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Existence and Multiplicity of Periodic Solutions for Second Order Hamiltonian Systems Depending on a Parameter

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The existence of at least one nontrivial periodic solution for a class of second order Hamiltonian systems depending on a parameter is obtained, under an algebraic condition on the nonlinearity G and without requiring any asymptotic behavior neither at zero nor at infinity. The existence is still deduced in the particular case when G is subquadratic at zero. Finally, two multiplicity results are proved if G, in addition, is required to fulfill some different Ambrosetti-Rabinowitz type superquadratic conditions at infinity. The approach is fully variational.

Keywords: Second order Hamiltonian systems, periodic solutions, critical points

1. Introduction

This paper is devoted to the study of the existence and the multiplicity of periodic solutions of a differential problem that is included in the following class of second order Hamiltonian system

$$-\ddot{u} = \nabla F(t, u) \quad \text{a.e. in } [0, T], \tag{1}$$

where $T > 0$, $N \geq 1$ is an integer and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a function smooth enough.

There is a large number of authors that treated problem (1), see for instance [19–27]. In particular, in [7], via min-max methods, the existence of at least two nontrivial periodic solutions is proved when $F(t, \cdot)$, in addition to a coercive condition, satisfies at zero a suitable behavior that, indeed, implies that zero is a local maximum, namely

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There exist $p \in \mathbb{N}$ and $r > 0$ such that, for every $|\xi| \le r$ and a.e. in $[0,T]$, put $\omega = 2\pi/T$,

$$-\frac{1}{2}(p+1)\omega^2|\xi|^2 \le F(t,\xi) - F(t,0) \le -\frac{1}{2}p\,\omega^2|\xi|^2. \tag{2}$$

Some other more general coercive conditions are exploited in [14, 20, 21, 26, 27] in order to still assure meaningful existence results. While, the multiplicity of periodic solutions is obtained again in [20, 21, 26] requiring (2).

On the other hand, starting from the Rabinowitz's paper ([17]), the well known Ambrosetti-Rabinowitz condition, namely

There exist $\mu > 2$, $r > 0$ such that, for every $|\xi| \ge r$ and for every $t \in [0,T]$

$$0 < \mu F(t,\xi) \le \nabla F(t,\xi)\cdot\xi, \tag{3}$$

has been fruitfully exploited in order to establish the existence of at least one nontrivial and non constant periodic solution (among the others, see also [9, 12, 22]). In particular, different types of superquadratic conditions, that are inspired from (3), play a crucial role in proving the existence theorems contained in [11, 19, 24, 25, 29].

Our aim is to consider problem (1) when

$$F(t,\xi) = \frac{1}{2}A(t)\xi\cdot\xi - \lambda b(t)G(\xi),$$

where $A : [0,T] \to \mathbb{R}^{N\times N}$ ($N \ge 1$) is a suitable matrix-valued function, $b \in L^1([0,T],\mathbb{R})$, $G \in C^1(\mathbb{R}^N)$ and λ is a positive real parameter, so that (1) reduces to

$$\begin{cases} -\ddot{u} + A(t)u = \lambda b(t)\nabla G(u) \text{ a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0. \end{cases} \tag{P$_\lambda$}$$

More precisely, making use of an abstract critical point theorem obtained in [6], in Section 3, we first assure the existence of a precise interval of parameters λ for which problem (P$_\lambda$) admits at least one nontrivial periodic solution (see Theorems 3.1 and 3.3), without any asymptotic condition, neither at zero nor at infinity, on G. As a special case, an existence result is obtained simply requiring a subquadratic behavior at zero of the nonlinearity G (see Theorem 3.6). In Section 4, two different multiplicity theorems are proved (Theorems 4.1 and 4.5) when coercivity assumptions as well as (2) are avoided, while some conditions of Ambosetti-Rabinowitz type are required (see assumptions (39) and (46)). Finally, some examples are given. Precisely, an example (Example 3.9) where the nonlinearity has a completely arbitrary behavior at infinity and the corresponding problem, owing to Theorem 3.6, admits at least one nontrivial solution, is pointed out. Moreover, some examples (Examples 4.3, 4.4, 4.6) where the nonlinearity is superquadratic at infinity are presented, so that the corresponding problems, owing to Theorems 4.1 and 4.5, admit at least two nontrivial solutions, while the results of the literature cannot be applied (see Remarks 4.2 and 4.7).

For the notation adopted we refer to [15] that represents an excellent monograph on this topic. Further papers where the multiplicity of solutions to problem (P_λ) is studied, are [3, 4, 10].

2. Preliminaries

Let H^1_T the Sobolev space of functions $u \in L^2([0,T],\mathbb{R}^N)$ having a weak derivative $\dot{u} \in L^2([0,T],\mathbb{R}^N)$. Let us recall that, in this framework, the weak derivative of $u \in L^1([0,T],\mathbb{R}^N)$ is a function $\dot{u} \in L^1([0,T],\mathbb{R}^N)$ such that

$$\int_0^T u(t)\cdot\dot{v}(t)\,dt = -\int_0^T \dot{u}(t)\cdot v(t)\,dt, \tag{4}$$

for every $v \in C^\infty_T$, being C^∞_T the space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ into $\mathbb{R}^N$. Moreover, whenever $u \in L^1([0,T],\mathbb{R}^N)$ admits a weak derivative $\dot{u}$ one has that

$$u(t) = \int_0^t \dot{u}(s)\,ds + c \quad \text{and} \quad u(0) = u(T). \tag{5}$$

It is easy to verify that, if we consider H^1_T endowed with the usual norm

$$\|u\|_{H^1_T} := \left(\int_0^T |u(t)|^2\,dt + \int_0^T |\dot{u}(t)|^2\,dt\right)^{1/2}, \tag{6}$$

then H^1_T is a reflexive Banach space which is compactly embedded in $C^0([0,T],\mathbb{R}^N)$ and $C^\infty_T \subset H^1_T$. Moreover, it is well known that

$$H^1_T = X^+ \oplus X^-, \tag{7}$$

where X^+ is the subspace of function with mean value zero in H^1_T and X^- is the subspace of constant functions in H^1_T (for more details see [15, Chapter 1] and [1]). Let $A : [0,T] \to \mathbb{R}^{N\times N}$ be a matrix-valued function such that

(A)$_1$ *$A(t) = (a_{ij}(t))$ is a symmetric matrix with $a_{ij} \in L^\infty([0,T])$ for every $t \in [0,T]$.*

(A)$_2$ *There exists a positive constant ν such that*

$$A(t)\xi\cdot\xi \geq \nu|\xi|^2$$

for every $\xi \in \mathbb{R}^N$ and a.e. in $[0,T]$.

A simple computation shows that

$$\nu|\xi|^2 \leq A(t)\xi\cdot\xi \leq \sum_{i,j=1}^N \|a_{ij}\|_\infty\,|\xi|^2, \tag{8}$$

for every $t \in [0,T]$ and $\xi \in \mathbb{R}^N$. Hence, if we put

$$\langle u, v\rangle := \int_0^T A(t)u(t)\cdot v(t)\,dt + \int_0^T \dot{u}(t)\cdot\dot{v}(t)\,dt, \tag{9}$$

for every $u, v \in H^1_T$, in view of $(\mathrm{A})_1$, $\langle \cdot, \cdot \rangle$ is an inner product on H^1_T inducing the following norm

$$\|u\| := \left(\int_0^T A(t)u(t) \cdot u(t)\ dt + \int_0^T |\dot{u}(t)|^2\ dt \right)^{1/2}. \tag{10}$$

Obviously, by (8), we conclude

$$\sqrt{m}\|u\|_{H^1_T} \le \|u\| \le \sqrt{M}\|u\|_{H^1_T}, \tag{11}$$

where $m = \min\{1, \nu\}$ and $M = \max\left\{1, \sum_{i,j=1}^N \|a_{ij}\|_\infty\right\}$, namely $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^1_T}$. Hence, $(H^1_T, \|\cdot\|)$ is still compactly embedded in $C^0([0,T], \mathbb{R}^N)$ and, if we put

$$k := \sqrt{\frac{2}{m}} \max\left\{\sqrt{T}, \frac{1}{\sqrt{T}}\right\}, \tag{12}$$

it can be shown that

$$\bar{k} := \sup_{u \in H^1_T \setminus \{0\}} \frac{\|u\|_{C^0}}{\|u\|} \le k. \tag{13}$$

An esaustive computation of (13) is given in [10].

In order to point out the variational structure of problem (P_λ), we introduce the following two functionals $\Phi, \Psi : H^1_T \to \mathbb{R}$ by putting

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_0^T b(t)G(u(t))\ dt, \tag{14}$$

for every $u \in H^1_T$. It is easy to verify that Φ is a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional, being

$$\Phi'(u)(v) = \int_0^T \dot{u}(t) \cdot \dot{v}(t)\ dt + \int_0^T A(t)u(t) \cdot v(t)\ dt, \tag{15}$$

for every $u, v \in H^1_T$. Moreover, from (13) we can derive that for every $r > 0$ one has that

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{ u \in C^0([0,T], \mathbb{R}^N) : \ \|u\|_{C^0} \le k\sqrt{2r} \right\}. \tag{16}$$

Since $H^1_T \hookrightarrow C^0$ compactly, Ψ turns out well-defined, sequentially weakly semicontinuous and continuously Gâteaux differentiable, with

$$\Psi'(u)(v) = \int_0^T b(t)\nabla G(u(t)) \cdot v(t)\ dt, \tag{17}$$

for every $u, v \in H^1_T$, being Ψ' a compact operator. By (16) we can observe that, for every $r > 0$

$$\sup_{u \in \Phi^{-1}]-\infty, r]} \Psi(u) \le \|b\|_1 \max_{|\xi| \le k\sqrt{2r}} G(\xi). \tag{18}$$

At this point, if we recall that a solution of problem (P_λ) is any function $u_0 \in C^1([0,T],\mathbb{R}^N)$ such that $\dot{u}_0$ is absolutely continuous

$$-\ddot{u}_0(t) + A(t)u_0 = \lambda b(t)\nabla G(u_0) \quad \text{a.e. in } [0,T], \tag{19}$$

and

$$u_0(T) - u_0(0) = \dot{u}_0(T) - \dot{u}_0(0) = 0, \tag{20}$$

we can claim that

$$\text{a critical point of the functional } I_\lambda := \Phi - \lambda\Psi \text{ is a solution of } (\mathrm{P}_\lambda). \tag{21}$$

In fact, if $u_0 \in H^1_T$ is such that $I'_\lambda(u_0) = 0$, taking in mind (15) and (17), one has

$$\int_0^T \dot{u}_0(t)\cdot\dot{v}(t)\ dt = \int_0^T -\left[A(t)u_0(t) - \lambda b(t)\nabla G(u_0(t))\right]\cdot v(t)\ dt$$

for every $v \in H^1_T$, namely, in view of (4), $\dot{u}_0$ admits a weak derivative $\ddot{u}_0$ satisfying (19). Moreover, since $C^\infty_T \subset H^1_T$ and reading (5) with u_0 or $\dot{u}_0$ in place of u, we conclude that (20) holds true.

In order to assure the multiplicity results, we will use the notion of Palais-Smale condition and as well of the more general Cerami condition, namely:

If X is a Banach space and $I : X \to \mathbb{R}$ is a Gâteaux differentiable function, we say that I satisfies the Palais-Smale condition (briefly (PS)) if

(PS) every sequence $\{x_n\}$ in X such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in X^* admits a strongly convergent subsequence.

Moreover, we say that I satisfies the Cerami condition (briefly (C)) if

$(\mathrm{C})_1$ every bounded sequence $\{x_n\}$ in X such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in X^* admits a strongly convergent subsequence;

$(\mathrm{C})_2$ for every $d \in \mathbb{R}$ there exist constants $\delta,\ R,\ \gamma > 0$ such that, for every $x \in I^{-1}([d-\delta, d+\delta])$ with $\|x\| \geq R$ one has

$$\|I'(x)\|_{X^*}\|x\| \geq \gamma.$$

Finally, we recall an abstract critical point theorem obtained in [6], that is inspired by the Ricceri Variational Principle (see [18]) and which represents our main tool in order to assure the existence of at least one nontrivial solution to problem (P_λ).

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2), \tag{22}$$

where

$$\beta(r_1, r_2) := \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}.$$

Then, for each $\lambda \in \left]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$ *there is* $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ *such that* $I_\lambda(u_{0,\lambda}) \leq I(u)$ *for all* $u \in \Phi^{-1}(]r_1, r_2[)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$.

Some others papers related to Theorem 2.1 are [2, 5].

3. Existence results

Here is our first main result.

Theorem 3.1. *Let* $A : [0, T] \to \mathbb{R}^{N\times N}$ *be satisfying* $(\mathrm{A})_1 - (\mathrm{A})_2$, $G \in C^1(\mathbb{R}^N)$ *and assume that there exist* $c_1, c_2 \in \mathbb{R}$, *with* $0 \leq c_1 < c_2$, *and* $\bar{\xi} \in \mathbb{R}^N$ *such that*

$$\frac{c_1}{k\sqrt{\nu T}} < |\bar{\xi}| < \frac{c_2}{k\sqrt{T \sum_{i,j=1}^N \|a_{ij}\|_\infty}} \tag{23}$$

and

$$\gamma_1 := \frac{\max_{|\xi|\leq c_2} G(\xi) - G(\bar{\xi})}{c_2^2 - k^2 \int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\, dt} < \frac{G(\bar{\xi}) - \max_{|\xi|\leq c_1} G(\xi)}{k^2 \int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\, dt - c_1^2} := \gamma_2, \tag{24}$$

where k *is defined in* (12).
Then, for every $b \in L^1([0, T], \mathbb{R}) \setminus \{0\}$ *a.e. nonnegative and for every* $\lambda \in \Lambda := \frac{1}{2k^2\|b\|_1}\left]\frac{1}{\gamma_2}, \frac{1}{\gamma_1}\right[$ *there exists at least one solution* $u_\lambda \in H^1_T$ *of problem* (P_λ) *such that* $\frac{c_1}{k} < \|u_\lambda\| < \frac{c_2}{k}$.

Proof. Let us begin observing that, putting together (8) and assumption (23), one has

$$c_1^2 < k^2\nu T|\bar{\xi}|^2 \leq k^2 \int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\, dt \leq k^2 T \sum_{i,j=1}^N \|a_{ij}\|_\infty |\bar{\xi}|^2 < c_2^2. \tag{25}$$

Moreover, since

$$k\sqrt{T \sum_{i,j=1}^N \|a_{ij}\|_\infty} \geq k\sqrt{\nu T} \geq \sqrt{2},$$

in view of (23), one has $|\bar{\xi}| < c_2$ so that

$$\max_{|\xi|\leq c_2} G(\xi) \geq G(\bar{\xi}). \tag{26}$$

By (25) and (26) it follows that $\gamma_2 > \gamma_1 \geq 0$, as well as

$$G(\bar{\xi}) > \max_{|\xi|\leq c_1} G(\xi). \tag{27}$$

Fix now $b \in L^1([0,T],\mathbb{R})\setminus\{0\}$ which is a.e. nonnegative and $\lambda \in \Lambda$. We will apply Theorem 2.1 where $X = H^1_T$, while Φ and Ψ are as defined in (14). In Section 2 we already observed that the functionals Φ and Ψ satisfy all the regularity assumptions required in Theorem 2.1. Let $u \in X$ be a function defined by

$$u(t) := \bar{\xi},$$

for every $t \in [0,T]$, and put

$$r_1 := \frac{c_1^2}{2k^2}, \qquad r_2 := \frac{c_2^2}{2k^2}.$$

Since $\Phi(u) = \frac{1}{2}\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\,dt$, from condition (25) it follows that

$$r_1 < \Phi(u) < r_2. \tag{28}$$

Hence, in view of (18) one has

$$\beta(r_1,r_2) \le \|b\|_1 \frac{\max_{|\xi|\le k\sqrt{2r_2}} G(\xi) - G(\bar{\xi})}{r_2 - \frac{1}{2}\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\,dt} = 2k^2\|b\|_1\gamma_1. \tag{29}$$

On the other hand, arguing as in the previous inequality, one has

$$\rho(r_1,r_2) \ge \|b\|_1 \frac{G(\bar{\xi}) - \max_{|\xi|\le k\sqrt{2r_1}} G(\xi)}{\frac{1}{2}\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\,dt - r_1} = 2k^2\|b\|_1\gamma_2. \tag{30}$$

From (29) and (30) it follows that (22) holds. Finally, if we observe that $\Lambda \subseteq \left]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$ we can conclude the proof applying Theorem 2.1 and taking in mind claim (21). □

Remark 3.2. When assumption (23) is replaced by the more restrictive condition

$$\frac{c_1}{k\sqrt{\nu T}} < |\bar{\xi}| \le \frac{c_2}{k\sqrt{2T\sum_{i,j=1}^N \|a_{ij}\|_\infty}}, \tag{31}$$

assumption (24) is verified provided

$$\max_{|\xi|\le c_1} G(\xi) + \max_{|\xi|\le c_2} G(\xi) < 2G(\bar{\xi}). \tag{32}$$

In fact, by (31), reasoning as in (25), it follows that

$$\frac{c_1^2}{k^2} < \int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\,dt \le \frac{c_2^2}{2k^2} \le \frac{c_1^2 + c_2^2}{2k^2},$$

namely

$$0 < k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\,dt - c_1^2 \le c_2^2 - k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\,dt. \tag{33}$$

At this point, since (32) implies that

$$\max_{|\xi|\le c_2} G(\xi) - G(\bar{\xi}) < G(\bar{\xi}) - \max_{|\xi|\le c_1} G(\xi), \tag{34}$$

putting together (26), (27), (33) and (34) we achive (24).

Theorem 3.3. *Let $A : [0,T] \to \mathbb{R}^{N\times N}$ be satisfying* $(\mathrm{A})_1 - (\mathrm{A})_2$, $G \in C^1(\mathbb{R}^N)$, *with* $G(0) = 0$, *and assume that there exist a positive number $c \in \mathbb{R}$ and $\bar{\xi} \in \mathbb{R}^N \setminus \{0\}$ such that*

$$|\bar{\xi}| < \frac{c}{k\sqrt{T\sum_{i,j=1}^{N}\|a_{ij}\|_\infty}} \tag{23'}$$

and

$$\frac{\max_{|\xi|\le c} G(\xi)}{c^2} < \frac{1}{k^2}\frac{G(\bar{\xi})}{\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt}, \tag{24'}$$

where k is defined in (12).
Then, for every $b \in L^1([0,T],\mathbb{R}) \setminus \{0\}$ which is a.e. nonnegative and for every $\lambda \in \Lambda' := \frac{1}{2k^2\|b\|_1}\left]\frac{k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt}{G(\bar{\xi})}, \frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$ there exists at least one nontrivial solution $u_\lambda \in H_T^1$ of problem (P_λ) *such that $|u_\lambda(t)| < c$ for every $t \in [0,T]$.*

Proof. Since (24') produces

$$-G(\bar{\xi}) < -k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt\frac{\max_{|\xi|\le c} G(\xi)}{c^2}, \tag{35}$$

we obtain

$$\begin{aligned}\frac{\max_{|\xi|\le c} G(\xi) - G(\bar{\xi})}{c^2 - k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt} &< \frac{\max_{|\xi|\le c} G(\xi) - k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt\frac{\max_{|\xi|\le c} G(\xi)}{c^2}}{c^2 - k^2\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt}\\ &= \frac{\max_{|\xi|\le c} G(\xi)}{c^2}\\ &< \frac{1}{k^2}\frac{G(\bar{\xi})}{\int_0^T A(t)\bar{\xi}\cdot\bar{\xi}\; dt}\end{aligned}$$

and (24) is satisfied, where $c_1 = 0$, $c_2 = c$ and $G(0) = 0$ is used. Fix $b \in L^1([0,T],\mathbb{R}) \setminus \{0\}$ which is a.e. nonnegative and $\lambda \in \Lambda'$. We can apply Theorem 3.1 and, because $\Lambda' \subset \Lambda$, there exists a solution $u_\lambda \in H_T^1$ of (P_λ) such that $0 < \|u_\lambda\| < \frac{c}{k}$, namely u_λ is nontrivial and, taking in mind (13), the proof is complete. $\square$

Remark 3.4. We explicitly observe that if c is a positive real number such that $\max_{|\xi|\le c} G(\xi) > 0$, applying Theorem 2.1 in the case $r_1 < 0$ or Theorem 2.5 of [18], it is possible to assure the existence of a solution u_λ of problem (P_λ) for every $\lambda \in \frac{1}{2k^2\|b\|_1}\left]0, \frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$. Anyway, following this way, one can simply conclude that $\|u_\lambda\| < \frac{c}{k}$ so that u_λ could be trivial.

Remark 3.5. We point out that the assumptions of the previous Theorem 3.3 are satisfied whenever G is not identically zero and such that

$$\lim_{c\to+\infty}\frac{\max_{|\xi|\le c} G(\xi)}{c^2} = 0. \tag{36}$$

In particular, if (36) holds, a simple computation shows that the interval Λ^* becomes $]\lambda^*, +\infty[$, where

$$\lambda^* := \frac{1}{2\|b\|_1} \inf_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{\int_0^T A(t)\xi \cdot \xi dt}{G(\xi)}.$$

Theorem 3.6. *Let $A : [0,T] \to \mathbb{R}^{N\times N}$ be satisfying* (A)$_1$–(A)$_2$, *$G \in C^1(\mathbb{R}^N)$, with $G(0) = 0$, and assume that*

$$\limsup_{\xi \to 0} \frac{G(\xi)}{|\xi|^2} = +\infty. \tag{37}$$

Then, for every $b \in L^1([0,T], \mathbb{R}) \setminus \{0\}$ which is a.e. nonnegative and for every $\lambda \in \Gamma' := \frac{1}{2k^2\|b\|_1} \left]0, \sup_{c>0} \frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$ problem (P$_\lambda$) *admits at least one nontrivial solution u_λ.*

Proof. First of all, it is simple to verify that, in view of (37) one has that $\sup_{c>0} \frac{c^2}{\max_{|\xi|\le c} G(\xi)} > 0$. Fix $b \in L^1([0,T], \mathbb{R}) \setminus \{0\}$ which is a.e. nonnegative and $\lambda \in \frac{1}{2k^2\|b\|_1} \left]0, \sup_{c>0} \frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$. Put

$$M := \frac{T \sum_{i,j=1}^N \|a_{ij}\|_\infty}{2\lambda \|b\|_1}$$

and pick $c > 0$ such that $\lambda < \frac{1}{2k^2\|b\|_1} \frac{c^2}{\max_{|\xi|\le c} G(\xi)}$.
Assumption (37) assure that there exists $\bar{\xi} \in \mathbb{R}^N \setminus \{0\}$, with $|\bar{\xi}| < \frac{c}{k\sqrt{T \sum_{i,j=1}^N \|a_{ij}\|_\infty}}$ such that

$$\frac{G(\bar{\xi})}{|\bar{\xi}|^2} > M.$$

Hence, by the choice of c, taking in mind (8), one has

$$\begin{aligned}\frac{\max_{|\xi|\le c} G(\xi)}{c^2} &< \frac{1}{2\lambda k^2 \|b\|_1} = \frac{M}{k^2 T \sum_{i,j=1}^N \|a_{ij}\|_\infty} \\ &< \frac{1}{k^2 T \sum_{i,j=1}^N \|a_{ij}\|_\infty} \frac{G(\bar{\xi})}{|\bar{\xi}|^2} \le \frac{1}{k^2} \frac{G(\bar{\xi})}{\int_0^T A(t)\bar{\xi} \cdot \bar{\xi}\, dt},\end{aligned}$$

that is all the assumptions of Theorem 3.3 are satisfied and $\lambda \in \Lambda'$. This completes the proof. □

Remark 3.7. An alternative proof of Theorem 3.6 goes on as follows. As observed in Remark 3.4, it is possible to claim that, fixed $\lambda \in \Gamma'$ and chosen $c > 0$ with $\lambda \in \frac{1}{2k^2\|b\|_1} \left]0, \frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$, there exists a solution u_λ, possibly equal to zero, with $\|u_\lambda\| < c/k$. More precisely, a carefully reading shows that u_λ is obtained as a particular critical point of the functional $I_\lambda = \Phi - \lambda\Psi$, namely u_λ is a local minimum of I_λ. This fact provides a non zero solution of (P_λ) if we verify that assumption (37) does not allow 0 to be a local minimum of I_λ. In fact, if we put

$$M := \frac{T\sum_{i,j=1}^{N}\|a_{ij}\|_\infty}{\lambda\|b\|_1},$$

by (37) we obtain the existence of a sequence $\{\xi_n\} \in \mathbb{R}^N \setminus \{0\}$ such that $\xi_n \to 0$ and

$$G(\xi_n) > M|\xi_n|^2$$

for every $n \in \mathbb{N}$. At this point, let us define

$$u_n(t) := \xi_n$$

for every $t \in [0, T]$ and $n \in \mathbb{N}$. Hence, since the sequence $\{u_n\} \subset H^1_T$ converges to 0 and, for every $n \in \mathbb{N}$ one has

$$\begin{aligned}\Phi(u_n) - \lambda\Psi(u_n) &\leq \frac{T}{2}\sum_{i,j=1}^{N}\|a_{ij}\|_\infty|\xi_n|^2 - \lambda\|b\|_1 G(\xi_n)\\ &< \ -\frac{T}{2}\sum_{i,j=1}^{N}\|a_{ij}\|_\infty|\xi_n|^2 < 0 = \Phi(0) - \lambda\Psi(0),\end{aligned}$$

we conclude.

Remark 3.8. From the proof of Theorem 3.6 it follows that, considering an interval possibly smaller than Γ', namely the interval $\frac{1}{2k^2\|b\|_1}\left]0, \frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[$, for some $c > 0$ such that $\frac{c^2}{\max_{|\xi|\leq c} G(\xi)} > 0$, the nontrivial solution obtained can be estimated in norm, uniformly with respect to λ, by the inequality $|u_\lambda(t)| < c$ for every $t \in [0, T]$.

In force of the previous Remark, it is possible to point out the following

Example 3.9. Let $H : \mathbb{R}^N \to \mathbb{R}$ be a function and c be a positive real number. Put

$$\tilde{G}(\xi) := \begin{cases} |\xi|^{3/2} & \text{if } |\xi| \leq c \\ H(\xi) & \text{otherwise.}\end{cases}$$

Then, for every $A : [0, 1] \to \mathbb{R}^{N\times N}$ satisfying $(\mathrm{A})_1$–$(\mathrm{A})_2$ with $\nu \geq 1$, for every $b \in L^1([0, 1], \mathbb{R})$ that is a.e. nonnegative and for every $\lambda \in \left]0, \frac{\sqrt{c}}{4\|b\|_1}\right[$ the following problem

$$\begin{cases} -\ddot{u} + A(t)u = \lambda b(t)\nabla\tilde{G}(u) \text{ a.e. in } [0, 1] \\ u(1) - u(0) = \dot{u}(1) - \dot{u}(0) = 0. \end{cases} \tag{38}$$

admits at least a nontrivial solution. Indeed, we can apply Theorem 3.6 to the function $G(\xi) = |\xi|^{3/2}$ for every $\xi \in \mathbb{R}^N$, when $T = 1$, taking in mind Remark 3.8, namely $\|u_\lambda\|_\infty < c$ and observing that $\tilde{G} \equiv G$ on $\bar{B}(0, c)$, as well as $k = \sqrt{2}$.

4. Multiplicity results

This section is devoted to show how the previous results can be employed in order to pass from the existence of at least one nontrivial solution to the existence of at lest two nontrivial solutions. This goal will be achieved making use of the particular nature of the first solution found, namely it is a local minimum. In particular, in a natural way, this information will be used to assure the existence of a second solution as a critical point of Mountain Pass type. In this direction, we begin with the following theorem, where the classical Ambrosetti-Rabinowitz condition is required.

Theorem 4.1. *Let $A : [0, T] \to \mathbb{R}^{N\times N}$ be satisfying* $(\mathrm{A})_1 - (\mathrm{A})_2$, $G \in C^1(\mathbb{R}^N)$, *with $G(0) = 0$ and $\nabla G(0) \neq 0$, in addition to* (37). *Moreover, assume that there are two positive constants $\mu > 2$ and r, such that, for every $|\xi| \geq r$ one has*

$$0 < \mu G(\xi) \leq \nabla G(\xi) \cdot \xi. \tag{39}$$

Then, for every $\lambda \in \Gamma' := \frac{1}{2k^2T}\left]0, \sup_{c>0} \frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[$ problem

$$\begin{cases} -\ddot{u} + A(t)u = \lambda \nabla G(u) \text{ a.e. in } [0, T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{cases} \tag{40}$$

admits at least two nontrivial solutions.

Proof. First of all, we can observe that, since $\nabla G(0) \neq 0$, problem (40) does not admits the trivial solution. Fix $\lambda \in \Gamma'$ and pick $c > 0$ such that $\lambda < \frac{1}{2k^2T} \frac{c^2}{\max_{|\xi|\leq c} G(\xi)}$. Applying Theorem 3.6, with $b \equiv 1$, it is possible to assure the existence of a nontrivial solution u_1 that, in particular, is obtained as a local minimum of the functional I_λ defined in (21). In view of assumption (39), reasoning in a standard way, it is possible to verify that every Palais-Smale sequence is bounded. This, leads to the fact that I_λ satisfies the classical Palais-Smale condition by the fact that Φ' is a linear isomorphism, while Ψ' is compact (see [12, Proposition 3.8]). Moreover, it is well known that, again by (39), there exist suitable positive constants C_1, C_2 with

$$G(\xi) \geq C_1|\xi|^\mu - C_2 \quad \forall \xi \in \mathbb{R}^N. \tag{41}$$

Let $\{\xi_n\}$ be a sequence in $\mathbb{R}^N$ such that $|\xi_n| \to +\infty$ and consider the related sequence $w_n(t) := \xi_n$ for every $t \in [0, T]$ which belongs to H^1_T. Obviously, in view of (41), one has

$$\begin{aligned} I_\lambda(w_n) &= \frac{1}{2}\int_0^T A(t)\xi_n \cdot \xi_n dt - \lambda T G(\xi_n) \\ &\leq \frac{T\sum_{i,j=1}^N \|a_{ij}\|_\infty |\xi_n|^2}{2} - \lambda(C_1 T|\xi_n|^\mu - C_2), \end{aligned}$$

that is

$$\liminf_{\|u\|\to+\infty} I_\lambda(u) = -\infty,$$

and there exists some $u_0 \in H^1_T$ such that

$$I_\lambda(u_0) < I_\lambda(u_1). \tag{42}$$

At this point, we can assume that u_1 is a strict local minimum of I_λ, otherwise there exist infinitely many nontrivial critical points of I_λ. Hence, we can apply the classical Mountain Pass Theorem (briefly MPT) and obtain a second critical point $u_2 \in H^1_T$ such that $I_\lambda(u_2) > I_\lambda(u_1)$, that is $u_1 \neq u_2$ and the proof is complete. □

Remark 4.2. In [7] it is studied a more general Hamiltonian system

$$\ddot{u} = \nabla F(t,u) \ \text{ a.e. in } [0,T], \qquad u(0)-u(T) = \dot{u}(0)-\dot{u}(T) = 0,$$

that gives back our problem (40) when

$$F(t,\xi) = \frac{1}{2}A(t)\xi\cdot\xi - \lambda G(\xi). \tag{43}$$

In particular, in [7] it is proved the existence of at least two nontrivial solutions under a set of assumptions which implies the coercivity of $F(t,\cdot)$, uniformly with respect to t, as well as that $F(t,\cdot)/|\cdot|^2$ is negative and bounded near to zero (see also (2)). Both these conditions are completely independent from those required in our Theorem 4.1. Indeed, as pointed out in the proof, assumption (39) forces the function $F(t,\cdot)$ defined in (43) to be anti coercive, as well as condition (37) leads to

$$\liminf_{|\xi|\to 0} \frac{F(t,\xi)}{|\xi|^2} = -\infty,$$

uniformly with respect to t. It is also worth noticing that in [12, Theorem 5.16] it is reported a personal communication of Nirenberg, where the following general system is treated

$$\ddot{u} + \nabla V(t,u) = 0, \tag{44}$$

which reduces to our (40) if $V(t,u) = \lambda G(u) - \frac{1}{2}A(t)u\cdot u$. In particular, V is considered to be a nonnegative smooth function satisfying the Ambrosetti-Rabinowitz condition, but the existence of at least only one nonconstant periodic solution is proved.

Example 4.3. Let $a \in \mathbb{R}^N \setminus \{0\}$ be a vector and let $A : [0,1] \to \mathbb{R}^{N\times N}$ be a function satisfying $(A)_1$–$(A)_2$, with $\nu \geq 1$. Then, for every $\lambda \in \left]0, \frac{1}{4|a|+1}\right[$ the following Hamiltonian system

$$\begin{cases} -\ddot{u} + A(t)u = \lambda\big(a + u|u|^2\big) \text{ a.e. in } [0,1] \\ u(0)-u(1) = \dot{u}(0)-\dot{u}(1) = 0, \end{cases}$$

admits at least two nontrivial solutions.
Indeed, it is possible to apply Theorem 4.1, with $T = 1$ and

$$G(\xi) := a \cdot \xi + \frac{|\xi|^4}{4} \quad \forall \xi \in \mathbb{R}^N.$$

Hence, $\nabla G(0) = a$, while,

$$\nabla G(\xi) = a + \xi|\xi|^3$$

and

$$\nabla G(\xi) \cdot \xi - 3G(\xi) = \frac{|\xi|^4}{4} - 2a \cdot \xi$$

so that (37) and (39) hold with $\mu = 3$. Moreover, since for every $c > 0$

$$\max_{|\xi| \leq c} G(\xi) \leq c|a| + \frac{c^4}{4}$$

clearly

$$\frac{c^2}{\max_{|\xi| \leq c} G(\xi)} \geq \frac{4c}{4|a| + c^3}$$

Hence,

$$\sup_{c>0} \frac{c^2}{\max_{|\xi| \leq c} G(\xi)} \geq \frac{4}{4|a| + 1}$$

as well as $k = \sqrt{2}$ and

$$\frac{1}{2k^2 T} \sup_{c>0} \frac{c^2}{\max_{|\xi| \leq c} G(\xi)} > \frac{1}{4|a| + 1}.$$

Example 4.4. Let $a \in \mathbb{R}^N \setminus \{0\}$ be a vector and let $A : [0, 1] \to \mathbb{R}^{N \times N}$ be a function satisfying $(A)_1$–$(A)_2$, with $\nu \geq 1$. Then, for every $\lambda \in \left]0, \frac{1}{2(|a|+e)}\right[$ the following Hamiltonian system

$$\begin{cases} -\ddot{u} + A(t)u = \lambda\big(a + ue^{|u|^2}\big) \text{ a.e. in } [0,1] \\ u(0) - u(1) = \dot{u}(0) - \dot{u}(1) = 0, \end{cases}$$

admits at least two nontrivial solutions.
Indeed, also in this case, it is possible to apply Theorem 4.1, with $T = 1$ and

$$G(\xi) := a \cdot \xi + \frac{1}{2} e^{|\xi|^2} - 1$$

for every $\xi \in \mathbb{R}^2$. Hence, $\nabla G(0) = a$, while,

$$\nabla G(\xi) = a + \xi e^{|\xi|^2}$$

and

$$\nabla G(\xi)\cdot\xi - 3G(\xi) = \left(|\xi|^2 - \frac{3}{2}\right)e^{|\xi|^2} - 2a\cdot\xi + 3$$

so that (37) and (39) hold with $\mu = 3$. Moreover, since for every $c > 0$

$$\max_{|\xi|\le c} G(\xi) \le c|a| + \frac{1}{2}e^{c^2}$$

clearly

$$\frac{c^2}{\max_{|\xi|\le c} G(\xi)} \ge \frac{2c}{2|a| + e^{c^2}}$$

Hence,

$$\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c} G(\xi)} \ge \frac{2}{2|a| + e},$$

as well as $k = \sqrt{2}$ and

$$\frac{1}{2k^2T}\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c} G(\xi)} > \frac{1}{2(|a| + e)}.$$

As usual, condition (39) plays a crucial role in proving that every Palais-Smale sequence is bounded, as well as that the so called 'mountain pass geometry' is satisfied. However, even dealing with different problems than ours, several authors studied more general or different assumptions that still allow to apply min-max methods in order to assure the existence of critical points ([13, 11, 28, 16]). Here we show a result that moves in this direction.

Theorem 4.5. *Let $G \in C^1(\mathbb{R}^N)$, with $G(0) = 0$ and $\nabla G(0) \ne 0$, in addition to* (37)*. Moreover, assume that*

$$\lim_{|\xi|\to+\infty}\frac{G(\xi)}{|\xi|^2} = +\infty \tag{45}$$

and that there exist $1 < \alpha \le \beta$, r, c_1, $c_2 > 0$ such that, for every $|\xi| \ge r$ one has

$$\nabla G(\xi)\cdot\xi - 2G(\xi) \ge c_1|\xi|^\beta, \tag{46}$$

and

$$|\nabla G(\xi)| \le c_2|\xi|^\alpha, \quad \text{or} \quad |G(\xi)| \le c_2|\xi|^{\alpha+1}. \tag{47}$$

Then, for every $\lambda \in \Gamma' := \frac{1}{2k^2T}\left]0, \sup_{c>0}\frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$ problem

$$\begin{cases} -\ddot{u} + u = \lambda\nabla G(u) \text{ a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{cases} \tag{48}$$

admits at least two nontrivial solutions.

Proof. Fix $\lambda \in \Gamma'$ and argue as in the proof of Theorem 4.1 when $A(t) = I_{N\times N}$, in order to assure the existence of a first nontrivial solution u_1. We explicitly observe that, in this case, $\|\cdot\| = \|\cdot\|_{H^1_T}$.
In view of assumption (45), it is possible to verify that

$$\liminf_{\|u\|\to+\infty} I_\lambda(u) = -\infty. \tag{49}$$

Indeed, for $\eta > \frac{1}{2\lambda}$ there exists $R > 0$ such that $G(\xi) > \eta|\xi|^2$ for every $\xi \in \mathbb{R}^N \setminus \bar{B}(0,R)$. Hence, if $\{\xi_n\}$ is a sequence in $\mathbb{R}^N$ with $|\xi_n| \to +\infty$ and we consider the sequence in H^1_T defined by putting $w_n(t) = \xi_n$ for every $t \in [0,T]$ one has that

$$I_\lambda(w_n) = \frac{T}{2}|\xi_n|^2 - \lambda T G(\xi_n) < \left(\frac{T}{2} - \lambda T\eta\right)|\xi_n|^2,$$

for every $n \in \mathbb{N}$ large enough and (49) follows from the choice of η. Hence, there exists $u_0 \in H^1_T$ such that (42) is true.
To complete the proof, we will verify that I_λ satisfies the (C) condition and apply the version of the MPT where (PS) is replaced by (C) (see [8]).

Let $\{v_n\}$ be a bounded sequence in H^1_T such that $\{I_\lambda(v_n)\}$ is bounded and $I'_\lambda(v_n) \to 0$. Since Φ' is a linear isomorphism and Ψ' is compact, passing to a subsequence if necessary, $\{v_n\}$ is strongly convergent (see [12, Proposition 3.8]), and $(\mathrm{C})_1$ holds.

Finally, by contradiction, assume that $(\mathrm{C})_2$ does not hold, namely there exists $d \in \mathbb{R}$ and a sequence $\{v_n\}$ in H^1_T such that

$$\|I_\lambda(v_n)\|_{H^{1*}_T}\|v_n\| \to 0, \quad I_\lambda(v_n) \to d \quad \text{and} \quad \|v_n\| \to +\infty. \tag{50}$$

Let us verify that $\{v_n\}$ is bounded in $L^\beta([0,T],\mathbb{R}^N)$. From (46) there exists $c_3 > 0$ such that

$$\nabla G(\xi)\cdot\xi - 2G(\xi) \geq c_1|\xi|^\beta - c_3 \tag{51}$$

for every $\xi \in \mathbb{R}^N$. Hence, a simple computation shows that for all $n \in \mathbb{N}$ one has

$$\begin{aligned} \|v_n\|^\beta_\beta &= \frac{1}{c_1}\int_0^T (c_1|v_n(t)|^\beta - c_3)dt + \frac{Tc_3}{c_1} \\ &\leq \frac{1}{c_1}\int_0^T (\nabla G(v_n(t))\cdot v_n(t) - 2G(v_n(t)))dt + \frac{Tc_3}{c_1}. \end{aligned} \tag{52}$$

Moreover, from (50) there exists $M > 0$ such that

$$\|v_n\|^2 - \lambda\int_0^T 2G(v_n(t))dt = 2I_\lambda(v_n) \leq M \tag{53}$$

as well as

$$-\|v_n\|^2 + \lambda\int_0^T \nabla G(v_n(t))\cdot v_n(t)dt = -I'_\lambda(v_n)(v_n) \leq \|I'_\lambda(v_n)\|_{H^{1*}_T}\|v_n\| \leq M \tag{54}$$

for every $n \in \mathbb{N}$. At this point (53) and (54) imply that

$$\lambda \int_0^T (\nabla G(v_n(t)) \cdot v_n(t) - 2G(v_n(t)))dt \leq 2M$$

and taking in mind (52) we achieve the boundness of $\{v_n\}$ in L^β. Since $1 < \alpha \leq \beta$, we conclude that $\{v_n\}$ is bounded in L^α too, as well as in L^1. In particular we can emphasize that

$$\frac{\|v_n\|_\alpha^\alpha}{\|v_n\|} \to 0 \quad \text{as } n \to +\infty. \tag{55}$$

Condition (47) assures the existence of some $c_4 > 0$ such that

$$|\nabla G(\xi)| \leq c_2|\xi|^\alpha + c_4, \quad \text{or} \quad |G(\xi)| \leq c_2|\xi|^{\alpha+1} + c_4, \tag{56}$$

for every $\xi \in \mathbb{R}^N$. From (7), it possible to write $v_n = \tilde{v}_n + \bar{v}_n$ with $\bar{v}_n = (1/T)\int_0^T v_n(t)dt$ and $\langle \tilde{v}_n, \bar{v}_n \rangle = 0$, so that

$$\|v_n\|^2 = \|\tilde{v}_n\|^2 + \|\bar{v}_n\|^2 \tag{57}$$

for every $n \in \mathbb{N}$. Hence, there exists $L_1 > 0$ such that, for every $n \in \mathbb{N}$,

$$|I'_\lambda(v_n)(\tilde{v}_n)|^2 \leq \|I'_\lambda(v_n)\|^2_{H_T^{1*}} \|\tilde{v}_n\|^2 \leq \|I'_\lambda(v_n)\|^2_{H_T^{1*}} \|v_n\|^2 < L_1^2.$$

Exploiting the first condition of (56) and taking in mind (13) we obtain

$$\begin{aligned} L_1 \geq I'_\lambda(v_n)(\tilde{v}_n) &= \langle \tilde{v}_n + \bar{v}_n, \tilde{v}_n \rangle - \lambda \int_0^T \nabla G(v_n(t)) \cdot \tilde{v}_n(t)dt \\ &\geq \|\tilde{v}_n\|^2 - \lambda \|\tilde{v}_n\|_{C^0} \int_0^T (c_2|v_n(t)|^\alpha + c_4)dt \\ &\geq \|\tilde{v}_n\|^2 - \lambda \bar{k} \|\tilde{v}_n\| \int_0^T (c_2|v_n(t)|^\alpha + c_4)dt \end{aligned} \tag{58}$$

Hence, thanks to (55) we conclude

$$\frac{\|\tilde{v}_n\|^2}{\|v_n\|^2} \leq \lambda \bar{k} \frac{\|\tilde{v}_n\|}{\|v_n\|} \left(c_2 \frac{\|v_n\|_\alpha^\alpha}{\|v_n\|} + \frac{T c_4}{\|v_n\|} \right) + \frac{L_1}{\|v_n\|^2}$$

so that

$$\frac{\|\tilde{v}_n\|^2}{\|v_n\|^2} \to 0. \tag{59}$$

Analogously, if the second condition in (56) holds, reasoning as in (58), there exists

$L_3 > 0$ such that

$$\begin{aligned} L_3 > I_\lambda(v_n) &= \frac{1}{2}\|v_n\|^2 - \lambda \int_0^T G(v_n(t))dt \\ &= \frac{1}{2}(\|\tilde{v}_n\|^2 + \|\bar{v}_n\|^2) - \lambda \int_0^T G(v_n(t))dt \\ &\geq \frac{1}{2}\|\tilde{v}_n\|^2 - \lambda \int_0^T \left(c_2|v_n(t)|^{\alpha+1} + c_4\right) dt \\ &\geq \frac{1}{2}\|\tilde{v}_n\|^2 - \lambda \left(\bar{k}\|v_n\| \int_0^T c_2|v_n(t)|^\alpha dt + Tc_4\right). \end{aligned}$$

Hence, also in this case, condition (59) occurs.

On the other hand, since $L^2 \hookrightarrow L^1$, from the Wirtinger's inequality it follows that

$$\|\tilde{v}_n\|_1 \leq \sqrt{T}\|\tilde{v}_n\|_2 \leq \frac{T\sqrt{T}}{2\pi}\|\dot{\tilde{v}}_n\|_2 \leq \frac{T\sqrt{T}}{2\pi}\|\tilde{v}_n\|.$$

Observe that $\|\bar{v}_n\| = (\int_0^T |\bar{v}_n|^2 dt)^{1/2} = \sqrt{T}|\bar{v}_n| = (1/\sqrt{T})\|\bar{v}_n\|_1$. Hence, because $|\bar{v}_n| \leq |v_n(t)| + |\tilde{v}_n(t)|$ for every $t \in [0, T]$, there exist $L_2,\ c_5,\ c_6 > 0$ such that

$$\begin{aligned} L_2 &\geq \int_0^T |v_n(t)|dt \geq \int_0^T |\bar{v}_n|dt - \int_0^T |\tilde{v}_n(t)|dt \\ &\geq c_5\|\bar{v}_n\| - c_6\|\tilde{v}_n\|, \end{aligned} \tag{60}$$

that is, for some $c_7,\ c_8 > 0$

$$\frac{\|\bar{v}_n\|}{\|v_n\|} \leq c_7\frac{\|\tilde{v}_n\|}{\|v_n\|} + \frac{c_8}{\|v_n\|}$$

for every $n \in \mathbb{N}$. Passing to the limit in the previous condition and taking in mind (59) one has that

$$\frac{\|\bar{v}_n\|^2}{\|v_n\|^2} \to 0. \tag{61}$$

Let us consider $w_n := \frac{v_n}{\|v_n\|}$ for every $n \in \mathbb{N}$. Putting together (57), (59) and (61) one has

$$\|w_n\|^2 = \frac{\|\tilde{v}_n\|^2 + \|\bar{v}_n\|^2}{\|v_n\|^2} \to 0,$$

so that $\{w_n\}$ is a sequence in H_T^1 of functions of unit norm that strongly converges to zero and we obtained a contradiction that concludes the proof. □

Example 4.6. Fix $a \in \mathbb{R}^N \setminus \{0\}$. For every $\lambda \in \left]0, \frac{e^2}{4(|a|+1)}\right[$ the following Hamiltonian system

$$\begin{cases} -\ddot{u} + u = \lambda\left(a + 2u\ln^2|u| + 2u\ln|u|\right) \text{ a.e. in } [0,1] \\ u(0) - u(1) = \dot{u}(0) - \dot{u}(1) = 0, \end{cases}$$

admits at least two nontrivial solutions.
Indeed, we can apply Theorem 4.5 with $T = 1$ if we consider the function

$$G(\xi) := \begin{cases} a \cdot \xi + |\xi|^2 \ln^2 |\xi| & \text{if } \xi \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } \xi = 0. \end{cases}$$

it is easy to verify that $\nabla G(0) = a$, as well as, for $\xi \neq 0$

$$\nabla G(\xi) = a + 2\xi \ln^2 |\xi| + 2\xi \ln |\xi|.$$

Hence,

$$\xi \cdot \nabla G(\xi) - 2G(\xi) = 2|\xi|^2 \ln |\xi| - a \cdot \xi.$$

Therefore, (37), (45), (46) and (47) hold with $\alpha = \beta = 2$. Moreover,

$$\max_{|\xi| \leq 1} G(\xi) \leq |a| + e^{-2},$$

so that, since $k = \sqrt{2}$,

$$\frac{1}{2k^2 T} \sup_{c>0} \frac{c^2}{\max_{|\xi| \leq c} G(\xi)} \geq \frac{e^2}{4(e^2|a| + 1)}$$

and we achieve the conclusion.

Remark 4.7. Although assumptions (45)–(47) remind those introduced in [11], it is worth noting that our Theorem 4.5 is independent from [11, Theorem 1.2] where the nonlinearity is required to be superquadratic at zero.

Remark 4.8. Theorem 4.1 and Theorem 4.5 are mutually independent. In fact, it is easy to verify that the function G introduced in Example 4.6 does not satisfy condition (39) because, for every $\mu > 2$

$$\begin{aligned} &\nabla G(\xi) \cdot \xi - \mu G(\xi) \\ &= (1 - \mu)\xi \cdot \bar{\xi} + (2 - \mu)|\xi|^2 \ln^2 |\xi| + 2|\xi|^2 \ln |\xi| \to -\infty \quad \text{as } |\xi| \to +\infty. \end{aligned}$$

On the other hand, in view of (41), although it is simple to show that condition (39) considered in Theorem 4.1 implies both (45) and (46) of Theorem 4.5, with $\beta = \mu$, the nonlinearity considered in Example 4.4, namely $G(\xi) = a \cdot \bar{\xi} + \frac{1}{2}e^{|\xi|^2} - 1$ for every $\xi \in \mathbb{R}^N$, where $a \in \mathbb{R}^N \setminus \{0\}$, represents an example of function that satisfies all the assumptions of Theorem 4.1, where $\mu > 2$ can be chosen arbitrarily, but, of course, there not exist $1 < \alpha \leq \beta$ and $r > 0$ satisfying (47).

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