steiner trees and generalizations (see also references therein) are given in [5], [3], [4], [9], [10] and a classical study for a distorted Steiner minimal tree (Fermat-Torricelli problem) is given in Chapter II of [2].

An evolutionary structure of convex quadrilaterals on surfaces of constant Gaussian curvature has been studied in [19] by introducing the notion of plasticity of weighted quadrilaterals on surfaces of constant Gaussian curvature.

Let M be a complete C^2 surface, a subset of $\mathbb{R}^3$ with variable Gaussian curvature K such that:

1. If $K_b < K$, for a positive constant number K_b then the shortest path (geodesic arc) of length not greater than $\frac{\pi}{\sqrt{K_b}}$ is unique and an estimate of the **injectivity radius** $r_i = \inf(r_i(P) : P \in M)$ of M is given by the inequality (see [13], [14, Theorem 3.5.2, Theorem 3.5.3]):

$$r_i \leq \frac{\pi}{\sqrt{K_b}}.$$

2. If M is simply connected and $K < 0$, then $r_i = \infty$ and every two points on M can be joined by a shortest path which is a geodesic (see [14, Theorem 3.7.2, 3.7.3, 3.7.4]).

We give the following three definitions which refer to the degree of plasticity (*Degplasticity*$\{A_1A_2A_3A_4A_5\}$) of weighted Steiner trees that correspond to a convex pentagon $A_1A_2A_3A_4A_5$ on M : (see for a general definition of degree of plasticity for the case of $\mathbb{R}^2$ in [21]):

Definition 1.1. The degree of plasticity (*Degplasticity*$\{A_1A_2A_3A_4A_5\}$) of the weighted (full) Steiner minimal tree of a convex pentagon on M which consists of exactly three weighted Fermat-Torricelli points at the interior of the pentagon is zero.

Definition 1.2. The degree of plasticity (*Degplasticity*$\{A_1A_2A_3A_4A_5\}$) of the weighted Steiner minimal tree of a convex pentagon on M which consists of exactly two weighted Fermat-Torricelli points at the interior of the pentagon is one.

Definition 1.3. The degree of plasticity (*Degplasticity*$\{A_1A_2A_3A_4A_5\}$) of the weighted Steiner minimal tree of a convex pentagon on M which consists of exactly one weighted Fermat-Torricelli point at the interior of the pentagon is two.

In this paper, we derive some new evolutionary structures of convex pentagons on a C^2 surface M in the three dimensional Euclidean Space and prove a creation principle for weighted dendrites of order three.

First, we give a concise study of the weighted Fermat-Torricelli problem for convex pentagons on M (Problem 2.1, Propositions 2.2, 2.3, 2.5, Section 2).

Second, we solve the problem of evolution of convex pentagons on M with respect to the 5-inverse weighted Fermat-Torricelli problem (Problem 3.1, Section 3) by applying the plasticity principle of weighted pentagons (Theorem 3.2, Corollary 3.3, Theorem 3.4).

Third, we study a weighted distorted Steiner tree for convex pentagons on M by assuming that two points with valency 3 and 4, respectively, are located at the interior of the convex pentagon on M with equal positive weights (Theorem 4.1, Section 4). A specific case is the derivation of a weighted (full) Steiner minimal tree for convex quadrilaterals on M (Proposition 4.3, Corollary 4.4, Section 4) which is a generalization of the study given in [23] for a weighted (full) steiner tree for convex quadrilaterals on the two dimensional K-plane.

Fourth, we introduce the 5-inverse distorted weighted Steiner tree for convex pentagons on M, (Problem 5.1, Section 5) which is a generalization of the 5-inverse weighted Fermat-Torricelli problem on M. The plasticity equations of the 5 weighted distorted Steiner tree on M (Theorem 5.2, Corollary 5.3, Section 5) provide a dynamic plasticity which is derived by reducing the degree of plasticity of convex pentagons from 2 to 1.

This reduction creates a weighted dendrite of order 3 on M (Definition 5.6, Section 5).

A creation principle of weighted dendrites of order three states that (Theorem 5.7, Subsection 5.1):

Consider the 3-inverse weighted Fermat-Torricelli problem for $\triangle A_1A_2A_3$ on M. At time $t = 0$, two geodesic arcs $A_0A_{4'}$ $A_0A_{5'}$ grow simultaneously from A_0 and move via parallel translation along the geodesic arc $A_0A_{0'}$. This evolutionary process creates a weighted dendrite of order three which corresponds to $\triangle A_4A_{0'}A_5$, such that a 5-inverse weighted distorted Steiner tree occurs and the points A_0 and $A_{0'}$ remain the same.

The creation principle of weighted dendrites of order three and the generalized plasticity which has been introduced in [20, Proposition 6, Corollary 4, Subsection 5.2] characterize the process of creation of various leaf tree types on surfaces (Propositions 5.10, 5.11, Corollary 5.12, Subsection 5.2). We note that our approach on the creation of leaf tree types on surfaces is different than the approach given in [16] which deals with the formation of tree leaves on the two dimensional Euclidean Space.

Finally, we mention the definition of an exponential mapping and the normal geodesic coordinate system on M at the appendix (see also [1], pages 117–119 and a detailed appendix in [17], [18]).

2. A plasticity principle of convex pentagons on a C^2 complete surface.

We state the weighted Fermat-Torricelli problem on M for pentagons.

Problem 2.1. *Let $A_1A_2A_3A_4A_5 \subset M$ be a convex pentagon whose perimeter is less than $\frac{2\pi}{\sqrt{K_b}}$ for $K > K_b$. Suppose that a positive number (weight) B_i, corresponds to the vertex A_i, for $i = 1, 2, 3, 4, 5$. Find the weighted Fermat-Torricelli point A_0*

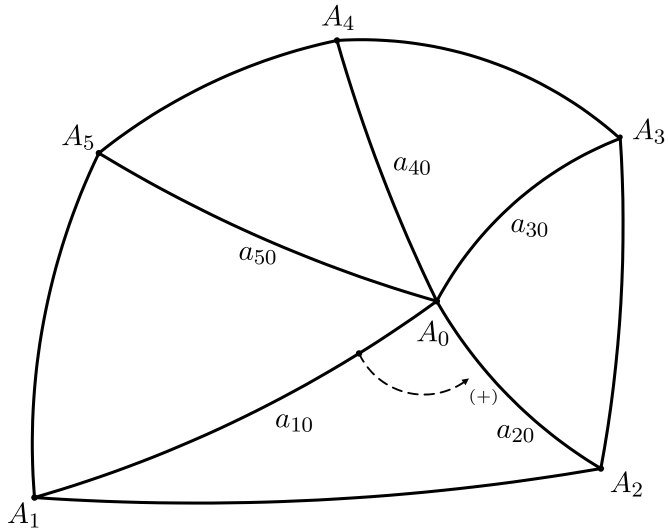

Figure 2.1.

such that

$$f(A_0) = B_1 a_{10} + B_2 a_{20} + B_3 a_{30} + B_4 a_{40} + B_5 a_{50} \rightarrow \min \tag{1}$$

where a_{i0} *is the length of the geodesic (shortest) arc from the weighted Fermat-Torricelli point* A_0 *to the vertex* A_i*, for* $i = 1, 2, 3, 4, 5$ *(Fig.* 2.1*).*

We extend the results given in [17, Proposition 1, page 45, Proposition 8, page 56], [18] for the case of a $\triangle ABC$ on a C^2 surface M for a convex pentagon $A_1A_2A_3A_4A_5$ on M.

Proposition 2.2. *The weighted Fermat-Torricelli point (Problem* 2.1*)* A_0 *exists and is unique.*

Proof of Proposition 2.2 (Existence). Let W be a compact subset of $\mathbb{R}^3$, the convex pentagon $A_1A_2A_3A_4A_5$ is a subset of W and $f : W \rightarrow \mathbb{R}$ is a continuous function. Then by applying Weirsstrass theorem f attains its minimum on W. $\square$

Proof of Proposition 2.2 (Uniqueness). It is proved in [17, Proposition 1, pages 45–47], [18], by applying results from comparison geometry of A. D. Aleksandrov that the length of a geodesic (shortest) arc a_{0i} is a strictly convex function. Thus, the objective function $f(A_0)$ which is the weighted sum of five strictly convex functions a_{0i}, for $i = 1, 2, 3, 4, 5$ is a strictly convex function and has at most one minimum point on W ([15, page 60, Corollary 7.4, page 91], [12], page 263). $\square$

We note that the characterization of the weighted Fermat-Torricelli point for triangles on a C^2 surface M is given by Proposition 6, page 53 (floating case) and Proposition 7, page 55 (absorbed case) in [17], [18] and for quadrilaterals is given in [22]. An extension of the characterization of the weighted Fermat-Torricelli point for convex pentagons on M are given by the following two Propositions 2.3 (Floating Case) and 2.5 (Absorbed case).

Proposition 2.3 (Floating Case). *If $\vec{U}_{RS}$ is the unit tangent vector of the geodesic arc RS at R and D is the domain of a C^2 surface M bounded by $A_1A_2A_3A_4A_5$, for $R, S \in \{A_1, A_2, A_3, A_4, A_5\}$ and*

$$\vec{U}_{RS} = (\varphi_\star)_R^{-1}\left(\frac{\exp_R^{-1}(S)}{a_{RS}}\right) = (\varphi_\star)_R^{-1}\left(\frac{\tilde{X}_{RS}}{\left\|\tilde{X}_{RS}\right\|}\right), \tag{2}$$

(see Appendix for the definition of the mapping $(\varphi_\star)_R^{-1}$ and the exponential mapping) then the following (I), (II), (III) conditions are equivalent:

(I) All the following inequalities are satisfied simultaneously:

$$\left\|B_2\vec{U}_{A_1A_2} + B_3\vec{U}_{A_1A_3} + B_4\vec{U}_{A_1A_4} + B_5\vec{U}_{A_1A_5}\right\| > B_1, \tag{3}$$

$$\left\|B_1\vec{U}_{A_2A_1} + B_3\vec{U}_{A_2A_3} + B_4\vec{U}_{A_2A_4} + B_5\vec{U}_{A_2A_5}\right\| > B_2, \tag{4}$$

$$\left\|B_1\vec{U}_{A_3A_1} + B_2\vec{U}_{A_3A_2} + B_4\vec{U}_{A_3A_4} + B_5\vec{U}_{A_3A_5}\right\| > B_3, \tag{5}$$

$$\left\|B_1\vec{U}_{A_4A_1} + B_2\vec{U}_{A_4A_2} + B_3\vec{U}_{A_4A_3} + B_5\vec{U}_{A_4A_5}\right\| > B_4, \tag{6}$$

$$\left\|B_1\vec{U}_{A_5A_1} + B_2\vec{U}_{A_5A_2} + B_3\vec{U}_{A_5A_3} + B_4\vec{U}_{A_5A_4}\right\| > B_5, \tag{7}$$

(II) The weighted Fermat-Torricelli point A_0 is an interior point of $A_1A_2A_3A_4A_5$ and does not belong to the geodesic arcs A_1A_2, A_2A_3, A_3A_4, A_4A_5 and A_5A_1.

(III) $B_1\vec{U}_{A_0A_1} + B_2\vec{U}_{A_0A_2} + B_3\vec{U}_{A_0A_3} + B_4\vec{U}_{A_0A_4} + B_5\vec{U}_{A_0A_5} = \vec{0}$.

Proof. $(II) \rightarrow (III)$. By applying the same method of differentiation of a length of geodesic arc with respect to an unknown length of a geodesic arc that was used in [17], [18], we will prove that $B_1\vec{U}_{A_0A_1}+B_2\vec{U}_{A_0A_2}+B_3\vec{U}_{A_0A_3}+B_4\vec{U}_{A_0A_4}+B_5\vec{U}_{A_0A_5} = \vec{0}$.

We assume that, the point A_i, the parameterized curve $c(s)$ and the point A_0 that belongs in $c(s)$ are given in the neighborhood W_{A_0} and the orientation of the angles with respect to A_0 taken counterclockwise (see Fig. 2.1). We join the two points A_i and A_0 by the geodesic $\gamma_{A_i}(s,t) = \gamma_{A_iA_0}(t)$ where t is a canonical parameter, counting from the point A_i such that for $t = 0$, $\gamma_{A_iA_0}(0) = A_i$ and for $t = 1$, $\gamma_{A_iA_0}(1) = A_0$. The function $a_{i0} = a_{i0}(A_0) = a_{i0}(\gamma_{A_iA_0}(t)) = a_{i0}(\gamma_{A_i}(s,t)) = a_{i0}(s)$ is differentiable with respect to s. We denote by $\alpha_{j0i}(s)$ the angle between $\frac{dc}{ds}$ and $\frac{d\gamma_{A_i}}{ds}$ at A_0 which is the same angle between the geodesic arcs A_jA_0 and A_jA_0, for $i, = 1, 2, 3, 4, 5$ and $i \neq j$.

We follow the same parametrization technique that was used in [17, Proof of Proposition 2, Pages 48–50], [18]:

From [14, Lemma 3.5.1 and Remark 3.5.1], we get:

$$\frac{da_{i0}(s)}{ds} = \cos(\alpha_{j0i}(s)). \tag{8}$$

We proceed by setting the parametrization:

$$a_{10}(s)=\int_0^s\left\|\frac{d\gamma_{A_1A_0}(t)}{dt}\right\|dt=\int_0^s\left\|\frac{d\gamma_{A_1}(s,t)}{dt}\right\|dt=s,$$

or

$$a_{10}(s)=a_{10}=s. \tag{9}$$

Suppose that a_{20}, a_{30} and a_{40} and a_{50} can be expressed as functions of a_{10}:

$$a_{20}=a_{20}(a_{10}),\quad a_{30}=a_{30}(a_{10}),\quad a_{40}=a_{40}(a_{10})\quad\text{and}\quad a_{50}=a_{50}(a_{10}). \tag{10}$$

From (10) and (1), we have:

$$B_1a_{10}+B_2a_{20}(a_{10})+B_3a_{30}(a_{10})+B_4a_{40}(a_{10})+B_5a_{50}(a_{10})=\text{minimum}. \tag{11}$$

By differentiation of (11) with respect to the variable a_{10} and taking into account (9), we get

$$B_1+B_2\frac{da_{20}}{da_{10}}+B_3\frac{da_{30}}{da_{10}}+B_4\frac{da_{40}}{da_{10}}+B_5\frac{da_{50}}{da_{10}}=0, \tag{12}$$

Taking into account that the arc length parameter counts from A_1 to A_0 and from (9), (8), we obtain:

$$\frac{da_{20}}{da_{10}}=\cos(\alpha_{102}(s)), \tag{13}$$

$$\frac{da_{30}}{da_{10}}=\cos(\alpha_{103}(s)), \tag{14}$$

$$\frac{da_{40}}{da_{10}}=\cos(\alpha_{104}(s)). \tag{15}$$

and

$$\frac{da_{50}}{da_{10}}=\cos(\alpha_{105}(s)). \tag{16}$$

By replacing (13), (14), (15), (16) and (9) in (12), we derive:

$$\begin{aligned}B_1+B_2\cos(\alpha_{102}(a_{10}))+B_3\cos(\alpha_{103}(a_{10}))\\+B_4\cos(\alpha_{104}(a_{10}))+B_5\cos(\alpha_{105}(a_{10}))=0.\end{aligned} \tag{17}$$

Similarly, by working cyclically and setting the parametrization

$$a_{20}(s')=\int_0^{s'}\left\|\frac{d\gamma_{A_2}(s',t)}{dt}\right\|dt=s',$$

$a_{30}(s'') = \int_0^{s''} \left\| \frac{d\gamma_{A_3}(s'',t)}{dt} \right\| dt = s''$, $a_{40}(s''') = \int_0^{s'''} \left\| \frac{d\gamma_{A_4}(s''',t)}{dt} \right\| dt = s'''$, $a_{50}(s'''') = \int_0^{s''''} \left\| \frac{d\gamma_{A_5}(s'''',t)}{dt} \right\| dt = s''''$ and by differentiating (1) with respect to a_{20}, for $s' = a_{20}$, a_{30}, for $s'' = a_{30}$, a_{40}, for $s''' = a_{40}$, a_{50}, for $s'''' = a_{50}$, we get respectively:

$$\begin{aligned} &B_1 \cos(\alpha_{201}(a_{20})) + B_2 + B_3 \cos(\alpha_{203}(a_{20})) \\ &\quad + B_4 \cos(\alpha_{204}(a_{20})) + B_5 \cos(\alpha_{205}(a_{20})) = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &B_1 \cos(\alpha_{301}(a_{30})) + B_2 \cos(\alpha_{302}(a_{30})) \\ &\quad + B_3 + B_4 \cos(\alpha_{304}(a_{30})) + B_5 \cos(\alpha_{305}(a_{30})) = 0. \end{aligned} \tag{19}$$

$$\begin{aligned} &B_1 \cos(\alpha_{401}(a_{40})) + B_2 \cos(\alpha_{402}(a_{40})) \\ &\quad + B_3 \cos(\alpha_{403}(a_{40})) + B_4 + B_5 \cos(\alpha_{405}(a_{40})) = 0. \end{aligned} \tag{20}$$

and

$$\begin{aligned} &B_1 \cos(\alpha_{501}(a_{50})) + B_2 \cos(\alpha_{502}(a_{50})) \\ &\quad + B_3 \cos(\alpha_{503}(a_{50})) + B_4 \cos(\alpha_{504}(a_{50})) + B_5 = 0. \end{aligned} \tag{21}$$

From the uniqueness of the weighted Fermat-Torricelli point A_0, we get:

$$\alpha_{j0i}(a_{j0}) = \alpha_{j0i},$$

for $i, j = 1, 2, 3, 4, 5$ and $j \neq i$.

By solving the linear system (17)–(18) with respect to B_1 and B_2, we obtain:

$$B_1 \sin(\alpha_{201}) + B_3 \sin(\alpha_{203}) + B_4 \sin(\alpha_{204}) + B_5 \sin(\alpha_{205}) = 0, \tag{22}$$

$$B_2 \sin(\alpha_{102}) + B_3 \sin(\alpha_{103}) + B_4 \sin(\alpha_{104}) + B_5 \sin(\alpha_{105}) = 0. \tag{23}$$

Similarly, by working cyclically and by solving the linear system (17) and (19) with respect to B_1, B_3, (17) and (20) with respect to B_1, B_4 and (17) and (21) with respect to B_1, B_5, we get, correspondingly:

$$B_1 \sin(\alpha_{301}) + B_2 \sin(\alpha_{302}) + B_4 \sin(\alpha_{304}) + B_5 \sin(\alpha_{305}) = 0, \tag{24}$$

$$B_1 \sin(\alpha_{401}) + B_2 \sin(\alpha_{402}) + B_4 \sin(\alpha_{403}) + B_5 \sin(\alpha_{405}) = 0, \tag{25}$$

and

$$B_1 \sin(\alpha_{501}) + B_2 \sin(\alpha_{502}) + B_3 \sin(\alpha_{503}) + B_4 \sin(\alpha_{504}) = 0. \tag{26}$$

We need the following relation of the tangent vector $\vec{U}_{A_0 A_i}$ at A_0 (see appendix):

$$\vec{U}_{A_0 A_i} = (\phi_\star)^{-1}_{A_0} \left(\frac{\exp^{-1}_{A_0}(A_i)}{a_{i0}} \right), \tag{27}$$

for $i = 1, 2, 3, 4, 5$, where $\exp_{A_0}(0) = A_0$.

By defining the coordinates of $\vec{U}_{A_0A_1}$ in $\mathbb{R}^2$ to be $(1,0) = \frac{\exp_{A_0}^{-1}(A_1)}{a_{10}}$, we derive that the coordinates of $\vec{U}_{A_0A_i}$ (by taking a counterclockwise orientation of angles with respect to A_0) are $(\cos(\alpha_{10i}), \sin(\alpha_{10i})) = \frac{\exp_{A_0}^{-1}(A_i)}{a_{i0}}$, for $i = 1,2,3,4,5$.

Therefore, by using (17) and (23), we get:

$$\begin{aligned}
&B_1\vec{U}_{A_0A_1} + B_2\vec{U}_{A_0A_2} + B_3\vec{U}_{A_0A_3} + B_4\vec{U}_{A_0A_4} + B_5\vec{U}_{A_0A_5} \\
&= \sum_{i=1}^{5} B_i(\phi_\star^{-1})_{A_0}(\cos\alpha_{10i}, \sin\alpha_{10i}) \\
&= (\phi_\star^{-1})_{A_0}\left(\sum_{i=1}^{5} B_i\cos\alpha_{10i}, \sum_{i=1}^{5} B_i\sin\alpha_{10i}\right) = (\phi_\star^{-1})_{A_0}(0,0) = \vec{0}.
\end{aligned}$$

□

Remark 2.4. The system of equations (17), (18), (19), (20), (21) may also be written in a unified form:

$$\sum_{i=1}^{5} B_i\cos\alpha_{j0i} = 0,$$

for $j = 1,2,3,4,5$.

From the 5-inverse weighted Fermat-Torricelli problem (see the definition in Section 3), we could also derive the system of equations (22), (23), (24), (25), (26):

$$\sum_{i=1}^{5} B_i\sin\alpha_{j0i} = 0,$$

for $j = 1,2,3,4,5$.

By combining the unified equations of "cosine" and "sine", we have:

$$\sum_{i=1}^{5} B_i\exp(\alpha_{j0i}) = 0,$$

for $j = 1,2,3,4,5$.

Proposition 2.5 (Absorbed Case). *The following (I), (II) conditions are equivalent.*

(I) One of the following inequalities is satisfied:

$$\left\| B_2\vec{U}_{A_1A_2} + B_3\vec{U}_{A_1A_3} + B_4\vec{U}_{A_1A_4} + B_5\vec{U}_{A_1A_5} \right\| \le B_1, \tag{28}$$

or

$$\left\| B_1\vec{U}_{A_2A_1} + B_3\vec{U}_{A_2A_3} + B_4\vec{U}_{A_2A_4} + B_5\vec{U}_{A_2A_5} \right\| \le B_2, \tag{29}$$

or

$$\left\| B_1\vec{U}_{A_3A_1} + B_2\vec{U}_{A_3A_2} + B_4\vec{U}_{A_3A_4} + B_5\vec{U}_{A_3A_5} \right\| \le B_3, \tag{30}$$

or

$$\left\| B_1\vec{U}_{A_4A_1} + B_2\vec{U}_{A_4A_2} + B_3\vec{U}_{A_4A_3} + B_5\vec{U}_{A_4A_5} \right\| \leq B_4, \tag{31}$$

or

$$\left\| B_1\vec{U}_{A_5A_1} + B_2\vec{U}_{A_5A_2} + B_3\vec{U}_{A_5A_3} + B_4\vec{U}_{A_5A_4} \right\| \leq B_5, \tag{32}$$

(II) *The weighted Fermat-Torricelli point A_0 is attained at A_1 or A_2 or A_3 or A_4 or A_5, respectively.*

Remark 2.6. By following the proof of Proposition 6, page 53 (floating case) and Proposition 7, page 55 (absorbed case) in [17], [18] for triangles $\triangle ABC$ on a C^2 surface line by line, we could easily extend these proofs to convex pentagons on a C^2 surface.

Remark 2.7. We note that the existence, uniqueness of the weighted Fermat-Torricelli point is also proved in [6] on positively curved manifolds. Furthermore, P. Fletcher, S. Venkatasubramanian and S. Joshi have also succeeded in calculating the weighted Fermat-Torricelli point by generalizing the famous Weiszfeld algorithm on Riemannian manifolds (see [6]).

3. The 5-inverse weighted Fermat-Torricelli problem on a complete surface M with variable Gaussian Curvature

Problem 3.1. *Given a point A_0 which belongs to the interior of $A_1A_2A_3A_4A_5$ on M whose perimeter is less than $\frac{2\pi}{\sqrt{K_b}}$ for $K > K_b$, does there exist a unique set of positive weights B_i, such that*

$$B_1 + B_2 + B_3 + B_4 + B_5 = c = const,$$

for which A_0 minimizes

$$f(A_0) = B_1a_{10} + B_2a_{20} + B_3a_{30} + B_4a_{40} + B_5a_{50}$$

where a_{i0} is the length of the geodesic arc (shortest arc) from A_0 to the vertex A_i for $i = 1, 2, 3, 4, 5$.

This is the 5-inverse weighted Fermat-Torricelli problem on M (Fig. 2.1).

For $B_5 = 0$, we derive the 4-inverse weighted Fermat-Torricelli problem. A negative answer to this problem is given in [22] and a positive answer is given in [17], [18] for the 3-inverse weighted Fermat-Torricelli problem on a C^2 complete surface M.

A negative answer is also given for the 5-inverse weighted Fermat-Torricelli problem, because three weights depend on two variable weights (Theorem 3.2 and Corollary 3.3).

The evolution of convex pentagons on M is given by the invariance of the weighted Fermat-Torricelli point for given convex pentagons (geometric plasticity) and the plasticity of convex pentagons which are convex pentagons which satisfy Theorem 3.2 (dynamic plasticity).

Theorem 3.2. *Consider the 5-inverse weighted Fermat-Torricelli problem on M in $\mathbb{R}^3$. The plasticity of convex pentagons on M is given by the following three equations:*

$$\left(\frac{B_2}{B_1}\right)_{12345} = \left(\frac{B_2}{B_1}\right)_{123}\left[1 - \left(\frac{B_4}{B_1}\right)_{12345}\left(\frac{B_1}{B_4}\right)_{134} - \left(\frac{B_5}{B_1}\right)_{12345}\left(\frac{B_1}{B_5}\right)_{135}\right] \quad (33)$$

$$\left(\frac{B_3}{B_1}\right)_{12345} = \left(\frac{B_3}{B_1}\right)_{123}\left[1 - \left(\frac{B_4}{B_1}\right)_{12345}\left(\frac{B_1}{B_4}\right)_{124} - \left(\frac{B_5}{B_1}\right)_{12345}\left(\frac{B_1}{B_5}\right)_{125}\right] \quad (34)$$

and

$$(B_1)_{12345} + (B_2)_{12345} + (B_3)_{12345} + (B_4)_{12345} + (B_5)_{12345} = const. \quad (35)$$

Proof of Theorem 3.2. Considering the 5-inverse weighted Fermat-Torricelli problem, the variable weights $(B_i)_{12345}$ for $i = 1, 2, 3, 4, 5$, satisfy equations (17)–(21) and (22)–(26).

From (24), we have:

$$\left(\frac{B_2}{B_1}\right)_{12345} = -\frac{\sin(\alpha_{301})}{\sin(\alpha_{302})}\left[1 + \left(\frac{B_4}{B_1}\right)_{12345}\frac{\sin(\alpha_{304})}{\sin(\alpha_{301})} + \left(\frac{B_5}{B_1}\right)_{12345}\frac{\sin(\alpha_{305})}{\sin(\alpha_{301})}\right].$$

By replacing $(B_4)_{12345} = (B_5)_{12345} = 0$, in (22), (23) and (24), we obtain a solution of the 3-inverse weighted Fermat-Torricelli problem (see [17], for a solution of this problem on a C^2-complete surface).

Specifically, we have:

$$-\frac{\sin(\alpha_{301})}{\sin(\alpha_{302})} = \left(\frac{B_2}{B_1}\right)_{123}. \quad (36)$$

Similarly, by replacing $(B_2)_{12345} = (B_5)_{12345} = 0$, in (22)–(24), we obtain a solution of the 3-inverse weighted Fermat-Torricelli problem for the triangle $\triangle A_1A_3A_{4^\star}$, where ($A_4^\star$ is the symmetric point of A_4 with respect to A_0. Hence, we have:

$$-\left(\frac{B_1}{B_4}\right)_{134^\star} = \frac{\sin(\alpha_{304^\star})}{\sin(\alpha_{301})} = -\frac{\sin(\alpha_{304})}{\sin(\alpha_{301})} = \left(\frac{B_1}{B_4}\right)_{134}, \quad (37)$$

Similarly, by replacing $(B_2)_{12345} = (B_4)_{12345} = 0$, in (22)–(24), we obtain a solution of the 3-inverse weighted Fermat-Torricelli problem for the triangle $\triangle A_1A_3A_{5^\star}$, where ($A_5^\star$ is the symmetric point of A_5 with respect to A_0. Hence, we have:

$$-\left(\frac{B_1}{B_5}\right)_{135^\star} = \frac{\sin(\alpha_{305^\star})}{\sin(\alpha_{301})} = -\frac{\sin(\alpha_{305})}{\sin(\alpha_{301})} = \left(\frac{B_1}{B_5}\right)_{135}, \quad (38)$$

(36), (37) and (38) yield (33).

By working cyclically and exchanging the indices $2 \to 3$ and $3 \to 2$, we derive (34). $\square$

A direct consequence of Theorem 3.2 is given by the following Corollary which is an extension of Corollary 4.5 in [19], page 418.

Corollary 3.3. *Set* $\sum_{12345} B := (B_1)_{12345}(1+\frac{B_2}{B_1}+\frac{B_3}{B_1}+\frac{B_4}{B_1}+\frac{B_5}{B_1})_{12345}$. *If* $\sum_{12345} B = \sum_{123} B = \sum_{124} B = \sum_{134} B = \sum_{125} B = \sum_{135} B$, *then*

$$(B_i)_{12345} = x_i(B_4)_{12345} + y_i(B_5)_{12345} + z_i, \quad i = 1,2,3:$$

$$(x_1,y_1,z_1) = \left(\frac{\left(\frac{B_1}{B_4}\right)_{134}\left(\frac{B_2}{B_1}\right)_{123} + \left(\frac{B_1}{B_4}\right)_{124}\left(\frac{B_3}{B_1}\right)_{123} - 1}{1+\left(\frac{B_2}{B_1}\right)_{123}+\left(\frac{B_3}{B_1}\right)_{123}}, \right.$$
$$\left. \frac{\left(\frac{B_1}{B_5}\right)_{135}\left(\frac{B_2}{B_1}\right)_{123} + \left(\frac{B_1}{B_5}\right)_{125}\left(\frac{B_3}{B_1}\right)_{123} - 1}{1+\left(\frac{B_2}{B_1}\right)_{123}+\left(\frac{B_3}{B_1}\right)_{123}}, (B_1)_{123} \right)$$

$$(x_2,y_2,z_2) = \left(x_1\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{134}\left(\frac{B_2}{B_1}\right)_{123}, \right.$$
$$\left. y_1\left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B_5}\right)_{135}\left(\frac{B_2}{B_1}\right)_{123}, (B_2)_{123} \right)$$

$$(x_3,y_3,z_3) = \left(x_1\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_4}\right)_{124}\left(\frac{B_3}{B_1}\right)_{123}, \right.$$
$$\left. y_1\left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B_5}\right)_{125}\left(\frac{B_3}{B_1}\right)_{123}, (B_3)_{123} \right).$$

Proof. Taking into account that

$$\sum_{12345} w := (B_1)_{12345}\left(1+\sum_{j=2}\frac{B_2}{B_1}+\frac{B_3}{B_1}+\frac{B_4}{B_1}+\frac{B_5}{B_1}\right)_{12345} = \sum_{ijk} B$$

for $i,j,k \in \{1,2,3,4,5,\}$ and by solving with respect to $(B_i)_{12345}$ we derive the plasticity equations for $i = 1,2,3$ which depend on 2 independent variables $(B_4)_{12345}$ and $(B_5)_{12345}$. □

Consider the 3-inverse weighted Fermat-Torricelli problem on M. We assume at time $t = 0$ that two geodesic arcs (fourth and fifth branch) grow simultaneously from the weighted Fermat-Torricelli point A_0 which is located at the interior of the triangle $\triangle A_1A_2A_3$. We derive the following plasticity principle of weighted convex pentagons on M which generalizes the result given in [19] for weighted convex quadrilaterals on surfaces of constant Gaussian curvature and the result given in [22] regarding weighted convex quadrilaterals on a complete convex surface.

Theorem 3.4. *Given five geodesic (shortest) arcs which meet at the weighted Fermat-Torricelli point A_0 and their endpoints form a convex pentagon on M and the weighted Fermat-Torricelli point belongs to the interior of this convex pentagon, an increase of the weight that corresponds to the fourth and fifth shortest arc causes*

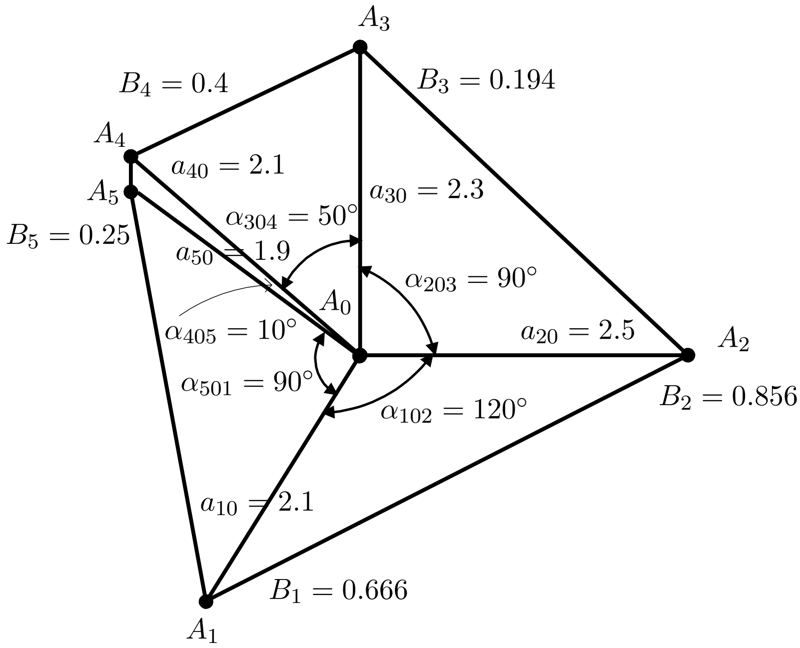

Figure 3.1.

a decrease to the two weights that correspond to the first and third shortest arc and an increase to the weight that corresponds to the second shortest arc.

Proof of Theorem 3.4: We need the following fundamental result which has been introduced in [19] on surfaces of constant Gaussian curvature and has been proved in [22, Theorem 2] on a complete convex surface of bounded specific curvature:

Given four shortest arcs which meet at the weighted Fermat-Torricelli point A_0 and their endpoints form a convex quadrilateral on M and the weighted Fermat-Torricelli point belongs to the interior of this convex quadrilateral, an increase of the weight that corresponds to a shortest arc causes a decrease to the two weights that correspond to the two neighboring shortest arcs and an increase to the weight that corresponds to the opposite shortest arc (plasticity principle of convex quadrilaterals).

By assuming that at time $t = 0$ two geodesic arcs (fourth and fifth branch) grow simultaneously from the weighted Fermat-Torricelli point A_0 which is located at the interior of the triangle $\triangle A_1A_2A_3$ and applying the plasticity principle of weighted convex quadrilaterals, we obtain the desired result. $\square$

Remark. Theorem 3.2, Corollary 3.3 and Theorem 3.4 have recently been derived in [24] as a limiting case of the dynamic plasticity of closed hexahedra in the three-dimensional Euclidean space.

Example 3.5. Consider the 5-inverse weighted Fermat-Torricelli problem for a convex pentagon in the two dimensional Euclidean Space as an extension of Examples 4.7 and 4.10 given in [19], pages 418–421.

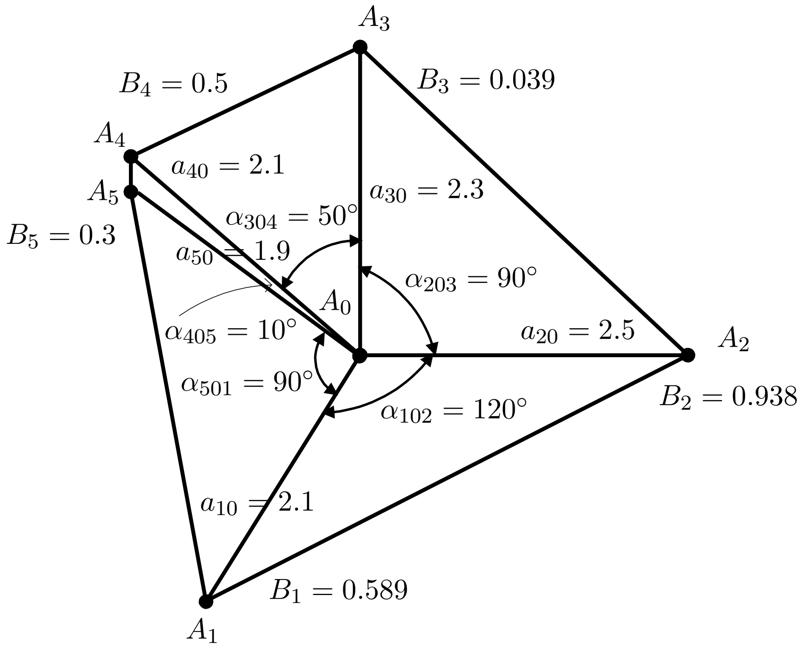

Figure 3.2.

Specifically, given the angles $\alpha_{102} = 120°$, $\alpha_{203} = 90°$, $\alpha_{304} = 50°$, $\alpha_{405} = 10°$, (α_{501} is calculated from the four given angles) $a_{10} = 2.1$, $a_{20} = 2.5$, $a_{30} = 2.3$, $a_{40} = 2.1$, $a_{50} = 1.9$, the positive weight $(B_1)_{123} = 1$ and the assumption that $\sum_{12345} B = \sum_{1234} B = \sum_{123} B = \sum_{124} B = \sum_{134} B = \sum_{125} B = \sum_{135} B = 2.366$ from the plasticity equations of Theorem 3.2 and Corollary 3.3 we derive the following relations:

$$\triangle 123(A_1A_2A_3) : (B_1)_{123} = 1, (B_2)_{123} = 0.50, (B_3)_{123} = 0.866$$

$$\sum_{123} B = 2.366$$

$$\triangle 124(A_1A_2A_4) : (B_1)_{124} = 0.609, (B_2)_{124} = 0.934, (B_4)_{124} = 0.82$$

$$\sum_{124} B = 2.366$$

$$\triangle 134(A_1A_3A_4) : (B_1)_{134} = 1.449, (B_3)_{134} = 1.862, (B_4)_{134} = -0.945$$

$$\sum_{134} B = 2.366$$

$$\triangle 125(A_1A_2A_5) : (B_1)_{125} = 0.5, (B_2)_{125} = 1, (B_5)_{125} = 0.866$$

$$\sum_{125} B = 2.366$$

$$\triangle 135(A_1A_3A_5) : (B_1)_{135} = 1.5, (B_3)_{135} = 1.732, (B_5)_{135} = -0.866$$

$$\sum_{135} B = 2.366,$$

$$(B_1)_{12345} - (B_1)_{123} = -0.475(B_4)_{12345} - 0.577(B_5)_{12345}, \tag{39}$$

$$(B_2)_{12345} - (B_2)_{123} = 0.53(B_4)_{12345} + 0.577(B_5)_{12345}, \tag{40}$$

$$(B_3)_{12345} - (B_3)_{123} = -1.054(B_4)_{12345} - (B_5)_{12345} \tag{41}$$

and $\sum_{12345} B = 2.366$. The inequalities that correspond to the plasticity of convex pentagon in the $\mathbb{R}^2$ are:

$$0 \leqslant (B_4)_{12345} \leqslant 0.82$$

$$0 \leqslant (B_5)_{12345} \leqslant 0.866$$

$$1 \geqslant (B_1)_{12345} \geqslant 0.11$$

$$0.5 \leqslant (B_2)_{12345} \leqslant 1.434$$

$$0.866 \geqslant (B_3)_{12345} \geqslant 0.$$

The weighted inequalities of the floating case must also be satisfied simultaneously such that the weighted Fermat-Torricelli point A_0 remains an interior point of $A_1A_2A_3A_4A_5$.

By replacing $(B_4)_{12345} = 0.4$, $(B_5)_{12345} = 0.25$ in (39), (40) and (41), we obtain:

$(B_1)_{12345} = 0.666$, $(B_2)_{12345} = 0.856$, $(B_3)_{12345} = 0.194$ (see Figure 3.1)

By replacing $(B_4)_{12345} = 0.5$, $(B_5)_{12345} = 0.3$ in (39), (40) and (41), we obtain:

$(B_1)_{12345} = 0.589$, $(B_2)_{12345} = 0.938$, $B_3 = 0.039$ (see Figure 3.2).

Therefore, an increase of $(B_4)_{12345}$ from $0.4 \to 0.5$ and $(B_5)_{12345}$ from $0.25 \to 0.3$, causes a decrease to $(B_1)_{12345}$ from $0.666 \to 0.589$, $(B_3)_{12345}$ from $0.194 \to 0.039$ and an increase of the value of $(B_2)_{12345}$ from $0.856 \to 0.938$, such that the weighted Fermat-Torricelli point A_0 of the convex pentagon $A_1A_2A_3A_4A_5$ of Figures 3.1 and 3.2 remains invariant (plasticity principle of convex pentagons).

4. A distorted weighted Steiner tree for convex pentagons on a C^2 surface.

Let $A_1A_2A_3A_4A_5$ be a convex pentagon on M whose perimeter is less than $\frac{2\pi}{\sqrt{K_b}}$ for $K > K_b$, and two points A_0 $A_{0'}$ are located at the interior convex domain on M.

Suppose that a positive number B_i(weight) corresponds to each vertex A_i, for $i \in \{0, 0', 1, 2, 3, 4, 5\}$, and $B_0 = B_{0'}$. We denote by a_{ij} the length of the geodesic arc that connects A_i with A_j, and α_{ijk}, the angle that is formulated between the geodesic arcs A_iA_j and A_jA_k, for $i, j, k \in \{0, 0', 1, 2, 3, 4, 5\}$ and $i \neq j \neq k$ (see Figure 4.1).

We assume the existence and uniqueness of a distorted weighted Steiner tree with boundary $A_1A_2A_3A_4A_5$ and we may consider as an open question to find a uniqueness condition for distorted weighted Steiner trees on surfaces in the spirit of Ivanov and Tuzhilin given in [8].

We mention the following theorem which is a generalization of Theorem 1 given in [23].

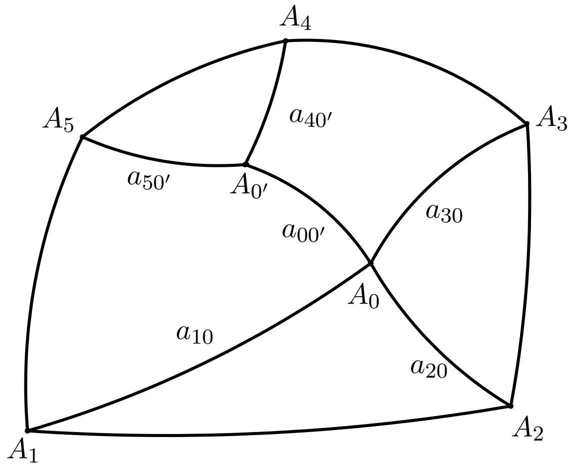

Figure 4.1.

Theorem 4.1. *A weighted distorted Steiner minimal tree of $A_1A_2A_3A_4A_5$ consists of two (weighted) Fermat-Torricelli points A_0 with valency three and A_0' with valency four which are located at the interior convex domain with corresponding weights $B_0 = B_{0'} = B_6$ and minimizes the objective function:*

$$B_1a_{10} + B_2a_{20} + B_3a_{30} + B_4a_{40'} + B_5a_{50'} + B_6a_{00'} = minimum, \tag{42}$$

such that:

$$\left\|B_j\vec{U}_{A_iA_j} + B_k\vec{U}_{A_iA_k}\right\| > B_i, \tag{43}$$

$$\left\|B_m\vec{U}_{A_lA_m} + B_3\vec{U}_{A_lA_n} + B_4\vec{U}_{A_lA_q}\right\| > B_l, \tag{44}$$

for $i, j, k \in \{0, 4, 5\}$, $l, m, n, q \in \{0', 1, 2, 3\}$ and $i \neq j \neq k$, $l \neq m \neq n \neq q$.

Proof of Theorem 4.1. We assume that, the point A_i, the parameterized curve $c(s)$ and the point A_k that belongs in $c(s)$ are given in a neighborhood W_{A_k}, for $k \in \{0, 0'\}$ and the orientation of the angles with respect to A_k taken counterclockwise (see Fig. 4.1). We join the two points A_i and A_k by the geodesic $\gamma_{A_i}(s,t) = \gamma_{A_iA_k}(t)$ where t is a canonical parameter, counting from the point A_i such that for $t = 0$, $\gamma_{A_iA_k}(0) = A_i$ and for $t = 1$, $\gamma_{A_iA_k}(1) = A_k$. The function $a_{ik} = a_{ik}(A_k) = a_{ik}(\gamma_{A_iA_k}(t)) = a_{ik}(\gamma_{A_i}(s,t)) = a_{ik}(s)$ is differentiable with respect to s. We denote by $\alpha_{jki}(s)$ the angle between $\frac{dc}{ds}$ and $\frac{d\gamma_{A_i}}{ds}$ at A_k which is the same angle between the geodesic arcs A_jA_k and A_jA_k, for $i, = 1, 2, 3, 4, 5$ and $i \neq j$.

We generalize the parametrization technique that was used in [17, Proof of Proposition 2, Pages 48–50], [18]:

From [14, Lemma 3.5.1 and Remark 3.5.1], we get:

$$\frac{da_{ik}(s)}{ds} = \cos(\alpha_{jki}(s)). \tag{45}$$

We set the parametrization:

$$a_{10'}(s) = \int_0^s \left\| \frac{d\gamma_{A_1A_{0'}}(t)}{dt} \right\| dt = \int_0^s \left\| \frac{d\gamma_{A_1}(s,t)}{dt} \right\| dt = s,$$

or

$$a_{10'}(s) = a_{10'} = s. \tag{46}$$

The length of the geodesic arcs $a_{40'}$, $a_{00'}$ can be expressed as functions of $a_{50'}$:

$$a_{40'} = a_{40'}(a_{50'}), \qquad a_{00'} = a_{00'}(a_{50'}), \tag{47}$$

From (47) and (42), we have:

$$B_1a_{10} + B_2a_{20} + B_3a_{30} + B_4a_{40'} + B_5a_{50'} + B_6a_{00'} = \text{minimum}. \tag{48}$$

By differentiation of (48) with respect to $a_{50'}$ and taking into account (46), we get

$$B_1\frac{da_{10}}{da_{50'}} + B_2\frac{da_{20}}{da_{50'}} + B_3\frac{da_{10}}{da_{50'}} + B_4\frac{da_{40'}}{da_{50'}} + B_5 + B_6\frac{da_{00'}}{da_{50'}} = 0, \tag{49}$$

or

$$B_4\frac{da_{40'}}{da_{50'}} + B_5 + B_6\frac{da_{00'}}{da_{50'}} = 0, \tag{50}$$

because

$$\frac{da_{10}}{da_{50'}} = 0,$$

$$\frac{da_{20}}{da_{50'}} = 0$$

and

$$\frac{da_{30}}{da_{50'}} = 0.$$

Taking into account that the arc length parameter counts from A_5 to $A_{0'}$ and from (46), (45), we obtain:

$$\frac{da_{40'}}{da_{50'}} = \cos(\alpha_{50'4}(s)), \tag{51}$$

$$\frac{da_{00'}}{da_{50'}} = \cos(\alpha_{00'5}(s)). \tag{52}$$

By replacing (51), (52) in (50), we get:

$$B_0\cos(\alpha_{50'0}(a_{50'})) + B_4\cos(\alpha_{50'4}(a_{50'})) = -B_5. \tag{53}$$

or

$$B_0\cos(\alpha_{50'0}) + B_4\cos(\alpha_{50'4}) = -B_5, \tag{54}$$

because from the assumption on the uniqueness of the distorted weighted Steiner tree of $A_1A_2A_3A_4A_5$, we have:

$$\alpha_{i0'j}(a_{k0'}) = \alpha_{i0'j},$$

for $i, j, k \in \{0, 4, 5\}$, $i \neq j \neq k$.

By following the same process with respect to the parameters $a_{00'}(s') = a_{00'} = s'$ and $a_{40'}(s'') = a_{40'} = s''$ we derive the following two equations, respectively:

$$B_5 \cos(\alpha_{00'5}) + B_4 \cos(\alpha_{00'4}) = -B_0 \tag{55}$$

and

$$B_5 \cos(\alpha_{40'5}) + B_0 \cos(\alpha_{40'0}) = -B_4. \tag{56}$$

By solving (54), (55) and (56) with respect to $\cos\alpha_{50'0}$, $\cos\alpha_{00'4}$ and $\cos\alpha_{50'4}$, we obtain:

$$\cos\alpha_{50'0} = \frac{B_4^2 - B_5^2 - B_0^2}{2B_5B_0}, \tag{57}$$

$$\cos\alpha_{00'4} = \frac{B_5^2 - B_4^2 - B_0}{2B_4B_0}, \tag{58}$$

$$\cos\alpha_{50'4} = \frac{B_0^2 - B_5^2 - B_4^2}{2B_5B_4}. \tag{59}$$

The relations (57), (58) and (59) show that A_0 is the weighted Fermat-Torricelli point of $\triangle A_0A_4A_5$ which minimizes the function $f_1 = B_0a_{00'} + B_4a_{40'} + B_5a_{50'}$.

Similarly, by following the same process with respect to the parameters $a_{10}(s_1) = a_{10} = s_1$, $a_{20}(s_2) = a_{20} = s_2$, $a_{30}(s_3) = a_{30} = s_3$, $a_{0'0}(s_4) = a_{0'0} = s_4$, we derive the following four equations, respectively:

$$B_1 + B_2 \cos(\alpha_{102}(a_{10})) + B_3 \cos(\alpha_{103}(a_{10})) + B_{0'} \cos(\alpha_{100'}(a_{10})) = 0. \tag{60}$$

$$B_1 \cos(\alpha_{201}(a_{20})) + B_2 + B_3 \cos(\alpha_{203}(a_{20})) + B_{0'} \cos(\alpha_{200'}(a_{20})) = 0, \tag{61}$$

$$B_1 \cos(\alpha_{301}(a_{30})) + B_2 \cos(\alpha_{302}(a_{30})) + B_3 + B_{0'} \cos(\alpha_{300'}(a_{30})) = 0. \tag{62}$$

$$B_1 \cos(\alpha_{0'01}(a_{0'0})) + B_2 \cos(\alpha_{0'02}(a_{0'0})) + B_3 \cos(\alpha_{0'03}(a_{0'0})) + B_{0'} = 0. \tag{63}$$

From the assumption on the uniqueness of the distorted weighted Steiner tree of $A_1A_2A_3A_4A_5$, we have:

$$\alpha_{i0j}(a_{k0}) = \alpha_{i0j},$$

for $i, j, k \in \{0', 1, 2, 3\}$, $i \neq j \neq k$.

Therefore, we have:

$$B_1 + B_2 \cos(\alpha_{102}) + B_3 \cos(\alpha_{103}) + B_{0'} \cos(\alpha_{100'}) = 0. \tag{64}$$

$$B_1 \cos(\alpha_{201}) + B_2 + B_3 \cos(\alpha_{203}) + B_{0'} \cos(\alpha_{200'}) = 0, \tag{65}$$

$$B_1 \cos(\alpha_{301}) + B_2 \cos(\alpha_{302}) + B_3 + B_{0'} \cos(\alpha_{300'}) = 0. \tag{66}$$

$$B_1 \cos(\alpha_{0'01}) + B_2 \cos(\alpha_{0'02}) + B_3 \cos(\alpha_{0'03}) + B_{0'} = 0. \tag{67}$$

The equations (64), (65), (66) and (67) show that A_0 is the weighted Fermat-Torricelli point of $A_{0'}A_1A_2A_3$ which minimizes the function $f_2 = B_{0'}a_{00'} + B_1a_{10} + B_2a_{20} + B_3a_{30}$. □

Remark 4.2. From (64), (65), (66) and (67) and taking into account the 4-inverse weighted Fermat-Torricelli problem on M (see [22, Proof of Theorem 1]) we obtain the corresponding equations at a sine form:

$$B_2\sin(\alpha_{102}) + B_3\sin(\alpha_{103}) + B_{0'}\sin(\alpha_{100'}) = 0, \tag{68}$$

$$B_1\sin(\alpha_{201}) + B_3\sin(\alpha_{203}) + B_{0'}\sin(\alpha_{200'}) = 0, \tag{69}$$

$$B_1\sin(\alpha_{301}) + B_2\sin(\alpha_{302}) + B_{0'}\sin(\alpha_{300'}) = 0 \tag{70}$$

and

$$B_1\sin(\alpha_{0'01}) + B_2\sin(\alpha_{0'02}) + B_3\sin(\alpha_{0'03}) = 0. \tag{71}$$

From (64)–(67) and (68)–(71) (see also the proof of Proposition 2.3, $(II) \to (III)$), the following balancing condition is satisfied: $B_1\vec{U}_{A_0A_1} + B_2\vec{U}_{A_0A_2} + B_3\vec{U}_{A_0A_3} + B_4\vec{U}_{A_0A_4} + B_5\vec{U}_{A_0A_5} = \vec{0}$. Thus, from Proposition 2 in [22] (floating case), we derive that A_0 is the corresponding weighted Fermat-Torricelli point of $A_{0'}A_1A_2A_3$.

A weighted full Steiner tree for convex quadrilaterals on M, is deduced from a weighted distorted Steiner tree for convex pentagons, because two weighted Fermat-Torricelli points are located at the interior convex domain.

Proposition 4.3. *A weighted (full) Steiner minimal tree of $A_1A_2A_3A_4$ on M consists of two (weighted) Fermat-Torricelli points A_0, A_0' of valency three which are located at the interior convex domain with corresponding weights $B_0{=}B_{0'}{=}B_{00'}$ and minimizes the objective function:*

$$B_1a_{10'} + B_2a_{20} + B_3a_{30} + B_4a_{40'} + B_{00'}a_{00'} = minimum, \tag{72}$$

such that:

$$\left\|B_j\vec{U}_{A_iA_j} + B_k\vec{U}_{A_iA_k}\right\| > B_i, \tag{73}$$

$$\left\|B_m\vec{U}_{A_lA_m} + B_k\vec{U}_{A_lA_n}\right\| > B_l, \tag{74}$$

for $i,j,k \in \{0,1,4\}$, $l,m,n \in \{0',2,3\}$ and $i \neq j \neq k$, $l \neq m \neq n$, where $\vec{U}_{A_iA_j}$ is a unit tangent vector of the geodesic arc A_iA_j at A_i.

Proof of Proposition 4.3. Consider, a distorted weighted Steiner tree of a convex pentagon $A_1A_2A_3A_4A_5$ on M. By setting $B_1 = 0$, we have two points A_0, $A_{0'}$ which exist at the interior convex domain of valency 3.

By setting $B_{00'} = B_0 = B_{0'}$, the objective function takes the form:

$$B_2a_{20} + B_3a_{30} + B_4a_{40'} + B_5a_{50'} + B_{00'}a_{00'} = \text{minimum}.$$

By changing the index $5 \to 1$, in (54), (55) and (56), we obtain:

$$B_0 \cos(\alpha_{10'0}) + B_4 \cos(\alpha_{10'4}) = -B_1. \tag{75}$$

$$B_1 \cos(\alpha_{10'0}) + B_4 \cos(\alpha_{00'4}) = -B_0 \tag{76}$$

and

$$B_1 \cos(\alpha_{10'4}) + B_0 \cos(\alpha_{00'4}) = -B_4. \tag{77}$$

By solving (75), (76) and (77) with respect to $\cos\alpha_{10'0}$, $\cos\alpha_{00'4}$ and $\cos\alpha_{10'4}$, we get:

$$\cos\alpha_{10'0} = \frac{B_4^2 - B_1^2 - B_0^2}{2B_1B_0}, \tag{78}$$

$$\cos\alpha_{00'4} = \frac{B_1^2 - B_4^2 - B_0^2}{2B_4B_0}, \tag{79}$$

$$\cos\alpha_{10'4} = \frac{B_0^2 - B_1^2 - B_4^2}{2B_1B_4}. \tag{80}$$

From (78), (79) and (80), we derive that A_0 is the weighted Fermat-Torricelli point of $\triangle A_1A_0A_4$ which minimizes the function $f_1 = B_1a_{10'} + B_0a_{00'} + B_4a_{40'}$.

Similarly, by following the same process we obtain:

$$\cos\alpha_{0'03} = \frac{B_2^2 - B_3^2 - B_{0'}^2}{2B_3B_{0'}}, \tag{81}$$

$$\cos\alpha_{0'02} = \frac{B_3^2 - B_{0'}^2 - B_2^2}{2B_{0'}B_2}, \tag{82}$$

$$\cos\alpha_{203} = \frac{B_{0'}^2 - B_2^2 - B_3^2}{2B_2B_3}. \tag{83}$$

From (81), (82) and (83) we derive that A_0 is the weighted Fermat-Torricelli point of $\triangle A_{0'}A_2A_3$ which minimizes the function $f_2 = B_0a_{00'} + B_2a_{20} + B_3a_{30}$. $\square$

Corollary 4.4. *If $B_i = 0.25$, for $i = 1, 2, 3, 4$ and $B_{00'} = 0.5$, then*

$$\alpha_{10'0} = \alpha_{00'4} = \alpha_{10'4} = \alpha_{0'02} = \alpha_{0'03} = \alpha_{203} = 120^\circ$$

(Unweighted full Steiner tree for convex quadrilaterals on M)

5. An inverse distorted weighted Steiner minimal tree for convex pentagons on a C^2 surface

Problem 5.1. *Given two points A_0 and $A_{0'}$ which belongs to the interior of $A_1A_2A_3A_4A_5$ on M, does there exist a unique set of positive weights B_i, B_0, $B_{0'}$ such that*

$$B_1 + B_2 + B_3 + B_{0'} = c = const,$$

$$B_0 = B_{0'},$$

for which A_0 *and* $A_{0'}$ *minimizes*

$$f(A_0, A_{0'}) = B_1 a_{10'} + B_2 a_{20} + B_3 a_{30} + B_4 a_{40} + B_5 a_{50'} + B_0 a_{00'}$$

where a_{ik} *is the length of the geodesic arc (shortest arc) from* A_k *to the vertex* A_i *for* $i = 1, 2, 3, 4, 5$*, and* $k \in \{0, 0'\}$*. This is the 5-inverse weighted distorted Steiner problem on* M*. (Fig. 4.1).*

Theorem 5.2. *The plasticity of the 5 distorted weighted Steiner minimal tree on* M *is given by the following equations:*

$$(B_4)_{045} = \frac{B_0 \sin(\alpha_{50'0})}{\sin(\alpha_{50'4})} \tag{84}$$

$$(B_5)_{045} = \frac{B_0 \sin(\alpha_{00'4})}{\sin(\alpha_{50'0})} \tag{85}$$

$$\left(\frac{B_2}{B_1}\right)_{1230'} = \left(\frac{B_2}{B_1}\right)_{123} \left[1 - \left(\frac{B_0'}{B_1}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{130'}\right], \tag{86}$$

$$\left(\frac{B_3}{B_1}\right)_{1230'} = \left(\frac{B_3}{B_1}\right)_{123} \left[1 - \left(\frac{B_0'}{B_1}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{120'}\right], \tag{87}$$

$$(B_0)_{045} = (B_{0'})_{1230'} \tag{88}$$

and

$$(B_1)_{1230'} + (B_2)_{1230'} + (B_3)_{1230'} + (B_{0'})_{1230'} = c_1 = const, \tag{89}$$

The weight $(B_i)_{1230'}$ *corresponds to the vertex* A_i *that lie on the geodesic arc* $A_0 A_i$*, for* $i \in \{1, 2, 3, 0'\}$ *and the weight* $(B_i)_{ijk}$ *corresponds to the vertex* A_i *that lie on the geodesic arc* $A_0 A_i$ *regarding the triangle* $\triangle A_i A_j A_k$*, for* $i, j, k \in \{1, 2, 3, 0'\}$ *and* $i \neq j \neq k$*.*

Proof of Theorem 5.2. We choose two counterclockwise orientations for angles with respect to the vertices A_0 and $A_{0'}$ on M at the corresponding tangent planes $T_{A_0}(M)$ and $T_{A_{0'}}(M)$.

From the 5 inverse distorted weighted Steiner tree on M and taking into account Theorem 4.1, we obtain the system of equations (with respect to the orientation at $A_{0'}$)

By solving (54) and (55) with respect to $B_4 = (B_4)_{045}$ and $B_5 = (B_5)_{045}$ and by setting $B_0 = (B_0)_{045}$, we obtain (84) and (85).

From the 5 inverse distorted weighted Steiner tree on M and taking into account Theorem 4.1, we obtain the system of equations (64), (65), (66) and (67) (with respect to the orientation at A_0).

By setting $(B_5)_{12345} = 0$ at the plasticity equations of Theorem 3.2 with respect to the 5 inverse weighted Fermat-Torricelli problem ,we derive the plasticity equations with respect to the 4 inverse weighted Fermat-Torricelli problem on M (see also Proposition 4.4 in [19] for surfaces of constant Gaussian curvature).

Then by setting $B_i = (B_i)_{1230'}$ at the plasticity equations of Theorem 3.2 and by changing the index $4 \to 0'$ and by eliminating the index 5, we obtain (86) and (87). From the 5 inverse distorted Steiner tree problem, we get (88) and (89). □

Corollary 5.3. *Set* $\sum_{1230'} B := (B_1)_{1230'}(1 + \frac{B_2}{B_1} + \frac{B_3}{B_1} + \frac{B'_0}{B_1})_{1230'}$. *If* $\sum_{1230'} B = \sum_{123} B = \sum_{120'} B = \sum_{130'} B$, *then*

$$(B_i)_{1230'} = x_i(B'_0)_{1230'} + y_i, \quad i = 1, 2, 3:$$

$$(x_1, y_1) = \left(\frac{\left(\frac{B_1}{B'_0}\right)_{130'} \left(\frac{B_2}{B_1}\right)_{123} + \left(\frac{B_1}{B'_0}\right)_{120'} \left(\frac{B_3}{B_1}\right)_{123} - 1}{1 + \left(\frac{B_2}{B_1}\right)_{123} + \left(\frac{B_3}{B_1}\right)_{123}}, (B_1)_{123} \right),$$

$$(x_2, y_2) = \left(x_1 \left(\frac{B_2}{B_1}\right)_{123} - \left(\frac{B_1}{B'_0}\right)_{130'} \left(\frac{B_2}{B_1}\right)_{123}, (B_2)_{123} \right),$$

$$(x_3, y_3) = \left(x_1 \left(\frac{B_3}{B_1}\right)_{123} - \left(\frac{B_1}{B'_0}\right)_{120'} \left(\frac{B_3}{B_1}\right)_{123}, (B_3)_{123} \right).$$

Proof of Corollary 5.3. By setting $(B_5)_{12345} = 0$ and by changing the index $4 \to 0'$ in Corollary 3.3, we derive the three equations $(B_i)_{1230'}$ for $i = 1, 2, 3$ which depend on $B_{0'}$. □

Remark 5.4. We need 3 angles α_{i0j} and 2 angles $\alpha_{k0'l}$ for $i, j \in \{1, 2, 3, 0'\}$ and $k, l \in \{0, 4, 5\}$ to calculate the plasticity equations of Theorem 5.2 and Corollary 5.3. The other two angles are calculated by the condition:

$$\alpha_{i0j} + \alpha_{j0k} + \alpha_{k0l} + \alpha_{l0m} = 2\pi$$

and

$$\alpha_{p0'q} + \alpha_{q0'r} + \alpha_{r0's} = 2\pi,$$

for $i, j, k, l, m \in \{1, 2, 3, 0'\}$ and $p, q, r, s \in \{0, 4, 5\}$

We mention an important application which connects the plasticity equations derived by the 5-inverse weighted Fermat-Torricelli problem with the plasticity equations derived by the 5-inverse distorted weighted Steiner tree in $\mathbb{R}^2$.

Proposition 5.5. *By performing a parallel translation of* $\triangle A_4 A_{0'} A_5$ *to* $A'_4 A_0 A'_5$, *(see Fig.* 5.1, $A'_4 A_0$ *is parallel to* $A_4 A_{0'}$ *and* $A'_5 A_0$ *is parallel to* $A_5 A_{0'}$*), the following equations are satisfied:*

$$B_1 \sin(2\pi - \alpha_{102}) + B_3 \sin(\alpha_{203}) + B_4 \sin(\alpha_{204'}) + B_5 \sin(2\pi - \alpha_{102} - \alpha_{0'01} + \alpha_{50'0} - \pi) = 0, \tag{90}$$

$$B_2 \sin(\alpha_{102}) + B_3 \sin(\alpha_{103}) + B_4 \sin(\alpha_{103} + \alpha_{300'} + \alpha_{00'4} - \pi) + B_5 \sin(3\pi - \alpha_{0'01} - \alpha_{50'0}) = 0. \tag{91}$$

$$B_1 \sin(\alpha_{301}) + B_2 \sin(\alpha_{302}) + B_4 \sin(\alpha_{300'} + \alpha_{00'4} - \pi) + B_5 \sin(\alpha_{300'} + \alpha_{00'4} - \pi + \alpha_{405}) = 0, \tag{92}$$

$$B_1 \sin(\alpha_{405} + \alpha_{0'01} + \alpha_{50'0} - \pi) + B_2 \sin(\alpha_{405} + \alpha_{0'01} + \alpha_{50'0} - \pi + \alpha_{102}) + B_4 \sin(3\pi - \alpha_{300'} - \alpha_{0'04}) + B_5 \sin(\alpha_{405}) = 0, \tag{93}$$

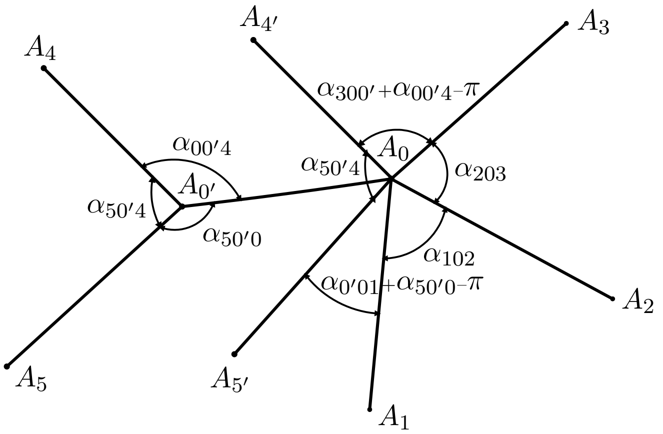

Figure 5.1.

and

$$B_1 \sin(\alpha_{0'01} + \alpha_{50'0} - \pi) + B_2 \sin(\alpha_{0'01} + \alpha_{50'0} - \pi + \alpha_{102}) \\ + B_3 \sin(\alpha_{0'01} + \alpha_{50'0} - \pi + \alpha_{103}) + B_4 \sin(2\pi - \alpha_{405}) = 0. \tag{94}$$

Proof of Proposition 5.5. By performing a parallel translation of$\triangle A_4 A_{0'} A_5$ to $A_4' A_0 A_5'$, the length of the line segment $a_{00'} \to 0$. By setting $a_{00'} = 0$, for $A_0 = A_{0'}$ in (42) we obtain:

$$B_1 a_{10} + B_2 a_{20} + B_3 a_{30} + B_4 a_{4'0'} + B_5 a_{5'0'}$$

or

$$B_1 a_{10} + B_2 a_{20} + B_3 a_{30} + B_4 a_{4'0} + B_5 a_{5'0} = \text{minimum}. \tag{95}$$

By changing the indices $4 \to 4'$ and $5 \to 5'$ in (1, we get (95).

Therefore, by changing the indices $4 \to 4'$ and $5 \to 5'$ in (22), (23), (24), (25) and (26), we get correspondingly:

$$B_1 \sin(\alpha_{201}) + B_3 \sin(\alpha_{203}) \\ + B_4 \sin(\alpha_{203} + \alpha_{300'} + \alpha_{00'4} - \pi) + B_5 \sin(\alpha_{205'}) = 0, \tag{96}$$

$$B_2 \sin(\alpha_{102}) + B_3 \sin(\alpha_{103}) + B_4 \sin(\alpha_{104'}) + B_5 \sin(\alpha_{105'}) = 0. \tag{97}$$

$$B_1 \sin(\alpha_{301}) + B_2 \sin(\alpha_{302}) + B_4 \sin(\alpha_{304'}) + B_5 \sin(\alpha_{305'}) = 0, \tag{98}$$

$$B_1 \sin(\alpha_{4'01}) + B_2 \sin(\alpha_{4'02}) + B_4 \sin(\alpha_{4'03}) + B_5 \sin(\alpha_{4'05'}) = 0, \tag{99}$$

and

$$B_1 \sin(\alpha_{5'01}) + B_2 \sin(\alpha_{5'02}) + B_3 \sin(\alpha_{5'03}) + B_4 \sin(\alpha_{5'04'}) = 0. \tag{100}$$

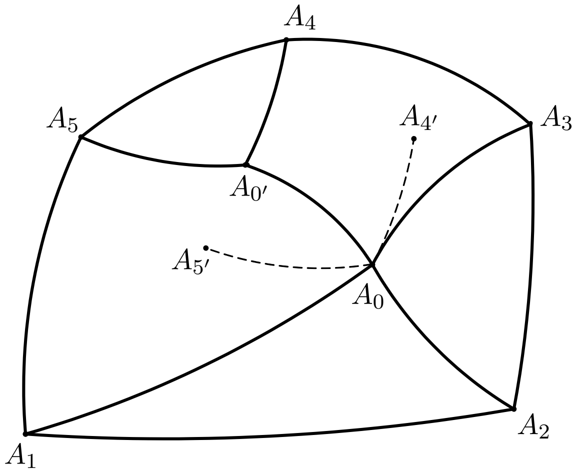

Figure 5.2.

where

$$\alpha_{4'05'} = \alpha_{40'5},$$

$$\alpha_{304'} = \alpha_{300'} + \alpha_{00'4} - \pi,$$

$$\alpha_{5'01} = \alpha_{0'01} + \alpha_{50'0} - \pi,$$

$$\alpha_{3'04'} + \alpha_{2'03'} = \alpha_{400'} + \alpha_{30'0} + \alpha_{20'3} - \pi,$$

$$\alpha_{204'} = \alpha_{203} + \alpha_{300'} + \alpha_{00'4} - \pi,$$

$$\alpha_{205'} = 2\pi - \alpha_{102} - \alpha_{0'01} + \alpha_{50'0} - \pi. \qquad \square$$

5.1. The genesis of a weighted dendrite of order three on a C^2 surface.

We start with the definition of a weighted dendrite of order three on M.

Definition 5.6. A weighted dendrite of order three on M is a distorted weighted Steiner tree of a geodesic triangle which consists of exactly one weighted Fermat-Torricelli point at the interior of the triangle and the degree of plasticity ($Degplasticity \triangle$) is zero.

We mention an evolutionary scheme of a weighted dendrite of order three with respect to the 5-inverse weighted distorted Steiner tree problem on M which we call a creation principle of weighted dendrites of order three on M :

Theorem 5.7. *Consider the 3-inverse weighted Fermat-Torricelli problem for $\triangle A_1A_2A_3$ on M. At time $t = 0$, two geodesic arcs $A_0A_{4'}$ $A_0A_{5'}$ grow simultaneously from A_0 and move via parallel translation along the geodesic arc $A_0A_{0'}$. This evolutionary process creates a weighted dendrite of order three which corresponds to $\triangle A_4A_{0'}A_5$, such that a 5-inverse weighted distorted Steiner tree occurs and the points A_0 and $A_{0'}$ remain the same.*

Proof of Theorem 5.7: From the 3-inverse weighted Fermat-Torricelli problem on M and by taking into account that $A_0A_{4'}$ $A_0A_{5'}$ grow simultaneously from A_0, we obtain the plasticity principle of $A_1A_2A_3A_{4'}A_{5'}$ by changing the indices $4 \to 4'$ and $5 \to 5'$ (see Theorem 3.4) which states that:

Given five geodesic (shortest) arcs which meet at the weighted Fermat-Torricelli point A_0 and their endpoints form a convex pentagon on M and the weighted Fermat-Torricelli point A_0 belongs to the interior of this convex pentagon, an increase of the weight that corresponds to the fourth and fifth shortest arc causes a decrease to the two weights that correspond to the first and third shortest arc and an increase to the weight that corresponds to the second shortest arc.

Therefore, an increase of $B_{4'}$ and $B_{5'}$ creates a decrease of B_1 and B_3 and an increase in B_2, such that the corresponding weighted Fermat-Torricelli point A_0 remains the same.

Let $\vec{U}_{A_0A_i}$ be the unit tangent vector of the geodesic arc A_0A_i at A_0 for $i \in \{0', 1, 2, 3, 4, 5\}$.

We have:

$$B_{0'}\vec{U}_{A_0A_{0'}} = B_{4'}\vec{U}_{A_0A_{4'}} + B_{5'}\vec{U}_{A_0A_{5'}}. \tag{101}$$

(101) yields a plasticity principle of the convex quadrilateral $A_1A_2A_3A_{0'}$ on M which states that: an increase of $B_{0'}$ creates a decrease of B_1 and B_3 and an increase in B_2, such that the corresponding weighted Fermat-Torricelli point A_0 remains the same.

By moving via parallel translation along the geodesic arc $A_0A_{0'}$ (see Section 3.3, 3.4.4 and Example 3.4.4 in [14] pages 153–161) the geodesic curves $A_0A_{4'} \to A_0A_4$ and $A_0A_{5'} \to A_0A_5$ (see Figure 5.2), the plasticity equations of the 5-inverse distorted Steiner tree (see Theorem 5.2) are satisfied by assuming that $B_0 = B_{0'}$. By setting $B_4 = B_{4'}$ and $B_5 = B_{5'}$, (84)–(89) are satisfied and the angles α_{i0j} and α_{k0l} remain the same for $i, j \in \{1, 2, 3, 0'\}$ and $k, l \in \{0, 4', 5'\}$. Furthermore, (84) and (85) shows that an increase in $B_0 = B_{0'}$ creates an increase in $B_{4'}$ and $B_{5'}$.

Thus, we derive the consistency of the plasticity of the 5-inverse weighted distorted Steiner tree which corresponds to $A_1A_2A_3A_4A_5$ on M with the plasticity principle of $A_1A_2A_3A_{4'}A_{5'}$. $\square$

5.2. The creation of a weighted leaf tree type on a C^2 surface M.

We start with the notion of generalized plasticity for convex pentagons on M whose perimeter is less than $\frac{2\pi}{\sqrt{K_b}}$ for $K > K_b$, which has been introduced in [20, Proposition 4.1] for pentahedra in the three dimensional Euclidean Space.

Proposition 5.8. *Let $A_1A_2A_3A_4A_5$ be a convex pentagon on M with non-negative weights B_i that correspond to each vertex A_i, respectively, which satisfy the weighted inequalities of the floating case (see Proposition 2.3) and A_0 is the corresponding weighted Fermat-Torricelli point. Assume that every non-negative weight B_i is* *split into n_i non-negative weights B_{ik} :*

$$\sum_{k=1}^{n_i} B_{ik} = B_i,$$

for $i = 1, 2, 3, 4, 5$. The weight $B_{i,k}$ corresponds to every vertex $A_{i,k}$ which belongs to the line segment A_0A_i, for every $k \neq n_i$ and the weight B_{i,n_i} corresponds to the vertex $A_i = A_{i,n_i}$. Then the weighted Fermat-Torricelli point of $\{A_{i,k}\}$ coincides with the generalized Fermat-Torricelli point of $\{A_1A_2A_3A_4A_5\}$, for $i = 1, 2, 3, 4, 5$.

Proof of Proposition 5.8. Let $g(X)$ be the objective function

$$g(X) = \sum_{i=1}^{5} \sum_{k=1}^{n_i} B_{i,k} a_{ik,x} \tag{102}$$

where $a_{ik,x}$ is the length of a geodesic arc from A_{ik} to X on M. We prove that the minimum of $g(X)$ is attained at $X = A_0$.

We recall from Proposition 2.3 the following two conditions are equivalent:

(II) The weighted Fermat-Torricelli point A_0 is an interior point of $A_1A_2A_3A_4A_5$ and does not belong to the geodesic arcs $A_{i,k}A_{j,m}$, for $i, j = 1, 2, 3, 4, 5$ $i \neq j$.

(III) $B_1\vec{U}_{A_0A_1} + B_2\vec{U}_{A_0A_2} + B_3\vec{U}_{A_0A_3} + B_4\vec{U}_{A_0A_4} + B_5\vec{U}_{A_0A_5} = \vec{0}$.

By following the same process given in the proof from $(II) \to (III)$ at Proposition 2.3, we obtain:

$$\operatorname{grad}(g(x)) = \sum_{i=1}^{5} \sum_{k=1}^{n_i} B_{i,k} \vec{U}_{X,A_{i,k}}. \tag{103}$$

By applying Proposition 2.3, we derive the following two equivalent conditions:

(II) The weighted Fermat-Torricelli point X is an interior point of $\{A_{i,k}\}$ and does not belong to the geodesic arcs A_1A_2, A_2A_3, A_3A_4, A_4A_5 and A_5A_1.

(III) $\sum_{i=1}^{5} \sum_{k=1}^{n_i} B_{i,k} \vec{U}_{X,A_{i,k}} = \vec{0}$.

By replacing $X = A_0$ in (103) we have:

$$\sum_{i=1}^{5} \sum_{k=1}^{n_i} B_{i,k} \vec{U}_{A_0,A_{i,k}} = \sum_{i=1}^{5} B_i \vec{U}_{A_0,A_i} = 0.$$

This result follows from the parallel translation of the unit vectors $\vec{U}_{A_0,A_{i,k}}$ along the geodesic curve A_0A_i to A_0, the uniqueness of the weighted Fermat-Torricelli point A_0 of $\{A_1A_2A_3A_4A_5\}$, which is derived by the strict convexity of the length of a geodesic arc on M(see Proposition 2.2, [17, Proposition 1], [18]). $\square$

From Proposition 5.8 and Theorem 3.2, we obtain the generalized plasticity equations for convex pentagons on M.

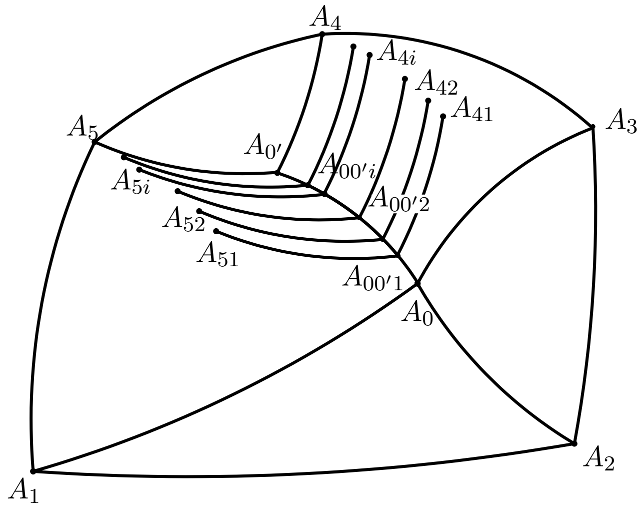

Figure 5.3.

Corollary 5.9.

$$\left(\frac{\sum_{k=1}^{n_2} B_{2k}}{\sum_{k=1}^{n_1} B_{1k}}\right)_{12345} = \left(\frac{B_2}{B_1}\right)_{123}\left[1-\left(\frac{\sum_{k=1}^{n_4} B_{4k}}{\sum_{k=1}^{n_1} B_{1k}}\right)_{12345}\left(\frac{B_1}{B_4}\right)_{134} - \left(\frac{\sum_{k=1}^{n_5} B_{5k}}{\sum_{k=1}^{n_1} B_{1k}}\right)_{12345}\left(\frac{B_1}{B_5}\right)_{135}\right]$$

$$\left(\frac{\sum_{k=1}^{n_3} B_{3k}}{\sum_{k=1}^{n_1} B_{1k}}\right)_{12345} = \left(\frac{B_3}{B_1}\right)_{123}\left[1-\left(\frac{\sum_{k=1}^{n_4} B_{4k}}{\sum_{k=1}^{n_1} B_{1k}}\right)_{12345}\left(\frac{B_1}{B_4}\right)_{124} - \left(\frac{\sum_{k=1}^{n_5} B_{5k}}{\sum_{k=1}^{n_1} B_{1k}}\right)_{12345}\left(\frac{B_1}{B_5}\right)_{125}\right]$$

and

$$\sum_{i=1}^{5}(B_i)_{12345} = c = const. \tag{104}$$

The plasticity of a weighted distorted Steiner tree with respect to the 5-inverse weighted distorted Steiner tree on M yield a generalized plasticity of a weighted leaf tree type by splitting $A_{0'}$ to $A_{00'i}$ with corresponding weights $B_{00'i}$ such that $\sum_{i=1}^{n} B_{00'i} = B_{0'} = B_0$ and the geodesic arcs $A_{4i}A_{00}$ are derived via parallel translation of $A_{4i}A_{00}$ and $A_{4i}A_{00}$ along the geodesic curve $A_0A_{0'}$ (see Figure 5.3).

Let $S_{A_1A_2A_3A_0A_4A_{0'}A_5}$ be a weighted distorted Steiner minimal tree of $A_1A_2A_3A_4A_5$ which consists of two (weighted) Fermat-Torricelli points A_0 with valency three and A_0' with valency four which are located at the interior convex domain with corresponding weights $B_0 = B_{0'}$.

Proposition 5.10. *By moving along the geodesic arc $A_0A_{0'}$ via parallel translation of geodesic arcs $A_4A_{0'} \to A_{4i}A_{00'i}$ and $A_5A_{0'} \to A_{5i}A_{00'i}$, for $i = 1, 2, 3, .., n$, we obtain a weighted Steiner leaf tree type on M (see Figure 5.3) which minimizes* *the objective function*

$$B_1 a_{10} + B_2 a_{20} + B_3 a_{30} + \sum_{i=1}^{n} B_{4i} a_{(4i)(00'i)} + \sum_{i=1}^{n} B_{5i} a_{(5i)(00'i)} + \sum_{i=1}^{n} B_{00'i} a_{0(00'i)} = minimum, \tag{105}$$

such that:

$$\left\| B_j \vec{U}_{A_i A_j} + B_k \vec{U}_{A_i A_k} \right\| > B_i, \tag{106}$$

$$\left\| B_m \vec{U}_{A_l A_m} + B_3 \vec{U}_{A_l A_n} + B_4 \vec{U}_{A_l A_q} \right\| > B_l, \tag{107}$$

for $i, j, k \in \{0r, 4r, 5r\}$, *and* $r = 1, 2, 3, .., n$, $l, m, n, q \in \{0', 1, 2, 3\}$ *and* $i \neq j \neq k$, $l \neq m \neq n \neq q$,

$$\sum_{q=1}^{n} \lambda_q = 1, \tag{108}$$

$$\sum_{i=1}^{n} B_{00'i} = B_0 = B_{0'} \tag{109}$$

$$\sum_{i=1}^{n} B_{4i} = B_4 \tag{110}$$

$$\sum_{i=1}^{n} B_{5i} = B_5 \tag{111}$$

A_0, $A_{0'}$ *remains the same with* $S_{A_1 A_2 A_3 A_0 A_4 A_{0'} A_5}$ *where* $A_{00'i}$ *belongs to the geodesic arc* $A_0 A_{0'}$ *with corresponding weight* $B_{00'i} = \lambda_i B_0$ *and the vertex* $A_j A_{00'i}$ *has a corresponding weight* $B_{ji} = \lambda_i B_j$ *for* $j = 4, 5$ *and* $i = 1, 2, 3, .., n$.

Proof of Proposition 5.10. From (108)–(111) and (105) and the parallel translation of geodesic arcs $A_4 A_{0'} \to A_{4i} A_{00'i}$ ($a_{40'} = a_{(4i)(00'i)}$) and $A_5 A_{0'} \to A_{5i} A_{00'i}$, ($a_{50'} = a_{(5i)(00'i)}$), we get:

$$\begin{aligned} & B_1 a_{10} + B_2 a_{20} + B_3 a_{30} + \sum_{i=1}^{n} B_{4i} a_{(4i)(00'i)} + \sum_{i=1}^{n} B_{5i} a_{(5i)(00'i)} + \sum_{i=1}^{n} B_{00'i} a_{0(00'i)} \\ &= B_1 a_{10} + B_2 a_{20} + B_3 a_{30} + \sum_{i=1}^{n} \lambda_i (B_4 a_{(4i)(00'i)} + B_5 a_{(5i)(00'i)} + B_0 a_{0(00'i)}) \\ &= \text{minimum}. \end{aligned}$$

By following the same parametrization technique that we used in Theorem 4.1, we derive that $A_{00'i}$ is the corresponding weighted Fermat-Torricelli point of

$\triangle A_{4i}A_0A_{5i}$, for $i = 1, 2, 3, ..., n$, where B_{ji} is the corresponding weight of the vertex A_{ji}, for $j = 4, 5$ and $B_{00'i} = \lambda_i B_0 = \lambda_i B_0$ is the corresponding weight of the vertex $A_{00'i}$.

Similarly, by applying the parametrization technique of Theorem 4.1 and by setting $B_5 = 0$ in Proposition 5.8, we derive that A_0 is the weighted Fermat-Torricelli point of $\{A_1A_2A_3\{A_{00'i}\}\}$ for $i = 1, 2, 3, .., n$, or

A_0 is the weighted Fermat-Torricelli point of $A_1A_2A_3A_{0'}\}$ because $\sum_{i=1}^{n} B_{00'i} = \sum_{i=1}^{n} \lambda_i B_{0'} = B_{0'}$. □

From Proposition 5.10 and by setting in Theorem 5.2 $B_{00'} = \lambda_i B_0$, we obtain the following plasticity equations of a weighted Steiner leaf tree type on M :

Proposition 5.11. *The plasticity of the weighted Steiner leaf tree type on M (see Figure* 5.3*) is given by the following equations:*

$$(B_{4i})_{045} = \frac{\lambda_i B_0 \sin(\alpha_{50'0})}{\sin(\alpha_{50'4})} \tag{112}$$

$$(B_{5i})_{045} = \frac{\lambda_i B_0 \sin(\alpha_{00'4})}{\sin(\alpha_{50'0})} \tag{113}$$

$$\left(\frac{B_2}{B_1}\right)_{1230'} = \left(\frac{B_2}{B_1}\right)_{123} \left[1 - \left(\frac{B_0'}{B_1}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{130'}\right], \tag{114}$$

$$\left(\frac{B_3}{B_1}\right)_{1230'} = \left(\frac{B_3}{B_1}\right)_{123} \left[1 - \left(\frac{B_0'}{B_1}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{120'}\right], \tag{115}$$

$$(B_0)_{045} = (B_{0'})_{1230'} \tag{116}$$

and

$$(B_1)_{1230'} + (B_2)_{1230'} + (B_3)_{1230'} + (B_{0'})_{1230'} = c_1 = const, \tag{117}$$

The weight $(B_i)_{1230'}$ corresponds to the vertex A_i that lie on the geodesic arc A_0A_i, for $i \in \{1, 2, 3, 0'\}$ and the weight $(B_i)_{ijk}$ corresponds to the vertex A_i that lie on the geodesic arc A_0A_i regarding the triangle $\triangle A_iA_jA_k$, for $i, j, k \in \{1, 2, 3, 0'\}$ and $i \neq j \neq k$.

Corollary 5.12. *By splitting the knot A_i for $i = 1, 2, 3$ to $A_{i,k}$ for $k = 1, 2, .., n_i$, such that $A_{i,k}$ belongs to the geodesic arc A_iA_0 with weights $B_{i,k} : \sum_{k=1}^{n_i} B_{i,k} = B_i$, we obtain the generalized plasticity equations of a weighted Steiner leaf tree type* *on* M :

$$(B_{4i})_{045} = \frac{\lambda_i B_0 \sin(\alpha_{50'0})}{\sin(\alpha_{50'4})} \tag{118}$$

$$(B_{5i})_{045} = \frac{\lambda_i B_0 \sin(\alpha_{00'4})}{\sin(\alpha_{50'0})} \tag{119}$$

$$\left(\frac{\sum_{k=1}^{n_2} B_{2,k}}{\sum_{k=1}^{n_1} B_{1,k}}\right)_{1230'} = \left(\frac{B_2}{B_1}\right)_{123} \left[1 - \left(\frac{\sum_{i=1}^{n} \lambda_i B_{0'}}{\sum_{k=1}^{n_1} B_{1,k}}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{130'}\right], \tag{120}$$

$$\left(\frac{\sum_{k=1}^{n_3} B_{3,k}}{\sum_{k=1}^{n_1} B_{1,k}}\right)_{1230'} = \left(\frac{B_3}{B_1}\right)_{123} \left[1 - \left(\frac{\sum_{i=1}^{n} \lambda_i B_{0'}}{\sum_{k=1}^{n_1} B_{1,k}}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{120'}\right], \tag{121}$$

$$(B_0)_{045} = (B_{0'})_{1230'} \tag{122}$$

and

$$B_1 + B_2 + B_3 + B_{0'} = c_1 = const, \tag{123}$$

Proof of Corollary 5.12. By setting $B_5 = 0$ and changing the index $4 \to 0'$ in the generalized plasticity equations Corollary 5.9, we obtain the generalized plasticity equations with respect to a 4-inverse weighted Fermat-Torricelli problem. Then by using the plasticity equations taken from Proposition 5.11, we derive (118)–(123). $\square$

Remark 5.13. By extending the plasticity of weighted Steiner trees from convex pentagons to polygons, we may derive various types of evolutionary structures of convex polygons on a C^2 complete surface and we may ask for the optimal bounds with respect to the evolution of the corresponding weighted Steiner Ratio which is the ratio of the length of a weighted Steiner tree with respect to a full weighted Steiner minimal tree. These optimal bounds will generalize the optimal bounds of the classical Steiner ratio conjectured by Gilbert and Pollak.

We may consider as a future work to prove the uniqueness of weighted distorted Steiner minimal trees for convex polygons on a surface M.

Finally, we may conclude with one question taking into consideration P. L. Maupertois mathematical inspiration which is connected with the discovery of the least action principle (see in [11]):

Do existing evolutionary structures behave randomly? How could we specify a degree of randomness such that existing evolutionary structures behave?

A. Appendix

We proceed by giving the definition of an exponential mapping and two results regarding the exponential mapping and the normal coordinate system (see in [1], pages 117–119, [17], [18]).

Let $A_0 \in M$, (U, ϕ) a local chart at A_0, such that $\varphi(A_0) = 0$ which is connected with a normal coordinate system $\{xi^k\}$ and we denote by $\tilde{X}_{A_0 A_i}$ a tangent vector in $T_{A_0}(M)$, for $k = 1, 2$.

Let $\gamma_{A_0}(t) = \gamma_{A_0}(t, \tilde{X}_{A_0A_i})$ be the geodesic arc from A_0 to A_i such that: $\gamma_{A_0}(0, \tilde{X}_{A_0A_i}) = A_0$, $\gamma_{A_0}(1, \tilde{X}_{A_i}) = A_i$ and $\gamma_{A_0}(t, \tilde{X}_{A_i}) = \gamma_{A_0}(1, t\tilde{X}_{A_0A_i})$.

We mention the following two results given in [1], Theorem 5.11 and Corollary 5.12, page 118, respectively:

1. The exponential mapping $\exp_{A_0}(X_{A_0A_i})$, defined by

$$\mathbb{R}^2 \supset \theta \ni X_{A_0A_i} \to \gamma_{A_0}(1, \tilde{X}_{A_0A_i}) \epsilon M,$$

is a diffeomorphism of θ (a neighborhood of zero, where the mapping is defined) onto a neighborhood Ω of A_0, where $\Omega = \exp_{A_0}(\theta)$. By definition, $\exp_{A_0}(0) = A_0$ and the identification of $\mathbb{R}^2$ with $T_{A_0}M$ is made by means of $(\phi_\star)_{A_0}$:

$$\tilde{X}_{A_0} = (\phi_\star)^{-1}_{A_0} X_{A_0}.$$

2. There exists a neighborhood Ω of A_0 such that every point $, Q_{A_i} \in \Omega$ can be joined to A_0 by a unique geodesic entirely included in Ω, where

$$Q_{A_i} = \gamma_{A_0}(t, \tilde{X}_{A_0A_i}) = \gamma_{A_0}(1, t\tilde{X}_{A_0A_i}) = \exp_{A_0} tX_{A_0A_i} = \{txi^k\}.$$

$(\Omega, \exp^{-1}_{A_0})$ is a local chart and the corresponding coordinate system $\{xi^k\}$ is called a normal geodesic coordinate system.

Let $\gamma_{A_0}(t) = \gamma^k_{A_0}(t)$ be the geodesic from A_0 to A_i lying in Ω.

From the equality

$$\gamma_{A_0}(1, t\tilde{X}_{A_0A_i}) = \gamma_{A_0}(t, \tilde{X}_{A_0A_i}),$$

we obtain that $\gamma^k_{A_0}(t) = txi^k$, where xi^1, xi^2 are the components of $\tilde{X}_{A_0A_i}$.

The arc length $s = \left\|\tilde{X}_{A_0A_i}\right\| t = \|X_{A_0A_i}\| t$.

For $t = 1$, we derive that $s = a_{0i}$. Thus, we have:

$$a_{0i} = \sqrt{g_{kl}(A_i) xi^k xi^l} = \sqrt{(xi^1)^2 + (xi^2)^2} = \|X_{A_0A_i}\|.$$

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