Journal of Convex Analysis
Volume 20 (2013), No. 4, 1095–1112

Proximal Subdifferential of the Bilateral Minimal Time Function and some Regularity Applications

Chadi Nour*
Department of Computer Science and Mathematics, Lebanese American University, Byblos Campus, P.O. Box 36, Byblos, Lebanon
cnour@lau.edu.lb

Received: November 21, 2012

We derive a formula for the proximal subdifferential of the bilateral minimal time function in the linear case. As a consequence, we prove some new regularity properties for this function, essentially, we provide suitable weak controllability assumption under which its epigraph is locally φ_0-convex.

Keywords: Bilateral and unilateral minimal time function, linear control system, φ-convexity, proximal analysis, nonsmooth analysis

1. Introduction

In this paper, we consider a control system governed by a linear differential inclusion. Let F be a linear multifunction mappings points x in $\mathbb{R}^n$ to subsets $F(x) := Ax + U$ of $\mathbb{R}^n$, where A is an $n \times n$ matrix and U is a convex and compact subset of $\mathbb{R}^n$ called the *control set*. Associated with F the differential inclusion:

$$\dot{x}(t) \in F(x(t)) \;\text{ a.e. } t \in [0,T],\; x(0) = x_0. \tag{1}$$

A solution to (1) is an absolutely continuous function $x(\cdot)$ defined on the interval $[0,T]$ with initial value $x(0) = x_0$, in which case we say that $x(\cdot)$ is a *trajectory* of F originating from x_0. For such trajectory, one can find a measurable function $u(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$, called *admissible control*, such that $u(t) \in U$ a.e. $t \in [0,T]$ and

$$x(t) = e^{At}\alpha + \int_0^t e^{A(t-\tau)}u(\tau)\,d\tau \;\text{ for all } t \in [0,T].$$

The *bilateral minimal time function* $T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,+\infty]$ associated to (1) can be defined as follows: For $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$, $T(\alpha,\beta)$ is the minimum time taken by a trajectory to go from α to β. When no such trajectory exists, $T(\alpha,\beta)$ is taken to be $+\infty$. The effective domain of $T(\cdot,\cdot)$ is denoted by $\mathcal{R}$, that is,

$$\mathcal{R} := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) < +\infty\}.$$

*This research was supported by the Lebanese American University under the SRDC grant number SRDC-t2012-11.

ISSN 0944-6532 / $ 2.50

The function $T(\cdot,\cdot)$ was introduced in Clarke and Nour [7], see also [16], where the Hamilton-Jacobi equation of the time-optimal control problem was studied in a domain which contains the target set. These authors used $T(\cdot,\cdot)$ in order to construct proximal solutions to this equation, and to the study the existence of time-semigeodesic trajectories. For $\beta \in \mathbb{R}^n$ fixed, the function $T(\cdot,\beta)$ coincides with the well-known *unilateral minimal time function* associated to the target $S = \{\beta\}$. The regularity of this function is a widely studied topic in control theory, see e.g. [1, 3, 5, 10, 11, 14, 22, 23] and the references therein. On the other hand, the regularity of $T(\cdot,\cdot)$ was studied by Nour in [15], where necessary and sufficient conditions were provided for $T(\cdot,\cdot)$ to be continuous and to be locally Lipschitz in $\mathcal{R}$. More precisely, it was proven that $\mathcal{R}$ is open and $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$ if and only if $T(\cdot,\cdot)$ is continuous at (α,α) for all $\alpha \in \mathbb{R}^n$, which is in turn equivalent to F and $-F$ to be *small-time locally controllable* at α for all $\alpha \in \mathbb{R}^n$. Since the small-time controllability of F and $-F$ at α, for all $\alpha \in \mathbb{R}^n$, is a quite strong hypothesis, in [15, Proposition 4.2] a sufficient condition for *local* continuity was provided as well. In particular, for $(\alpha,\beta) \in \mathcal{R}$, if F and $-F$ are small-time locally controllable at α *or* at β then $T(\cdot,\cdot)$ is continuous at (α,β). For the Lipschitz continuity, Nour proved in [15] that for $(\alpha,\beta) \in \mathcal{R}$, if $-A\alpha \in \operatorname{int} U$ *or* $-A\beta \in \operatorname{int} U$ then $T(\cdot,\cdot)$ is Lipschitz near (α,β). As a consequence, Nour deduced in [15, Proposition 4.6] that $\mathcal{R}$ is open and $T(\cdot,\cdot)$ is locally Lipschitz in $\mathcal{R}$ if and only if $-A\alpha \in \operatorname{int} U$ for all $\alpha \in \mathbb{R}^n$. Another useful result in [15] is the following theorem, concerning local *semiconvexity* of the bilateral minimal time function. It can be viewed as a generalization of a semiconvexity result due to Cannarsa and Sinestrati [4] in the unilateral setting.

Theorem 1.1 ([15, Corollary 4.8]). *Assume that* $-A\alpha \in \operatorname{int} U$ *or* $-A\beta \in \operatorname{int} U$, *where* $(\alpha,\beta) \in \mathcal{R}$ *with* $\alpha \neq \beta$. *Then* $T(\cdot,\cdot)$ *is semiconvex near* (α,β) (*that is, on an open set containing* (α,β)).

We recall that a function $f : U \longrightarrow \mathbb{R}^n$ is said to be semiconvex on $U \subset \mathbb{R}^n$ if for any convex $C \subset\subset U$ there exists $K_C > 0$ such that the function $x \mapsto f(x) + K_C\|x\|^2$ is convex on C, where $\|\cdot\|$ denotes the euclidean norm. The semiconvexity property can be seen as an intermediate property between Lipschitz continuity and continuous differentiability. More precisely, semiconvex functions are essentially a quadratic perturbation of convex functions and therefore inherit several regularity properties from convexity such as local Lipschitzianity and a.e. twice differentiability in the interior of their domain. Moreover, their epigraphs satisfy an external sphere condition with locally uniform radius; this property, for general sets, is often referred to *positive reach* [12], *proximal smoothness* [8] and *φ-convexity* [9]. *Lower semicontinuous* functions with φ-convex epigraph are studied in Colombo and Marigonda [9]. Such functions are semiconvex if and only if they are locally Lipschitz and then are good candidate to extend Theorem 1.1 under a weaker condition. In [9], Colombo and Marigonda proved that for a function with φ-convex epigraph, a.e. x admits a neighborhood where the function is semiconvex and as a consequence it is also twice differentiable almost everywhere.

The main result of this paper is the following generalization of Theorem 1.1 which asserts that under weaker condition, mainly F and $-F$ are small-time locally controllable near α or β, the epigraph of $T(\cdot,\cdot)$ is φ_0-convex near (α,β).

Theorem 1.2. *Assume that F and $-F$ are small-time locally controllable near α or β, where $(\alpha,\beta)\in\mathcal{R}$ with $\alpha\neq\beta$. Then the epigraph of $T(\cdot,\cdot)$ is φ_0-convex near (α,β).*

To prove the preceding theorem, we first find a representation formula for the proximal subdifferential of the function $T(\cdot,\cdot)$. This formula will help us to overpass the lack of Lipschitz continuity which can cause the emptiness of both the proximal subdifferential and superdifferential. Based on this formula and using an analytic characterization of the φ_0-convexity property, we prove that the epigraph of $T(\cdot,\cdot)$ is locally φ_0-convex. Some of our techniques can be seen as a generalization, to the bilateral case, of the techniques used in Colombo, Marigonda and Wolenski [10] where the unilateral case was studied. What is different here is that in our time-optimal control problem, both initial and end points are variables which will make the problem more difficult to be handled.

The outline of the paper is the following. In Section 2, we introduce our notations and some basic definitions and results from nonsmooth analysis and control theory. A formula for the proximal subdifferential of the bilateral minimal time function is given in Section 3. Section 4 is devoted to the proof of Theorem 1.2.

2. Preliminaries

Throughout the paper, concepts and basic results from nonsmooth analysis and control theory will be used. Although most definitions and results are classical, we list them in details, in order to fix the notations. The first subsection is devoted to our general notations, while the second and the third one to nonsmooth analysis and control theory, respectively.

2.1. General notations

We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, B and $\bar{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho>0$ and $x\in\mathbb{R}^n$, we set $B(x;\rho):=x+\rho B$ and $\bar{B}(x;\rho):=x+\rho\bar{B}$. For a set $S\subset\mathbb{R}^n$, S^c, $\operatorname{int} S$, $\operatorname{bdry} S$ and $\operatorname{cl} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of S, respectively. The distance from a point x to a set S is denoted by $d_S(x)$. We also denote by $\operatorname{proj}_S(x)$ the set of closest points in S to x, that is, the set of points $s\in S$ which satisfy $d_S(x)=\|s-x\|$. If S is compact and convex, then we recall that the *lower support function* $h_S(\cdot)$ of S is defined by

$$h_S(\zeta):=\min\{\langle\zeta,s\rangle : s\in S\}.$$

The graph and epigraph of a function $f:U\longrightarrow\mathbb{R}$ are denoted respectively by

$$\operatorname{gr} f:=\{(x,f(x)) : x\in U\}\quad\text{and}\quad\operatorname{epi} f:=\{(x,r) : x\in U, r\in\mathbb{R}\text{ and } f(x)\leq r\}.$$

2.2. Nonsmooth analysis

Now we provide certain definitions from proximal analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [6]; see also Rockafellar and Wets [20]. Let S be a nonempty closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to S at x* provided that there exists $\sigma = \sigma(x, \zeta) \geq 0$ such that

$$\langle \zeta, s - x \rangle \leq \sigma \|s - x\|^2 \quad \forall s \in S. \tag{2}$$

The relation (2) is commonly referred to as the *proximal normal inequality.* This proximal normal inequality can be taken to be a local one, that is, only for s near x in S as proved in [6, Proposition 1.5]. We note that no nonzero ζ satisfying (2) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \operatorname{bdry} S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0, 0)$. For such points, the only proximal normal is $\zeta = 0$. In view of (2), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$. Now let $x \in \operatorname{bdry} S$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset. \tag{3}$$

Then ζ is a proximal normal to S at x and we say that ζ is *realized by an r-sphere.* Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = \frac{1}{2r}$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{4}$$

If S is convex and $x \in \operatorname{bdry} S$, then we can easily see that a vector $0 \neq \zeta \in \mathbb{R}^n$ belongs in $N_S^P(S)$ if and only if it is realized by an r-sphere for all $r > 0$. Therefore, the proximal normal inequality (2) becomes

$$\langle \zeta, s - x \rangle \leq 0 \quad \forall s \in S.$$

This proves that in the convex case, the proximal normal cone to S at x coincides with the well known convex normal cone to S at x, denoted here by $N_S(x)$. Now we recall from [6] that for a function $f : U \longrightarrow \mathbb{R}$, a vector $\zeta \in \mathbb{R}^n$ is a *proximal subgradient* of a lower semicontinuous function f at x provided that

$$(\zeta, -1) \in N^P_{\operatorname{epi} f}(x, f(x)),$$

and this is equivalent to the existence of $\sigma = \sigma(x, \zeta) \geq 0$ and $\delta > 0$ such that the following *proximal subgradient inequality* holds:

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall y \in B(x; \delta) \cap U.$$

The *proximal subdifferential* of f at x, denoted $\partial_P f(x)$, is the set of all the proximal subgradients at x. It is an exercise using the proximal normal inequality to show that every proximal normal (ζ, t) to $\operatorname{epi} f$ at $(x, f(x))$ has $t \leq 0$.

We proceed to define the φ_0-convexity property. A detailed analysis of this property may be found in [9].

Definition 2.1. Let $S \subset \mathbb{R}^n$ be a nonempty closed set. We say that S is φ_0-convex if there exists a constant $\varphi_0 \geq 0$ such that for all $x \in \operatorname{bdry} S$ we have:

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \varphi_0 \|s - x\|^2 \text{ for all } s \in S \text{ and } 0 \neq \zeta \in N_S^P(x).$$

Using the equivalence between (3) and (4), we can easily see that S is φ_0-convex if and only if there exists a constant $\varphi_0 \geq 0$ such that for all $x \in \operatorname{bdry} S$ and $0 \neq \zeta \in N_S^P(x)$, we have:

- ζ is realized by a $\frac{1}{2\varphi_0}$-sphere, if $\varphi_0 \neq 0$,
- ζ is realized by an r-sphere for all $r > 0$, if $\varphi_0 = 0$.

Therefore, φ_0-convexity coincides with the positive reach property and the proximal smoothness property defined in [12] and [8], respectively. More precisely, a set S is φ_0-convex if and only if it is $\frac{1}{2\varphi_0}$-proximal smooth (or has positive reach with radius $\frac{1}{2\varphi_0}$). For other related properties, we refer the reader to [2, 17, 18, 19, 20, 21] and the references therein.

Now let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. In view of Definition 2.1, the epigraph of f is φ_0-convex if there exits a constant $\varphi_0 \geq 0$ such that for all $x \in \operatorname{dom} f$, we have:

$$\left\langle \frac{(\zeta, s)}{\|(\zeta, r)\|}, (y, \rho) - (x, f(x)) \right\rangle \leq \varphi_0 \|(y, \rho) - (x, f(x))\|^2, \tag{5}$$

for all $y \in \operatorname{dom} f$, $\rho \geq f(y)$ and $0 \neq (\zeta, s) \in N^P_{\operatorname{epi} f}(x, f(x))$. On the other hand, the epigraph of f is said to be φ_0-convex near a given point $\alpha \in \operatorname{dom} f$, if there exists an open neighborhood O of α such that the epigraph of the restriction of f to O, denoted by $f_{|O}$, is φ_0-convex. The following lemma will play an important role in the proof of our main result. It proves that to get the φ_0-convexity of the epigraph of a continuous function, only points of the form (y, ρ), where $\rho = f(y)$, need to satisfy the inequality (5).

Lemma 2.2. *Let $O \subset \mathbb{R}^n$ be an open and convex set and let $f : O \longrightarrow \mathbb{R}$ be a continuous function. The epigraph of f is φ_0-convex if and only if there exists a constant $\varphi_0 \geq 0$ such that for all $x \in O$, we have:*

$$\left\langle \frac{(\zeta, s)}{\|(\zeta, r)\|}, (y, f(y)) - (x, f(x)) \right\rangle \leq \varphi_0 \|(y, \rho) - (x, f(x))\|^2, \tag{6}$$

for all $y \in O$ and $0 \neq (\zeta, s) \in N^P_{\operatorname{epi} f}(x, f(x))$.

Proof. Since the necessary condition is trivial, we will only prove the sufficient one. Due to the equivalence between (3) and (4), we have that:

- The inequality (5) is equivalent to

$$B\left((x, f(x)) + \frac{1}{2\varphi_0} \frac{(\zeta, s)}{\|(\zeta, s)\|}; \frac{1}{2\varphi_0}\right) \cap \operatorname{epi} f = \emptyset. \tag{7}$$

- The inequality (6) is equivalent to

$$B\left((x,f(x))+\frac{1}{2\varphi_0}\frac{(\zeta,s)}{\|(\zeta,s)\|};\frac{1}{2\varphi_0}\right)\cap \operatorname{gr} f=\emptyset. \tag{8}$$

Therefore, we only need to prove that:

For every $x\in O$ and $0\neq(\zeta,s)\in N^P_{\operatorname{epi} f}(x,f(x))$, if (8) holds then (7) holds too.

Let $x\in O$ and let $0\neq(\zeta,s)\in N^P_{\operatorname{epi} f}(x,f(x))$, and assume that (8) holds. We proceed via contradiction. Assume that (7) does not hold, that is,

$$B\left((x,f(x))+\frac{1}{2\varphi_0}\frac{(\zeta,s)}{\|(\zeta,s)\|};\frac{1}{2\varphi_0}\right)\cap \operatorname{epi} f\neq\emptyset.$$

Then there exists a point (y,ρ) such that $y\in O$, $\rho>f(y)$ and

$$(y,\rho)\in B\left((x,f(x))+\frac{1}{2\varphi_0}\frac{(\zeta,s)}{\|(\zeta,s)\|};\frac{1}{2\varphi_0}\right).$$

Since $(\zeta,s)\in N^P_{\operatorname{epi} f}(x,f(x))$, there exists $0<r<\frac{1}{2\varphi_0}$ such that

$$B\left((x,f(x))+r\frac{(\zeta,s)}{\|(\zeta,s)\|};r\right)\cap \operatorname{epi} f=\emptyset.$$

Now let $(x',r')\in B\big((x,f(x))+r\frac{(\zeta,s)}{\|(\zeta,s)\|};r\big)$ such that $x'\in O$. Due to the convexity of O, the continuity of f and since $(x',r')\notin \operatorname{epi} f$, we have that the segment joining (y,ρ) to (x',r') contains a point on $\operatorname{gr} f$, that is, a point of the form $(y',f(y'))$ where $y'\in O$. But $B\big((x,f(x))+\frac{1}{2\varphi_0}\frac{(\zeta,s)}{\|(\zeta,s)\|};\frac{1}{2\varphi_0}\big)$ is convex, then

$$(y',f(y'))\in B\left((x,f(x))+\frac{1}{2\varphi_0}\frac{(\zeta,s)}{\|(\zeta,s)\|};\frac{1}{2\varphi_0}\right)\cap \operatorname{gr} f.$$

This contradicts (8) and then terminates the proof of the lemma. □

2.3. Control theory

As we mentioned in the introduction, the bilateral minimal time function $T(\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^n\longrightarrow[0,+\infty]$ associated to (1) is defined as follows:

$$T(\alpha,\beta):=\inf\{T\geq 0:\text{some trajectory } x(\cdot) \text{ of } F \text{ has } x(0)=\alpha \text{ and } x(T)=\beta\}.$$

If no trajectory between α and β exists, then $T(\alpha,\beta)=+\infty$. Clearly we have $T(\alpha,\alpha)=0$ for all $\alpha\in\mathbb{R}^n$. We define:

$$\mathcal{R}^\beta_+(t):=\{\alpha\in\mathbb{R}^n: T(\beta,\alpha)<t\},\quad t>0,$$

the set of points reachable from β in time less than t. Similarly, we introduce:

- $\bar{\mathcal{R}}_+^\beta(t) := \{\alpha \in \mathbb{R}^n : T(\beta,\alpha) \le t\}$,
- $\mathcal{R}_+^\beta := \bigcup_{t>0} \mathcal{R}_+^\beta(t) = \{\alpha \in \mathbb{R}^n : T(\beta,\alpha) < +\infty\}$,
- $\mathcal{R}_-^\beta(t) := \{\alpha \in \mathbb{R}^n : T(\alpha,\beta) < t\}$,
- $\bar{\mathcal{R}}_-^\beta(t) := \{\alpha \in \mathbb{R}^n : T(\alpha,\beta) \le t\}$,
- $\mathcal{R}_-^\beta := \bigcup_{t>0} \mathcal{R}_-^\beta(t) = \{\alpha \in \mathbb{R}^n : T(\alpha,\beta) < +\infty\}$,
- $\mathcal{R}(t) := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) < t\}$,
- $\bar{\mathcal{R}}(t) := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) \le t\}$,
- $\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t) = \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) < +\infty\}$.

We also define:

$$\mathcal{A}_+^\beta(t) := \{x(t) : x(\cdot) \text{ is a trajectory of } F \text{ on } [0,t] \text{ satisfying } x(0) = \beta\}, \quad t \ge 0,$$

the *attainable set* from β at time t. As above, we then introduce:

- $\mathcal{A}_+^\beta := \bigcup_{t\ge0} \mathcal{A}_+^\beta(t)$,
- $\mathcal{A}_-^\beta(t) := \{x(0) : x(\cdot)$ is a trajectory of F on $[0,t]$ satisfying $x(t) = \beta\}$,
- $\mathcal{A}_-^\beta := \bigcup_{t>0} \mathcal{A}_-^\beta(t)$,
- $\mathcal{A}(t) := \{(x(0), x(t)) : x(\cdot)$ is a trajectory of F on $[0,t]\}$.

The following are basic and known facts:

- We have $\mathcal{A}_+^\beta(t) \subset \bar{\mathcal{R}}_+^\beta(t)$.
- We have $\mathcal{A}_+^\beta(t) = e^{At}\alpha + \mathcal{A}_+^0(t)$ and the set $\mathcal{A}_+^\beta(t)$ is compact and convex.
- The set $\mathcal{A}(t)$ is compact and convex.
- We have $\operatorname{bdry} \bar{\mathcal{R}}(t) \subset \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) = t\}$.
- $T(\cdot,\cdot)$ and $T(\cdot,\beta)$ are lower semicontinuous.
- We have $T(\alpha,\beta) = 0$ if and only if $\alpha = \beta$. Moreover, If $T(\alpha,\beta) < +\infty$ then the infimum defining $T(\alpha,\beta)$ is attained.
- For all (α,β,γ) we have the following triangle inequality:

$$T(\alpha,\beta) \le T(\alpha,\gamma) + T(\gamma,\beta).$$

We proceed to recall some definitions and results related to the regularity of the bilateral minimal function. For more information about these results, we invite the reader to see [15].

Definition 2.3. Let $\beta \in \mathbb{R}^n$. We say that F is β-STLC, β-small-time locally controllable, if $\beta \in \operatorname{int} \mathcal{R}_-^\beta(t)$ $\forall t > 0$.

Remark 2.4. It is well known that the small-time local controllability of F at a point β is equivalent to the openness of $\mathcal{R}_-^\beta$ and the continuity of the unilateral minimal time function $T(\cdot,\beta)$ on $\mathcal{R}_-^\beta$. It also implies that $\langle A\beta, \zeta\rangle + h_U(\zeta) \le 0$ for any unit vector ζ, which is in turn equivalent to $-A\beta \in U$ (that is, $0 \in F(\beta)$).

The following proposition gives a local and a global continuity result for $T(\cdot,\cdot)$.

Proposition 2.5. *Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have:*

(a) *$T(\cdot,\cdot)$ is continuous at $(\alpha,\alpha) \Longleftrightarrow F$ and $-F$ are α-STLC.*

(b) *Assume that $(\alpha,\beta) \in \mathcal{R}$ and that one of the following conditions holds:*
 (i) *F is α-STLC and $-F$ is β-STLC.*
 (ii) *F and $-F$ are α-STLC.*
 (iii) *F and $-F$ are β-STLC.*
 Then $T(\cdot,\cdot)$ is continuous at (α,β).

(c) *The following assertions are equivalent:*
 (i) *$\mathcal{R}$ is open, $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$ and for any $(\alpha_0,\beta_0) \in \partial\mathcal{R}$ we have*

$$\lim_{(\alpha,\beta)\longrightarrow(\alpha_0,\beta_0)} T(\alpha,\beta) = +\infty.$$

 (ii) *$T(\cdot,\cdot)$ is continuous at (α,α) for all $\alpha \in \mathbb{R}^n$.*
 (iii) *F and $-F$ are β-STLC for all $\beta \in \mathbb{R}^n$.*

For the Lipschitz continuity of $T(\cdot,\cdot)$ we have the following.

Proposition 2.6. *Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have:*

(a) *If $(\alpha,\beta) \in \mathcal{R}$ then*

$$[-A\alpha \in U \text{ or } -A\beta \in U] \Longrightarrow T(\cdot,\cdot) \text{ is Lipschitz near } (\alpha,\beta).$$

(b) *$-A\alpha \in U \Longleftrightarrow T(\cdot,\cdot)$ is Lipschitz near (α,α).*

(c) *The following assertions are equivalent:*
 (i) *$\mathcal{R}$ is open and $T(\cdot,\cdot)$ is locally Lipschitz in $\mathcal{R}$.*
 (ii) *$T(\cdot,\cdot)$ is Lipschitz near (α,α) for all $\alpha \in \mathbb{R}^n$.*
 (iii) *$-A\beta \in U$ for all $\beta \in \mathbb{R}^n$.*

3. Proximal subdifferential formula

The main result of this section is the following theorem which gives a formula for the proximal subdifferential of $T(\cdot,\cdot)$ under the same assumptions of Theorem 1.2.

Theorem 3.1. *Assume that F and $-F$ are small-time locally controllable near α or β, where $(\alpha,\beta) \in \mathcal{R}$ with $T(\alpha,\beta) = r > 0$. Then a vector (ζ,θ) in $\mathbb{R}^n \times \mathbb{R}^n$ belongs to $\partial_P T(\alpha,\beta)$ if and only if the following assertions are satisfied:*

(i) $\langle A\alpha,\zeta\rangle + h_U(\zeta) = \langle A\beta,-\theta\rangle + h_U(-\theta) = -1.$

(ii) *$\zeta = -e^{A^T r}\theta$ and $\theta \in N_{\mathcal{A}_+^\alpha(r)}(\beta)$.*

Before giving the proof of the preceding theorem, we provide some technical lemmas. We note that a version of Lemma 3.4 already appeared in [15, Section 4.2]. We will repeat the proof here, in order to make the paper more self-contained.

Lemma 3.2. *Assume that F is small-time locally controllable near α or β, where $(\alpha,\beta) \in \mathcal{R}$ with $T(\alpha,\beta) = r > 0$. Then $\bar{\mathcal{R}}(r)$ is convex near (α,β) and hence $(\zeta,\theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$ if and only if*

$$\langle(\zeta,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle \leq 0 \text{ for all } (\alpha',\beta') \in \bar{\mathcal{R}}(r) \text{ and near } (\alpha,\beta).$$

Proof. Let $(\alpha, \beta) \in \mathcal{R}$ such that $T(\alpha, \beta) = r > 0$ and assume that F is small-time locally controllable near α (the case F is small-time locally controllable near β follows by a similar argument, we only need to interchange the roles of α and β). Then there exists $\rho > 0$ such that F is small-time locally controllable at α' for all $\alpha' \in B(\alpha; \rho)$. Now let $\lambda \in [0, 1]$ and let (α', β') and (α'', β'') in $\bar{\mathcal{R}}(r) \cap B((\alpha, \beta); \rho)$ and. We claim that

$$\lambda(\alpha', \beta') + (1 - \lambda)(\alpha'', \beta'') = (\lambda\alpha' + (1 - \lambda)\alpha'', \lambda\beta' + (1 - \lambda)\beta'') \in \bar{\mathcal{R}}(r).$$

Indeed, since (α', β') and (α'', β'') in $\bar{\mathcal{R}}(r)$ there exist two trajectories x' and x''' of F such that

$$x'(0) = \alpha', \quad x'(T(\alpha', \beta')) = \beta', \quad x''(0) = \alpha' \quad \text{and} \quad x''(T(\alpha'', \beta'')) = \beta''.$$

Let $\bar{T} := \max\{T(\alpha', \beta'), T(\alpha'', \beta'')\} \leq r$. Since F is small-time locally controllable at α' and α'', we get from Remark 2.4 that $0 \in F(\alpha')$ and $0 \in F(\alpha'')$. Therefore, we can extend the two trajectories x' and x'' to $[0, \bar{T}]$ and obtain two trajectories $\bar{x}'$ and $\bar{x}''$ of F satisfying

$$\bar{x}'(0) = \alpha', \quad \bar{x}'(\bar{T}) = \beta', \quad \bar{x}''(0) = \alpha' \quad \text{and} \quad x''(\bar{T}) = \beta''.$$

The convexity of U gives that $y := \lambda\bar{x}' + (1 - \lambda)\bar{x}''$ is a trajectory of F over $[0, \bar{T}]$ which satisfies

$$y(0) = \lambda\alpha' + (1 - \lambda)\alpha'' \quad \text{and} \quad y(\bar{T}) = \lambda\beta' + (1 - \lambda)\beta''.$$

Hence $T(\lambda\alpha' + (1 - \lambda)\alpha'', \lambda\beta' + (1 - \lambda)\beta'') \leq \bar{T} \leq r$. This proves the claim and then the convexity of $\bar{\mathcal{R}}(r) \cap B((\alpha, \beta); \rho)$. Now to prove the hence part of Lemma 3.2, it is sufficient to use the convexity of $\bar{\mathcal{R}}(r) \cap B((\alpha, \beta); \rho)$ combined with the proximal normal inequality. The details are left to the reader. $\square$

Lemma 3.3. *Assume that F is small-time locally controllable near α or β, where $(\alpha, \beta) \in \mathcal{R}$ with $T(\alpha, \beta) = r > 0$. Then*

$$N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta) = \{(-e^{A^T r}\theta, \theta) : \theta \in N_{\mathcal{A}^\alpha_+(r)}(\beta)\}.$$

Proof. We begin by the inclusion $\{(-e^{A^T r}\theta, \theta) : \theta \in N_{\mathcal{A}^\alpha_+(r)}(\beta)\} \subset N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$. Let $\theta \in N_{\mathcal{A}^\alpha_+(r)}(\beta)$. Then we have

$$\langle \theta, \beta' - \beta \rangle \leq 0 \ \text{ for all } \beta' \in \mathcal{A}^\alpha_+(r). \tag{9}$$

Now let ρ be as in the proof of Lemma 3.2 and let $(\alpha', \beta') \in \bar{\mathcal{R}}(r) \cap B((\alpha, \beta); \rho)$. Since $T(\alpha', \beta') \leq r$, $T(\alpha, \beta) = r$ and $0 \in F(\alpha')$, we get the existence of two admissible controls $u(\cdot)$ and $u'(\cdot)$ such that

$$\beta = e^{Ar}\alpha + \int_0^r e^{A(r-\tau)}u(\tau)\, d\tau \quad \text{and} \quad \beta' = e^{Ar}\alpha' + \int_0^r e^{A(r-\tau)}u'(\tau)\, d\tau.$$

We have

$$\begin{aligned}\langle(-e^{A^Tr}\theta,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle &= \langle -e^{A^Tr}\theta,\alpha'-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\ &= \langle -\theta,e^{Ar}(\alpha'-\alpha)\rangle+\langle\theta,\beta'-\beta\rangle\\ &= \langle\theta,(\beta'-e^{Ar}\alpha')-(\beta-e^{Ar}\alpha)\rangle\\ &= \left\langle\theta,\int_0^r e^{A(r-\tau)}u'(\tau)\,d\tau-\int_0^r e^{A(r-\tau)}u(\tau)\,d\tau\right\rangle\\ &= \left\langle\theta,e^{Ar}\alpha+\int_0^r e^{A(r-\tau)}u'(\tau)\,d\tau-\beta\right\rangle\le 0,\end{aligned}$$

where the last inequality holds by (9) and since $e^{Ar}\alpha+\int_0^r e^{A(r-\tau)}u'(\tau)\,d\tau\in\mathcal{A}_+^\alpha(r)$. This completes, due to Lemma 3.2, the proof of the desired inclusion. We proceed to prove the opposite inclusion, that is, $N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)\subset\{(-e^{A^Tr}\theta,\theta):\theta\in N_{\mathcal{A}_+^\alpha(r)}(\beta)\}$. Let $(\zeta,\theta)\in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$. Then by Lemma 3.2, there exists $\rho>0$ such that

$$\langle(\zeta,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle\le 0\ \text{ for all } (\alpha',\beta')\in\bar{\mathcal{R}}(r)\cap B((\alpha,\beta);\rho). \tag{10}$$

Since $\{\alpha\}\times\mathcal{A}_+^\alpha(r)\subset\bar{\mathcal{R}}(r)$, we get that

$$\langle\theta,\beta'-\beta\rangle\le 0\ \text{ for all } \beta'\in\mathcal{A}_+^\alpha(r)\cap B(\beta;\rho),$$

which proves that $\theta\in N^P_{\mathcal{A}_+^\alpha(r)}(\beta)=N_{\mathcal{A}_+^\alpha(r)}(\beta)$. Now we prove that $\zeta=-e^{A^Tr}\theta$. Let $0<\varepsilon<\frac{\rho}{1+\|e^{Ar}\|}$ and let $\alpha'\in B(\alpha;\varepsilon)$. Since $T(\alpha,\beta)=r$, we get the existence of an admissible control $u(\cdot)$ such that

$$\beta=e^{Ar}\alpha+\int_0^r e^{A(r-\tau)}u(\tau)\,d\tau.$$

We define

$$\beta':=e^{Ar}\alpha'+\int_0^r e^{A(r-\tau)}u(\tau)\,d\tau.$$

Clearly $(\alpha',\beta')\in\bar{\mathcal{R}}(r)$. Moreover,

$$\begin{aligned}\|(\alpha',\beta')-(\alpha,\beta)\| &\le \|\alpha'-\alpha\|+\|\beta-\beta'\|\\ &\le (1+\|e^{Ar}\|)\|\alpha'-\alpha\|\\ &\le (1+\|e^{Ar}\|)\varepsilon\\ &< \rho.\end{aligned}$$

Hence by (10) we get that

$$\begin{aligned}\langle(\zeta,\theta),(\alpha',\beta')-(\alpha,\beta)\rangle &= \langle\zeta,\alpha'-\alpha\rangle+\langle\theta,\beta'-\beta\rangle\\ &= \langle\zeta,\alpha'-\alpha\rangle+\langle\theta,e^{Ar}(\alpha'-\alpha)\rangle\\ &= \langle\zeta+e^{A^Tr}\theta,\alpha'-\alpha\rangle\\ &\le 0.\end{aligned}$$

Since the last inequality holds for all $\alpha' \in B(\alpha;\varepsilon)$, we get that $\zeta + e^{A^T r}\theta = 0$. This completes the proof of the opposite inclusion, and then the proof of the lemma. $\square$

Lemma 3.4. *We have the following:*

(*a*) *Let $\alpha \in \mathbb{R}^n$. Then a vector (ζ,θ) in $\mathbb{R}^n \times \mathbb{R}^n$ belongs to $\partial_P T(\alpha,\beta)$ if and only if the following assertions are satisfied:*
   (*i*) $\theta = -\zeta$.
   (*ii*) $\langle A\alpha,\zeta\rangle + h_U(\zeta) \geq -1$.

(*b*) *Let $(\alpha,\beta) \in \mathcal{R}$ such that $T(\alpha,\beta) = r > 0$. Then a vector (ζ,θ) in $\mathbb{R}^n \times \mathbb{R}^n$ belongs to $\partial_P T(\alpha,\beta)$ if and only if the following assertions are satisfied:*
   (*i*) $\langle A\alpha,\zeta\rangle + h_U(\zeta) = \langle A\beta,-\theta\rangle + h_U(-\theta) = -1$.
   (*ii*) $(\zeta,\theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha,\beta)$.

Proof. (*a*) Suppose $\alpha \in \mathbb{R}^n$ and $(\zeta,\theta) \in \partial_P T(\alpha,\alpha)$. Then there exist $\sigma,\nu > 0$ such that for all $(\alpha',\beta') \in B((\alpha,\alpha);\nu)$ we have

$$T(\alpha',\beta') \geq -\sigma\|(\alpha'-\alpha,\beta'-\alpha)\|^2 + \langle(\zeta,\theta),(\alpha'-\alpha,\beta'-\alpha)\rangle.$$

We take $\alpha' = \beta'$ and we get that for all $\alpha' \in B(\alpha;\nu)$

$$0 \geq -\sigma\|(\alpha'-\alpha,\alpha'-\alpha)\|^2 + \langle(\zeta,\theta),(\alpha'-\alpha,\alpha'-\alpha)\rangle.$$

Let $v \in \mathbb{R}^n$ and $\alpha_n = \alpha + \frac{v}{n}$ for all $n \in \mathbb{N}^*$. There exists n_0 such that for $n \geq n_0$ we have

$$0 \geq \frac{-\sigma}{n}\|(v,v)\|^2 + \langle(\zeta,\theta),(v,v)\rangle.$$

Since v is an arbitrary vector in $\mathbb{R}^n$, we get that $\theta = -\zeta$.
It is well-known, see [23, Theorem 3.2], that for $\zeta \in \partial_P T(\cdot,\alpha)(\alpha)$ we have

$$\langle A\alpha,\zeta\rangle + h_U(\zeta) \geq -1.$$

But one can easily prove that

$$\partial_P T(\alpha,\alpha) \subset \partial_P T(\cdot,\alpha)(\alpha) \times \partial_P T(\alpha,\cdot)(\alpha).$$

Hence $h(\alpha,\zeta) \geq -1$. This terminates the proof of the first implication. For the converse one, suppose now that $\zeta \in \mathbb{R}^n$ and $\langle A\alpha,\zeta\rangle + h_U(\zeta) \geq -1$. We will show that $(\zeta,-\zeta) \in \partial_P T(\alpha,\alpha)$. Suppose the contrary, then there exists a sequence (α_n,β_n) in $\mathbb{R}^n \times \mathbb{R}^n$ such that for all $n \in \mathbb{N}^*$ we have

$$(\alpha_n,\beta_n) \neq (\alpha,\alpha), \qquad (\alpha_n,\beta_n) \longrightarrow (\alpha,\alpha) \quad \text{and}$$

$$T_n = T(\alpha_n,\beta_n) < -n\|(\alpha_n-\alpha,\beta_n-\alpha)\|^2 + \langle(\zeta,-\zeta),(\alpha_n-\alpha,\beta_n-\alpha)\rangle. \tag{11}$$

Then we have

$$0 < T_n < 2\|\zeta\|\|(\alpha_n-\alpha,\beta_n-\alpha)\|. \tag{12}$$

Since $T_n < +\infty$ there exists a trajectory x_n of F on $[0,+\infty[$ such that $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$. Therefore

$$\beta_n - \alpha_n = \int_0^{T_n} \dot{x}_n(t)\,dt. \tag{13}$$

Let $p_n(t) := \mathrm{proj}_{F(\alpha)}(\dot{x}_n(t))$, then since $\langle A\alpha, \zeta\rangle + h_U(\zeta) \geq -1$ we have

$$\int_0^{T_n} \langle \zeta, p_n(t)\rangle \, dt > -T_n. \tag{14}$$

Since T_n is bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}^*$ and for all $t \in [0, T_n]$ we have

$$\|x_n(t) - \alpha\| \leq \|x_n(t) - \alpha_n\| + \|\alpha_n - \alpha\| \leq MT_n + \|\alpha_n - \alpha\|,$$

and then

$$\|x_n(t) - \alpha\| \leq MT_n + \|(\alpha_n - \alpha, \beta_n - \alpha)\|. \tag{15}$$

Moreover, the linearity of F gives that

$$\left\langle \zeta, \int_0^{T_n} (\dot{x}_n(t) - p_n(t))\, dt \right\rangle \geq -\|A\|\|\zeta\| \int_0^{T_n} \|x_n(t) - \alpha\| \, dt. \tag{16}$$

Using (12), (15) and (16) we can prove the existence of $K > 0$ such that

$$\left\langle \zeta, \int_0^{T_n} (\dot{x}_n(t) - p_n(t))\, dt \right\rangle \geq -K\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2. \tag{17}$$

Then by (13) and (14) we get that

$$T_n - \langle \zeta, \alpha_n - \beta_n\rangle \geq -K\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2,$$

and this contradicts (11) since $\langle (\zeta, -\zeta), (\alpha_n - \alpha, \beta_n - \alpha)\rangle = \langle \zeta, \alpha_n - \beta_n\rangle$. This finishes the proof of (a).

(b) Let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$, with $T(\alpha, \beta) = r > 0$, and let $(\zeta, \theta) \in \partial_P T(\alpha, \beta)$. Then there exists $\sigma > 0$ and $\delta > 0$ such that for all $(\alpha', \beta') \in B((\alpha, \beta); \delta)$ we have

$$T(\alpha', \beta') \geq r - \sigma\|(\alpha' - \alpha, \beta' - \beta)\|^2 + \langle (\zeta, \theta), (\alpha' - \alpha, \beta' - \beta)\rangle.$$

Hence by taking $(\alpha', \beta') \in \bar{\mathcal{R}}(r) \cup B((\alpha, \beta); \delta)$ we get that

$$0 \geq -\sigma\|(\alpha' - \alpha, \beta' - \beta)\|^2 + \langle (\zeta, \theta), (\alpha' - \alpha, \beta' - \beta)\rangle,$$

which gives $(\zeta, \theta) \in N^P_{\bar{\mathcal{R}}(r)}(\alpha, \beta)$. On the other hand, it is well-known, see [23, Theorem 3.2], that for $\zeta \in \partial_P T(\cdot, \beta)(\alpha)$ and $\alpha \neq \beta$, we have $\langle A\alpha, \zeta\rangle + h_U(\zeta) = -1$. Then the fact that $\partial_P T(\alpha, \beta) \subset \partial_P T(\cdot, \beta)(\alpha) \times \partial_P T(\alpha, \cdot)(\beta)$, gives that

$$\langle A\alpha, \zeta\rangle + h_U(\zeta) = \langle A\beta, -\theta\rangle + h_U(-\theta) = -1,$$

which finishes the proof of the first implication. For the converse one, we proceed exactly as in (a). The details are left for the reader. $\square$

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. The proof follows directly from the part (b) of Lemma 3.4 combined with Lemma 3.3. $\square$

4. Proof of our main result

The goal of this section is to prove the main result of this paper, that is, Theorem 1.2. Our proof relies on the characterization of the proximal subgradient of $T(\cdot,\cdot)$ given in Section 3, the Pontryagin's maximum principle and some techniques from nonsmooth analysis.

Proof of Theorem 1.2. Let $(\alpha,\beta) \in \mathcal{R}$ such that $T(\alpha,\beta) = r > 0$ and assume that F and $-F$ are small-time locally controllable near α. We need to prove the existence of an open neighborhood O of (α,β) and a constant $\varphi_0 \geq 0$ such that the epigraph of $T_{|O}$ is φ_0-convex. Since F and $-F$ are small-time locally controllable near α, we have the existence of $\delta > 0$ such that F and $-F$ are α'-STLC for all $\alpha' \in B(\alpha,\delta)$. Then by Remark 2.4 and Proposition 2.5, we get that:

- $\mathcal{R}_-^\alpha \times \mathcal{R}_+^\alpha$ is open.
- $T(\cdot,\cdot)$ is continuous at (α',β) for all $\alpha' \in B(\alpha;\delta)$ and $\beta \in \mathcal{R}_+^{\alpha'}$.

But $(\alpha,\beta) \in \mathcal{R}_-^\alpha \times \mathcal{R}_+^\alpha$, then there exists $0 < \gamma < \delta$ such that $B((\alpha;\beta);\gamma) \subset \mathcal{R}$. We define $O := B((\alpha,\beta);\gamma)$. We claim that $T(\cdot,\cdot)$ is continuous in O. Indeed, for $(\alpha',\beta') \in O$, we have $\alpha' \in B(\alpha;\gamma) \subset B(\alpha;\delta)$ and $\beta' \in \mathcal{R}_+^{\alpha'}$. Hence, $T(\cdot,\cdot)$ is continuous at (α',β') and therefore continuous in O. Since $T(\alpha,\beta) = r > 0$, we can assume that $T(\cdot,\cdot)$ is bounded and does not vanish in O. Therefore, we have the following:

- F and $-F$ are α'-STLC for all $(\alpha',\beta') \in O$.
- There exists $M > 0$ such that $0 < T(\alpha',\beta') \leq M$ for all $(\alpha',\beta') \in O$.
- $T(\cdot,\cdot)$ is continuous in O.

Now by Lemma 2.2, we deduce that to prove the φ_0-convexity of the epigraph of $T_{|O}$ it is sufficient to prove the existence of a constant $\varphi_0 \geq 0$ such that for (α',β') and (α'',β'') in O we have, for $r' := T(\alpha',\beta')$ and $r'' := T(\alpha'',\beta'')$, that

$$\left\langle \frac{((\zeta,\theta),s)}{\|((\zeta,\theta),s)\|}, ((\alpha'',\beta''),r'') - ((\alpha',\beta'),r') \right\rangle \leq \varphi_0 \|((\alpha'',\beta''),r'') - ((\alpha',\beta'),r')\|^2,$$

for all $0 \neq ((\zeta,\theta),s) \in N^P_{\operatorname{epi} T}((\alpha',\beta'),r')$.
Let (α',β'), $(\alpha'',\beta'') \in O$ and let $0 \neq ((\zeta,\theta),s) \in N^P_{\operatorname{epi} T}((\alpha',\beta'),r')$.

Lemma 4.1. *$\zeta = -e^{A^T r'}\theta$ and $\theta \in N_{\mathcal{A}_+^{\alpha'}(r')}(\beta')$.*

Proof. To prove the lemma, we consider the following two cases:

Case 1: $s < 0$. Then the vector $\frac{-(\zeta,\theta)}{s} \in \partial_P T(\alpha',\beta')$. By Theorem 3.1, we get that $\frac{-\zeta}{s} = -e^{A^T r'}\left(\frac{-\theta}{s}\right)$ and $\frac{-\theta}{s} \in N_{\mathcal{A}_+^{\alpha'}(r')}(\beta')$. Therefore,

$$\zeta = -e^{A^T r'}\theta \quad \text{and} \quad \theta \in N_{\mathcal{A}_+^{\alpha'}(r')}(\beta').$$

Case 2: $s = 0$. By the proximal normal inequality, there exists $\sigma > 0$ such that

$$\langle (\zeta,\theta), (\alpha''',\beta''') - (\alpha',\beta') \rangle \leq \sigma \|((\alpha''',\beta'''),\rho) - ((\alpha',\beta'),r')\|^2,$$

for all $(\alpha''', \beta''') \in O$ and $\rho > T(\alpha''', \beta''')$. Taking $(\alpha''', \beta''') \in O \cap \bar{\mathcal{R}}(r')$ and $\rho = r'$, we get that

$$(\zeta, \theta) \in N^P_{\bar{\mathcal{R}}(r')}(\alpha', \beta').$$

Now using Lemma 3.3, we get that $\zeta = -e^{A^T r'}\theta$ and $\theta \in N_{\mathcal{A}_+^{\alpha'}(r')}(\beta')$. $\square$

We proceed with the proof of Theorem 3.1 and we invoke Pontryagin's maximum principle, see [13], to get that for $u'(\cdot)$ an admissible control steering α' to β' we have

$$\langle \theta(t), u'(t) \rangle = \max_{u \in U} \langle \theta(t), u \rangle \quad \text{a.e. } t, \tag{18}$$

where $\theta(t) = e^{A^T(r'-t)}\theta$. By Lemma 2.2, we have $\zeta = -\theta(0)$. Then we have:

$$\begin{aligned}
&\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\
&= \langle (\zeta, \theta), (\alpha'' - \alpha', \beta'' - \beta') \rangle + s(r'' - r') \\
&= \langle (-\theta(0), \theta(r')), (\alpha'' - \alpha', \beta'' - \beta') \rangle + s(r'' - r') \\
&= \langle -\theta(0), \alpha'' - \alpha' \rangle + \langle \theta(r'), \beta'' - \beta' \rangle) + s(r'' - r').
\end{aligned} \tag{19}$$

Lemma 4.2. *Assume that $r'' \leq r'$, then there exists $\varphi_0 \geq 0$ such that*

$$\begin{aligned}
&\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\
&\leq \varphi_0 \|\theta\| (\|\alpha'' - \alpha'\|^2 + \|r'' - r'\|^2) + (r' - r'')(h_U(-\theta(0)) + \langle A\alpha', -\theta(0) \rangle - s).
\end{aligned}$$

Proof. By (19) we have:

$$\begin{aligned}
&\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\
&= \langle -\theta(0), \alpha'' - \alpha' \rangle + \langle \theta(r'), \beta'' - \beta' \rangle) + s(r'' - r').
\end{aligned}$$

We write

$$\begin{aligned}
&\langle -\theta(0), \alpha'' - \alpha' \rangle \\
&= \langle \theta(r' - r'') - \theta(0), \alpha'' - \alpha' \rangle \\
&\quad + \langle -\theta(r' - r''), \alpha'' - x(r' - r'') \rangle + \langle -\theta(r' - r''), x(r' - r'') - \alpha' \rangle,
\end{aligned} \tag{20}$$

where $x(\cdot)$ is a trajectory that realizes the minimum time between α' and β', that is,

$$x(t) := e^{At}\alpha' + \int_0^t e^{A(t-\tau)} u'(\tau)\, d\tau.$$

We have:

$$\langle \theta(r' - r'') - \theta(0), \alpha'' - \alpha' \rangle = \langle (e^{A^T r''} - e^{A^T r'})\theta, \alpha'' - \alpha' \rangle \tag{21}$$

$$\leq k_1 \|\theta\| (\|\alpha'' - \alpha'\|^2 + \|r'' - r'\|^2), \tag{22}$$

where k_1 is a positive constant that depends on A and M. On the other hand, we have:

$$
\begin{aligned}
&\langle -\theta(r'-r''), \alpha'' - x(r'-r'')\rangle \\
&= \left\langle -e^{A^T r''}\theta, \alpha'' - e^{A(r'-r'')}\alpha' - \int_0^{r'-r''} e^{A(r'-r''-\tau)} u'(\tau)\, d\tau \right\rangle \\
&= \left\langle -e^{A^T r''}\theta, \alpha'' - e^{-Ar''}\beta' + \int_{r'-r''}^{r'} e^{A(r'-r''-\tau)} u'(\tau)\, d\tau \right\rangle \\
&= \left\langle -e^{A^T r''}\theta, e^{-Ar''}(\beta''-\beta') + \int_{r'-r''}^{r'} e^{A(r'-r''-\tau)} (u'(\tau) - u''(\tau - r' + r''))\, d\tau \right\rangle \\
&= \left\langle -\theta, (\beta''-\beta') + \int_{r'-r''}^{r'} e^{A(r'-\tau)} (u'(\tau) - u''(\tau - r' + r''))\, d\tau \right\rangle \\
&= \langle -\theta, \beta''-\beta'\rangle + \int_{r'-r''}^{r'} \langle \theta(\tau), (u''(\tau) - u'(\tau - r' + r''))\rangle\, d\tau \\
&= \langle -\theta(r'), \beta''-\beta'\rangle + \int_{r'-r''}^{r'} \langle \theta(\tau), (u''(\tau) - u'(\tau - r' + r''))\rangle\, d\tau \\
&\le \langle -\theta(r'), \beta''-\beta'\rangle,
\end{aligned} \tag{23}
$$

where $u''(\cdot)$ is an admissible control steering α'' to β'' and where the last inequality holds due to (18). We continue with the proof of the lemma and we remark that

$$
\begin{aligned}
&\langle -\theta(r'-r''), x(r'-r'') - \alpha'\rangle \\
&= \left\langle -\theta(r'-r''), \int_0^{r'-r''} (Ax(t) + u'(t))\, dt \right\rangle \\
&= \int_0^{r'-r''} \langle -\theta(r'-r''), Ax(t) + u'(t)\rangle\, dt \\
&= \int_0^{r'-r''} \langle \theta(t) - \theta(r'-r''), Ax(t) + u'(t)\rangle\, dt + \int_0^{r'-r''} \langle \theta(0) - \theta(t), Ax(t)\rangle\, dt \\
&\qquad + \int_0^{r'-r''} \langle -\theta(0), A(x(t) - x(0))\rangle\, dt + \int_0^{r'-r''} \langle -\theta(0), A\alpha'\rangle\, dt \\
&\qquad + \int_0^{r'-r''} \langle \theta(0) - \theta(t), u'(t)\rangle\, dt + \int_0^{r'-r''} \langle -\theta(0), u'(t)\rangle\, dt.
\end{aligned}
$$

One can easily prove the existence of a positive constant k_2 depending on A, α, γ, M and U such that for $t \in [0, r'-r'']$ we have:

- $\langle \theta(t) - \theta(r'-r''), Ax(t) + u'(t)\rangle \le k_2\|\theta\|(r'-r'')$,
- $\langle \theta(0) - \theta(t), Ax(t)\rangle \le k_2\|\theta\|(r'-r'')$,
- $\langle -\theta(0), A(x(t) - x(0))\rangle \le k_2\|\theta\|(r'-r'')$ and

- $\langle \theta(0) - \theta(t), u'(t) \rangle \leq k_2 \|\theta\| (r' - r'')$.

Then

$$\begin{aligned} &\langle -\theta(r' - r''), x(r' - r'') - \alpha' \rangle \\ &\leq 4k_2 \|\theta\| (r' - r'')^2 + (r' - r'')(h_U(-\theta(0)) + \langle A\alpha', -\theta(0) \rangle. \end{aligned}$$

Now if we take $\varphi_0 := k_1 + 4k_2$ and we combine this inequality with (19), (20), (22) and (23), then we obtain that:

$$\begin{aligned} &\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\ &\leq \varphi_0 \|\theta\| (\|\alpha'' - \alpha'\|^2 + \|r'' - r'\|^2) + (r' - r'')(h_U(-\theta(0)) + \langle A\alpha', -\theta(0) \rangle - s). \end{aligned}$$

This terminates the proof of Lemma 4.2. $\square$

Now we terminate the proof of Theorem 1.2. First we note that a similar version of the preceding lemma also holds for $r'' > r'$. Indeed, it is sufficient to apply the following substitution technique on $\theta(\cdot)$ and $u'(\cdot)$ and then proceed as in the proof of Lemma 4.2 (the details are left to the reader):

- Define $\bar{\theta}(t) := e^{A^T(r''-t)}\theta$ for all $t \in [0, r'']$.
- Define $\bar{u}'(\cdot)$ on $[0, r'']$ by:
  - $\bar{u}'(t)$ realizes the maximum of $\max_{u \in U} \langle \bar{\theta}(t), u \rangle$ if $t \in [0, r'' - r']$,
  - $\bar{u}'(t) = u'(t - (r'' - r'))$ if $t \in]r'' - r', r'']$.

We proceed and we consider the following two cases:

Case 1: $s < 0$. Then the vector $\frac{-(\zeta,\theta)}{s} \in \partial_P T(\alpha', \beta')$. By Theorem 3.1, we get that

$$\left\langle A\alpha', \frac{-\zeta}{s} \right\rangle + h_U\left(\frac{-\zeta}{s}\right) = -1.$$

Hence

$$\langle A\alpha', -\theta(0)) \rangle + h_U(-\theta(0)) - s = 0,$$

and this gives with Lemma 4.2 that

$$\begin{aligned} &\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\ &\leq \varphi_0 \|\theta\| (\|\alpha'' - \alpha'\|^2 + \|r'' - r'\|^2) \\ &\leq \varphi_0 \|((\zeta, \theta), s)\| \|((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\|^2. \end{aligned}$$

Case 2: $s = 0$. Then by Lemma 4.2 we get that

$$\begin{aligned} &\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\ &\leq \varphi_0 \|\theta\| (\|\alpha'' - \alpha'\|^2 + \|r'' - r'\|^2) + (r' - r'')(h_U(-\theta(0)) + \langle A\alpha', -\theta(0) \rangle). \end{aligned}$$

Since F is α'-STLC we have, see Remark 2.4, that $h_U(-\theta(0)) + \langle A\alpha', -\theta(0) \rangle \leq 0$. Hence

$$\begin{aligned} &\langle ((\zeta, \theta), s), ((\alpha'', \beta''), r'') - ((\alpha', \beta'), r') \rangle \\ &\leq \varphi_0 \|\theta\| (\|\alpha'' - \alpha'\|^2 + \|r'' - r'\|^2) + (r' - r'')(h_U(-\theta(0)) + \langle A\alpha', -\theta(0) \rangle) \\ &\leq \varphi_0 \|((\zeta, \theta), s)\| \|((\alpha'', \beta''), r'') - ((\alpha', \beta'), r')\|^2. \end{aligned}$$

The proof of Theorem 1.2 is now completed. $\square$

References

[1] M. Bardi, I. Capuzzo-Dolcetta: Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston (1997).

[2] A. Canino: On p-convex sets and geodesics, J. Differ. Equations 75(1) (1988) 118–157.

[3] P. Cannarsa, H. Frankowska: Interior sphere property of attainable sets and time optimal control problems, ESAIM, Control Optim. Calc. Var. 12 (2006) 350–370.

[4] P. Cannarsa, C. Sinestrari: Convexity properties of the minimum time function, Calc. Var. Partial Differ. Equ. 3 (1995) 273–298.

[5] P. Cannarsa, C. Sinestrari: Semiconcave Functions, Hamilton-Jacobi Equations and Optimal Control, Birkhäuser, Boston (2004).

[6] F. H. Clarke, Yu. Ledyaev, R. Stern, P. Wolenski: Nonsmooth Analysis and Control Theory, Graduate Texts in Mathematics 178, Springer, New York (1998).

[7] F. H. Clarke, C. Nour: The Hamilton-Jacobi equation of minimal time control, J. Convex Analysis 11 (2004) 413–436.

[8] F. H. Clarke, R. Stern, P. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2 (1995) 117–144.

[9] G. Colombo, A. Marigonda: Differentiability properties for a class of non-convex functions, Calc. Var. Partial Differ. Equ. 25 (2005) 1–31.

[10] G. Colombo, A. Marigonda, P. Wolenski: Some new regularity properties for the minimal time function, SIAM J. Control Optim. 44(6) (2006) 2285–2299.

[11] G. Colombo, K. T. Nguyen: On the structure of the minimum time function, SIAM J. Control Optim. 48(7) (2010) 4776–4814.

[12] H. Federer: Curvature measures, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[13] H. Hermes, J. P. LaSalle: Functional Analysis and Time Optimal Control, Academic Press, New York (1969).

[14] K. T. Nguyen: Hypographs satisfying an external sphere condition and the regularity of the minimum time function, J. Math. Anal. Appl. 372(2) (2010) 611–628.

[15] C. Nour: The bilateral minimal time function, J. Convex Analysis 13(1) (2006) 61–80.

[16] C. Nour: Semigeodesics and the minimal time function, ESAIM, Control Optim. Calc. Var. 12(1) (2006) 120–138.

[17] C. Nour, R. J. Stern, J. Takche: Proximal smoothness and the exterior sphere condition, J. Convex Analysis 16(2) (2009) 501–514.

[18] R. A. Poliquin, R. T. Rockafellar: Prox-regular functions in variational analysis, Trans. Amer. Math. Soc. 348, (1996) 1805–1838.

[19] R. A. Poliquin, R. T. Rockafellar, L. Thibault: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[20] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Grundlehren der Mathematischen Wissenschaften 317, Springer, Berlin (1998).

[21] A. S. Shapiro: Existence and differentiability of metric projections in Hilbert spaces, SIAM J. Optim. 4 (1994) 231–259.

[22] C. Sinestrari: Semiconcavity of the value function for exit time problems with nonsmooth target, Commun. Pure Appl. Anal. 3(4) (2004) 757–774.

[23] P. Wolenski, Y. Zhuang: Proximal analysis and the minimal time function, SIAM J. Control Optim. 36 (1998) 1048–1072.