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Two Conditions for a Function to be Convex

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We present two sufficient conditions in order that a real function on a finite-dimensional normed space be convex (Theorems 1.1 and 1.2) and show some consequences of them. In particular, it comes out that a real function f on a finite-dimensional Hilbert space X is convex, provided that f has the property that for each point $y \in X$ and each $\lambda > 0$ the real function

$$X \ni x \to \lambda f(x) + \|x - y\|^2$$

has a unique global minimum.

1. Introduction and results

This paper deals with the following problem: "Let a real function f on a Hilbert space X have the property that for each point $y \in X$ and each $\lambda > 0$ the real function

$$X \ni x \to \lambda f(x) + \|x - y\|^2 \tag{1}$$

has a unique global minimum. Must f be convex?" Our understanding is that there is no complete solution to it. Actually, we know from X. Wang [5], Corollary 3.8, that if X is finite-dimensional a partial answer is "Yes, provided that f is lower semicontinuous". For a general Hilbert space there is a positive partial answer too (see [1], Corollary 5.2), but strong assumptions are made use of, apart from lower semicontinuity of f, concerning the map $(y, \lambda) \to x_{y,\lambda}$, the unique global minimum of function (1). On the other hand, it has been showed by B. Ricceri [3] that, for an infinite-dimensional, separable Hilbert spaces X, a qualified negative answer to the above stated question would yield the existence of some non-convex Chebyshev subset of the space $L^2([0,1], X)$. Thus, the problem appears to be of some interest in the best approximation theory.

Here, we will focus on the finite-dimensional case. Moreover, we will consider an equivalent form of the initially stated problem, namely: "Let a real function f on

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a real Hilbert space X have the property that for each $\alpha > 0$ and each point $z \in X$ the real function

$$X \ni x \to f(x) + \alpha\|x\|^2 + \langle z, x\rangle \tag{2}$$

has a unique global minimum. Must f be convex?"

In this connection we present two general sufficient conditions in order that a real function on a finite-dimensional normed space be convex (Theorems 1.1 and 1.2 are our main results here). Both of them are generalizations of the above-mentioned result by X. Wang [5]. In fact the wider setting of normed spaces, rather than of Hilbert spaces, is considered. Moreover, the role of the higher-order "perturbation" $\|x\|^2$ is played by a more general function satisfying the condition to tend to $+\infty$ as $\|x\| \to +\infty$, that is, either any strictly convex function (Theorem 1.1), or even any lower semicontinuous function (Theorem 1.2). Last but not least, in Theorem 1.1 the lower semicontinuity assumption on f is dropped. Thus, in the finite dimensional case the initial problem is completely solved in positive.

Henceforth, $(X, \|\ \|)$ will be a real normed space of finite dimension n, with dual space $(X', \|\ \|')$.

Theorem 1.1. *Let $f : X \to \mathbb{R}$ be a function satisfying the following assumption:*

(a) *There exist a strictly convex function $H : X \to \mathbb{R}$, with*

$$\lim_{\|x\|\to+\infty} H(x) = +\infty,$$

and a non-empty set $A \subseteq]0, +\infty[$, with $\inf A = 0$, such that for each $\alpha \in A$ and each $\varphi \in X'$ the real function

$$X \ni x \to f(x) + \alpha H(x) + \langle \varphi, x\rangle \tag{3}$$

has a unique global minimum.

Then f is a convex function.

Theorem 1.2. *Let $f : X \to \mathbb{R}$ be a lower semicontinuous function satisfying the following assumption:*

(b) *There exist a lower semicontinuous function $K : X \to \mathbb{R}$, with*

$$\lim_{\|x\|\to+\infty} K(x) = +\infty,$$

and a non-empty set $A \subseteq]0, +\infty[$, with $\inf A = 0$, such that for each $\alpha \in A$ and each $\varphi \in X'$ the real function

$$X \ni x \to f(x) + \alpha K(x) + \langle \varphi, x\rangle \tag{4}$$

has a unique global minimum.

Then f is a convex function.

The proofs of the previous theorems are postponed to Section 3. However, we would like to point out here that our arguments will involve only basic facts of Convex Analysis.

In the subsequent Section 2 we give Theorems 1.1 and 1.2 a more complete form, stating them as necessary and sufficient conditions. Also, we will show there some consequences of them.

2. Remarks and corollaries

An obvious question to ask about Theorems 1.1 and 1.2 is whether conditions (a) and (b) are also necessary in order that f be convex.

We consider first condition (a). After a short reflection, one realizes that if $f : X \to \mathbb{R}$ is a convex function, then it is sufficient to take any strictly convex function $H : X \to \mathbb{R}$ such that

$$\lim_{\|x\|\to+\infty} \frac{H(x)}{\|x\|} = +\infty,$$

in order that function (3) have a a unique global minimum for every choice of $\alpha > 0$ and $\varphi \in X'$. Indeed, function (3) is strictly convex and tends to $+\infty$ as $\|x\| \to +\infty$, thanks to the inequality

$$\begin{aligned} f(x) + \alpha H(x) + \langle \varphi, x\rangle &\geq S(x) + \alpha H(x) + \langle \varphi, x\rangle \\ &\geq c - \|\psi + \varphi\|'\|x\| + \alpha H(x) \quad \forall x \in X, \end{aligned}$$

where

$$X \ni x \to S(x) = c + \langle \psi, x\rangle,$$

$c \in \mathbb{R}$, $\psi \in X'$, is any support function of f (see [4], Theorem B on p. 108). Thus, we see that (a) is a necessary condition in order that f be convex. More precisely, we have the following theorem.

Theorem 2.1. *For any real function f on X the following statements are equivalent.*

(c) *f is a convex function.*

(a) *There exist a strictly convex function $H : X \to \mathbb{R}$, with*

$$\lim_{\|x\|\to+\infty} H(x) = +\infty,$$

and a non-empty set $A \subseteq]0, +\infty[$, with $\inf A = 0$, such that for each $\alpha \in A$ and each $\varphi \in X'$ the real function

$$X \ni x \to f(x) + \alpha H(x) + \langle \varphi, x\rangle \tag{3}$$

has a unique global minimum.

(a_1) *For every strictly convex function $H_1 : X \to \mathbb{R}$, with*

$$\lim_{\|x\|\to+\infty} \frac{H_1(x)}{\|x\|} = +\infty, \tag{5}$$

and every $\varphi \in X'$ the real function

$$X \ni x \to f(x) + H_1(x) + \langle \varphi, x \rangle \tag{6}$$

has a unique global minimum.

Proof. The implication $(c) \Longrightarrow (a_1)$ has just been proved before, the one $(a_1) \Longrightarrow (a)$ is trivial and the remaining one, $(a) \Longrightarrow (c)$, is precisely Theorem 1.1.

Since the composition $x \to H_1(x) = M(\|x\|)$ of any strictly convex, increasing function $M : [0, +\infty[\to \mathbb{R}$ with any strictly convex norm (see [2], p. 342) $\| \ \|$ on X is a strictly convex function on X, we get the following corollary.

Corollary 2.2. *Let $\| \ \|$ be a strictly convex norm on X. For any function $f : X \to \mathbb{R}$ the following statements are equivalent.*

(c) *f is a convex function.*

(n) *There exist an exponent $p > 1$ and and a non-empty set $A \subseteq]0, +\infty[$, with $\inf A = 0$, such that for each $\alpha \in A$ and each $\varphi \in X'$ the real function*

$$X \ni x \to f(x) + \alpha\|x\|^p + \langle \varphi, x \rangle \tag{7}$$

has a unique global minimum.

(n_1) *For every choice of $p > 1$, $\alpha > 0$ and $\varphi \in X'$ the real function* (7) *has a unique global minimum.*

Remark 2.3. In particular (taking $\| \ \|$ to be any Hilbertian norm on X and $p = 2$), we see from the above corollary that for a finite dimensional Hilbert space the answer to the initially asked question is "Yes. Unconditionally."

Remark 2.4. The implication $(c) \Longrightarrow (a_1)$ can also be read in a slightly different way. Denote by $\mathcal{H}$ the class of all functions $H_1 : X \to \mathbb{R}$ such that function (6) has a unique global minimum for every convex function $f : X \to \mathbb{R}$ and every $\varphi \in X'$. Then, the above mentioned implication says that $\mathcal{H}$ includes all strictly convex functions $H_1 : X \to \mathbb{R}$ that satisfy growth condition (5). It is another interesting consequence of Theorem 1.1 that the class $\mathcal{H}$ actually coincide with the class of all strictly convex functions $H_1 : X \to \mathbb{R}$ that satisfy (5).

Corollary 2.5. *A function $H_1 : X \to \mathbb{R}$ belongs to the class $\mathcal{H}$ if and only if it is strictly convex and satisfy condition* (5).

Proof. We have to prove just the "only if" part. Let $H_1 \in \mathcal{H}$ be given. Fix any strictly convex function $f : X \to \mathbb{R}$ such that $\lim_{\|x\| \to +\infty} f(x) = +\infty$. Then $\beta f : X \to \mathbb{R}$, $\beta > 0$, is a convex function too, so the real function

$$X \ni x \to \beta f(x) + H_1(x) + \langle \varphi, x \rangle$$

has a unique global minimum for each $\beta > 0$ and each $\varphi \in X'$. Thus, by Theorem 1.1, H_1 is a convex function. Now we observe that the assumption $H_1 \in \mathcal{H}$, taking the convex function $f = 0$, in particular says that

$$X \ni x \to H_1(x) + \langle \varphi, x \rangle$$

has a unique global minimum for each $\varphi \in X'$. From this we readily see that H_1 is in fact a strictly convex function. Moreover, as a consequence of Proposition 2.6 below, we have that H_1 satisfies (5). This accomplishes the proof.

Proposition 2.6. *Let a function $g : X \to \mathbb{R}$ have the property that for each $\psi \in X'$ the real function*

$$X \ni x \to g(x) + \langle \psi, x \rangle$$

has a finite lower bound. Then

$$\lim_{\|x\| \to +\infty} \frac{g(x)}{\|x\|} = +\infty.$$

Proof. We proceed by contradiction. We assume the existence of some sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \|x_n\| = +\infty$ and that

$$\lim_{n\to\infty} \frac{g(x_n)}{\|x_n\|} = \ell < +\infty.$$

We can also suppose, passing to a subsequence, if necessary, that

$$\left\{ \frac{x_n}{\|x_n\|} \right\}$$

converges to some $z \in X$, $\|z\| = 1$. Then, taking any $\overline{\psi} \in X'$ such that

$$\langle \overline{\psi}, z \rangle < -\ell,$$

we have

$$g(x_n) + \langle \overline{\psi}, x_n \rangle = \|x_n\| \left[\frac{g(x_n)}{\|x_n\|} + \left\langle \overline{\psi}, \frac{x_n}{\|x_n\|} \right\rangle \right] \to -\infty,$$

but this is a contradiction.

Now we consider condition (b). Since $(a) \Longrightarrow (b)$, we have that also (b) is a necessary condition in order that f be convex. On the other hand, it is apparent from Corollary 2.5 that it is not possible to give condition (b) an equivalent "strong" formulation, as we did in Theorem 2.1 for condition (a) by means of (a_1). Thus, we must limit ourselves to restate Theorem 1.2 in the following form.

Theorem 2.7. *In order that a real function f on X be convex, it is necessary and sufficient that f be lower semicontinuous and satisfy condition (b).*

Finally, it is worth to point out that Theorem 1.2 implies the following corollary.

Corollary 2.8. *Let $f : X \to \mathbb{R}$ be a lower semicontinuous function satisfying the following assumption:*

(d) *For each $\psi \in X'$ the real function*

$$X \ni x \to f(x) + \langle \psi, x \rangle$$

has a unique global minimum.

Then f is a strictly convex function.

Proof. We already know from Proposition 2.6 that assumption (d) implies that

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Hence, taking $K = f : X \to \mathbb{R}$, we have that K is a lower semicontinuous function such that

$$\lim_{\|x\|\to+\infty} K(x) = +\infty.$$

Moreover, by assumption (d), we have that for each $\alpha > 0$ and each $\varphi \in X'$ the real function

$$X \ni x \to f(x) + \alpha K(x) + \langle \varphi, x \rangle = (1+\alpha)\left[f(x) + \langle \tfrac{\varphi}{1+\alpha}, x \rangle\right]$$

has a unique global minimum. Thus, by Theorem 1.2, f is a convex function. At this point, assumption (d) guarantees that f actually is a strictly convex function.

Remark 2.9. The following example shows that in Corollary 2.8 the lower semicontinuity assumption on f cannot be erased.

Example 2.10. (*A non convex function f satisfying condition* (d)). Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{R} \setminus \{0,1,2\}, \\ x^2 + 1 & \text{if } x \in \{0, 2\}, \\ 0 & \text{if } x = 1. \end{cases}$$

It is apparent that f is not convex and it follows by elementary calculations that, for each $m \in \mathbb{R}$, the function $x \to f(x) + mx$ has a unique global minimum x_m, namely:

$$x_m = \begin{cases} 1 & \text{if } m \in [-4, 0], \\ -\frac{m}{2} & \text{if } x \in \mathbb{R} \setminus [-4, 0]. \end{cases}$$

3. Proofs

The terminology we will use is quite standard. An *affine functional* on X is a real function ζ on X of the form

$$\zeta(x) = c + \langle \varphi, x \rangle \quad \forall x \in X,$$

where $\varphi \in X'$ and $c \in \mathbb{R}$. An *affine subspace* W of X is the translation of some vector subspace L of X, that is

$$W = y + L$$

for some $y \in X$ (in fact $y \in W$). The vector subspace L is uniquely identified by W and we put, by definition, $\dim W = \dim L$. If W is an affine subspace of X, with $\dim W = m \geq 1$, we say that two subsets W^+, W^- of W constitute a *couple of closed complementary halfspaces* of W provided that they can be written as

$$W^+ = \{w \in W : \langle \varphi, w \rangle \geq k\}, \qquad W^- = \{w \in W : \langle \varphi, w \rangle \leq k\}$$

for some $\varphi \in X'$, φ not constant on W, and some $k \in \mathbb{R}$. In the sequel, for brevity, we will say "couple of complementary halfspaces" instead of "couple of closed complementary halfspaces". Clearly, the intersection $W^+ \cap W^-$ is an affine subspace of dimension $m-1$ and, conversely, given any affine subspace $V \subseteq W$ of dimension $m-1$, there is a unique couple of complementary halfspaces of W, whose intersection is just V.

We need two lemmas.

Lemma 3.1. *In order that a function $g : X \to \mathbb{R}$ be convex, it is necessary and sufficient that the following condition (c_1) be satisfied.*

(c_1) *For each point $\overline{x} \in X$, each one-dimensional affine subspace W_1 of X, that contains $\overline{x}$, and each affine functional ζ on X such that*

$$g(\overline{x}) + \zeta(\overline{x}) = 0,$$

the inequality

$$\max\left\{ \inf_{W_1^+}[g+\zeta]\,,\ \inf_{W_1^-}[g+\zeta] \right\} \geq 0$$

holds true, where W_1^+, W_1^- is the couple of complementary halfspaces of W_1 such that $W_1^+ \cap W_1^- = \{\overline{x}\}$.

Proof. We show that g fails to be convex if and only if the negation of (c_1) occurs to be true, that is:

$(\neg c_1)$ *There exist a one-dimensional affine subspace W_1 of X, a couple W_1^+, W_1^- of complementary halfspaces of W_1 and an affine functional ζ on X such that*

$$g(\overline{x}) + \zeta(\overline{x}) = 0, \tag{8}$$

where $\{\overline{x}\} = W_1^+ \cap W_1^-$, and that

$$\inf_{W_1^+}[g+\zeta] < 0, \qquad \inf_{W_1^-}[g+\zeta] < 0. \tag{9}$$

(As a matter of fact, in the sequel of the paper we will use only the implication "*f not convex* $\Rightarrow (\neg c_1)$".)

If g is not convex, there are $a, b \in X$ and $\overline{t} \in]0,1[$ such that, denoting

$$\overline{x} = (1-\overline{t})a + \overline{t}b,$$

one has the strict inequality

$$g(\overline{x}) > \Gamma := (1-\overline{t})g(a) + \overline{t}g(b).$$

Then, considering the one-dimensional linear subspace

$$L_1 = \{s(b-a) \,:\, s \in \mathbb{R}\},$$

we see that condition $(\neg c_1)$ is satisfied by taking the one dimensional affine subspace $W_1 := \overline{x} + L_1$, the couple of complementary halfspaces

$$W_1^+ := \{\overline{x} + s(b-a) \,:\, s \geq 0\}, \qquad W_1^- := \{\overline{x} + s(b-a) \,:\, s \leq 0\}$$

and the affine functional ζ given by

$$\zeta(x) = -g(\overline{x}) - \langle \psi, x - \overline{x} \rangle \ \ \forall x \in X,$$

where ψ is any linear extension to the whole X of the linear functional

$$L_1 \ni s(b-a) \to s(g(b) - g(a)) \in \mathbb{R}.$$

Indeed, condition (8) is trivially satisfied and, as far as (9) is concerned, we notice that

$$a = \overline{x} - \overline{t}(b-a), \qquad b = \overline{x} + (1-\overline{t})(b-a),$$

so that $a \in W_1^-$, $b \in W_1^+$; moreover, we have

$$\langle \psi, a - \overline{x} \rangle = -\overline{t}[g(b) - g(a)], \qquad \langle \psi, b - \overline{x} \rangle = (1-\overline{t})[g(b) - g(a)],$$

whence it follows

$$g(a) + \zeta(a) = g(b) + \zeta(b) = \Gamma - g(\overline{x}) < 0$$

and therefore inequalities (9) hold true.

Conversely, if condition $(\neg c_1)$ is satisfied, then by virtue of (9) we can take $a \in W_1^-$ and $b \in W_1^+$ in such a way that

$$\zeta(a) + g(a) < 0, \qquad \zeta(b) + g(b) < 0.$$

Since $a \in W_1^-$, $b \in W_1^+$, we have $\overline{x} = (1-\overline{t})a + \overline{t}b$ for some $\overline{t} \in [0,1]$. It follows by (8) that $\zeta + g$ is not a convex function, hence g is not convex too.

Lemma 3.2. *Let a function $h : X \to \mathbb{R}$ satisfy*

$$\inf_X h > -\infty \tag{10}$$

and

$$\lim_{\|x\| \to +\infty} [h(x) + \langle \varphi, x \rangle] = +\infty \ \ \forall \varphi \in X'. \tag{11}$$

Then for any two affine subspaces V, W of X, with $V \subset W$ and $\dim W = (\dim V) + 1$, there is an affine functional χ on X such that

$$\chi(v) = 0 \ \ \forall v \in V,$$

and

$$\inf_{W^+}[h + \chi] = \inf_{W^-}[h + \chi] \ \ \Big[= \inf_W [h + \chi] \Big],$$

where W^+, W^- is the couple of complementary halfspaces of W such that $W^+ \cap W^- = V$.

Proof. As a matter of fact, we will prove that the conclusion of the lemma is fulfilled by a suitable multiple $\chi = \bar{t}\zeta$ of any affine functional ζ on X such that $\zeta(v) = 0$ for every $v \in V$ and $\zeta(\overline{w}) \neq 0$ for some $\overline{w} \in W$, that is any affine functional ζ for which the equality

$$V = \{w \in W \,:\, \zeta(w) = 0\}$$

holds true, so that we can assume, with no loss of generality,

$$W^+ = \{w \in W \,:\, \zeta(w) \geq 0\}\,, \qquad W^- = \{w \in W \,:\, \zeta(w) \leq 0\}\,. \tag{12}$$

We have to prove that the following functions p and q:

$$\mathbb{R} \ni t \to p(t) := \inf_{x \in W^+} [h(x) + t\zeta(x)], \qquad \mathbb{R} \ni t \to q(t) := \inf_{x \in W^-} [h(x) + t\zeta(x)],$$

take the same value at some point $\bar{t} \in \mathbb{R}$. We first notice that these two functions are real-valued. Indeed, if (11) is true, then assumption (10) actually is equivalent to

$$\inf_{x \in X} [h(x) + \langle \varphi, x \rangle] > -\infty \quad \forall \varphi \in X'. \tag{10$'$}$$

Next, having in mind (12), we easily see that p is an increasing function, q is decreasing and we have

$$\lim_{t \to -\infty} p(t) = \lim_{t \to +\infty} q(t) = -\infty.$$

Therefore, the proof will be accomplished once we show that p and q are continuous functions. In fact, we have that p and q are concave functions on $\mathbb{R}$ (this is plainly seen: given any $t_1, t_2 \in \mathbb{R}$ and any $\theta \in [0, 1]$, we have, for each $x \in W^+$,

$$\begin{aligned} h(x) + [(1-\theta)t_1 + \theta t_2]\,\zeta(x) &= (1-\theta)\,[h(x) + t_1\zeta(x)] + \theta\,[h(x) + t_2\zeta(x)] \\ &\geq (1-\theta)p(t_1) + \theta p(t_2) \end{aligned}$$

and consequently

$$p\left((1-\theta)t_1 + \theta t_2\right) \geq (1-\theta)p(t_1) + \theta p(t_2)).$$

Proof of Theorem 1.1. In order to expedite the proof, we introduce the following notation.

Notation. Given any function $\gamma : X \to \mathbb{R}$, we denote by $\Sigma_H(\gamma)$ the set of all numbers $\sigma \in \mathbb{R}$ such that the real function

$$X \ni x \to \gamma(x) + \sigma H(x) + \langle \varphi, x \rangle$$

has a unique global minimum for each $\varphi \in X'$.

Remark 3.3. It is an obvious remark that we have $\Sigma_H(\gamma) = \Sigma_H(\gamma + \zeta)$ for every affine functional ζ on X.

Remark 3.4. Having in mind that $\lim_{\|x\|\to+\infty} H(x) = +\infty$, and hence $\inf_X H > -\infty$, it is easy to see that assumptions (10) and (11) of Lemma 3.2 are fulfilled by any function γ, whose set $\Sigma_H(\gamma)$ contains some negative number.

Now, we come back to the proof of Theorem 1.1. Proceeding by contradiction, we assume that f is not a convex function and fix, as it is possible, a sufficiently small number $\overline{\alpha} \in A$, with $\overline{\alpha} < \sup A$, in such a way that the function $g := f + \overline{\alpha}H$ is not convex either. It readily follows from assumption (a) that for this function g we have

$$A - \overline{\alpha} \subseteq \Sigma_H(g),$$

hence g satisfies the following condition:

$$\textit{The set } \Sigma_H(g) \textit{ contains some } \sigma^+ > 0, \textit{ some } \sigma^- < 0 \textit{ and zero as well.} \tag{13}$$

Since g is not convex, we know after Lemma 3.1 that there exist an affine subspace W_1 of X of dimension one, a couple W_1^+, W_1^- of complementary halfspaces of W_1, with $W_0 := \{\overline{x}\} = W_1^+ \cap W_1^-$, and an affine functional ζ on X, for which conditions (8) and (9) hold. We put $h := g + \zeta$ and, having in mind the above remarks, observe that $\Sigma_H(h) = \Sigma_H(g)$, thus also the function h satisfies condition (13) and, consequently, it satisfies assumptions (10) and (11) of Lemma 3.2 as well. Thus, we can apply Lemma 3.2 to the function h, taking $V = W_0$ and $W = W_1$, and get the existence of an affine functional χ_1 on X such that

$$\chi_1(\overline{x}) = 0, \tag{14}$$

$$\inf_{W_1^+}[h + \chi_1] = \inf_{W_1^-}[h + \chi_1] =: b_1. \tag{15}$$

Taking into account (14), it is not difficult to deduce from (9) that $b_1 < 0$.

Since also $h + \chi_1$ satisfies condition (13), we can apply Lemma 3.2 to this function, taking $V = W_1$ and $W = W_2$, where W_2 is any affine subspace of dimension 2 containing W_1, and so get the existence of an affine functional χ_2 on X such that

$$\chi_2(v) = 0 \quad \forall v \in W_1, \tag{16}$$

$$\inf_{W_2^+}[h + \chi_1 + \chi_2] = \inf_{W_2^-}[h + \chi_1 + \chi_2] =: b_2, \tag{17}$$

where W_2^+, W_2^- is the couple of complementary halfspaces of W_2 such that $W_2^+ \cap W_2^- = W_1$. Having in mind (16), we easily see from (15) that $b_2 \le b_1$. Next, it is possible to apply Lemma 3.2 to $h + \chi_1 + \chi_2$ and get the existence of an affine functional χ_3, and so on.

By iterating the application of Lemma 3.2 we find the affine subspaces

$$W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n,$$

of dimensions $0, 1, 2, \ldots, n$ (hence $W_n = X$), and the affine functionals on X

$$\chi_1, \chi_2, \ldots, \chi_n$$

such that the following two conditions are satisfied for every $i = 1, 2, \ldots, n$:

$$\chi_i(v) = 0 \quad \forall v \in W_{i-1}, \tag{18$_i$}$$

$$\inf_{W_i^+}[h + \chi_1 + \cdots + \chi_i] = \inf_{W_i^-}[h + \chi_1 + \cdots + \chi_i] =: b_i \tag{19$_i$}$$

$$\Big[= \inf_{W_i}[h + \chi_1 + \cdots + \chi_i] \Big],$$

where W_i^+, W_i^- is the couple of complementary halfspaces of W_i such that $W_i^+ \cap W_i^- = W_{i-1}$. By conditions $(18)_i$ and the definitions of the b_i's we have

$$b_n \le b_{n-1} \le \cdots \le b_1 < 0 =: b_0.$$

Let m be the maximum of those indexes $j \in \{1, \ldots, n\}$ for which the strict inequality $b_j < b_{j-1}$ holds, that is the index m such that $b_n = b_m < b_{m-1}$. Then, denoting

$$\ell = h + \chi_1 + \cdots + \chi_n,$$

so that also ℓ satisfies (13), and taking into account, if $m < n$, conditions $(18)_i$, $m < i \le n$, implying that

$$\chi_{m+1}(v) = \cdots = \chi_n(v) = 0 \quad \forall v \in W_m,$$

we have

$$\begin{aligned} \inf_X \ell = b_n = b_m = & \inf_{W_m^+}[h + \chi_1 + \cdots + \chi_m] \\ = & \inf_{W_m^+}[h + \chi_1 + \cdots + \chi_m + \cdots + \chi_n] = \inf_{W_m^+} \ell \end{aligned}$$

and similarly

$$\inf_X \ell = \inf_{W_m^-} \ell;$$

moreover, again by $(18)_i$, $m \le i \le n$,

$$\begin{aligned} \inf_X \ell = b_n < b_{m-1} = & \inf_{W_{m-1}}[h + \chi_1 + \cdots + \chi_{m-1}] \\ = & \inf_{W_{m-1}}[h + \chi_1 + \cdots + \chi_{m-1} + \chi_m + \cdots + \chi_n] = \inf_{W_{m-1}} \ell. \end{aligned}$$

By the previous inequality we are sure that x^*, the unique global minimum point of ℓ, does not belong to $W_{m-1} = W_m^+ \cap W_m^-$. Assume, for instance, that $x^* \notin W_m^+$ and consider a sequence $\{\xi_k\}$ in W_m^+ such that

$$\lim_{k\to\infty} \ell(\xi_k) = \inf_{W_m^+} \ell = \inf_X \ell = \ell(x^*).$$

This sequence is bounded because, owing to condition (13), ℓ satisfies assumption (11), and in particular we have $\lim_{\|x\|\to+\infty} \ell(x) = +\infty$. Hence, taking a subsequence if necessary, we can assume that

$$\lim_{k\to\infty} \xi_k = x^{**} \in W_m^+.$$

Since H is strictly convex, it has strict support at x^{**}, that is there is an affine functional T on X such that

$$H(x^{**}) = T(x^{**})$$

and

$$H(x) > T(x) \quad \forall x \in X \setminus \{x^{**}\}.$$

At this point we get a contradiction taking an element $\overline{\sigma}$ of $\Sigma_H(\ell)$, with $\overline{\sigma} > 0$, and considering the function $\mu := \ell + \overline{\sigma}(H - T)$. This function has a unique global minimum, since $\overline{\sigma} \in \Sigma_H(\ell)$. On the other hand we have

$$\mu(x) > \ell(x^*) \quad \forall x \in X,$$

because

$$\mu(x) = \ell(x) + \overline{\sigma}\,[H(x) - T(x)] > \ell(x) \geq \ell(x^*) \quad \forall x \in X \setminus \{x^{**}\}$$

and

$$\mu(x^{**}) = \ell(x^{**}) > \ell(x^*).$$

Moreover we have

$$\begin{aligned} \lim_{k\to\infty} \mu(\xi_k) &= \lim_{k\to\infty} \ell(\xi_k) + \overline{\sigma} \lim_{k\to\infty} [H(\xi_k) - T(\xi_k)] \\ &= \ell(x^*) + \overline{\sigma}\,[H(x^{**}) - T(x^{**})] = \ell(x^*). \end{aligned}$$

Hence we conclude that μ does not have global minimum and the proof is accomplished.

Proof of Theorem 1.2. We proceed again by contradiction. Just as in the above proof of Theorem 1.1, we construct a real function ℓ on X, satisfying condition (13′) below:

$$\text{\textit{The set } } \Sigma_K(\ell) \text{ \textit{contains some} } \sigma^+ > 0, \text{ \textit{some} } \sigma^- < 0 \text{ \textit{and zero as well}} \tag{13′}$$

(the definition of Σ_K is just like that of Σ_H, with K in place of H; moreover, Remarks 3.3 and 3.4 still hold for Σ_K), and two affine subspaces W_m, W_{m-1} of X, with $W_m \supset W_{m-1}$ and $\dim W_m = \dim W_{m-1} + 1$, such that

$$\inf_X \ell = \inf_{W_m^+} \ell = \inf_{W_m^-} \ell < \inf_{W_{m-1}} \ell,$$

where W_m^+, W_m^- is the couple of complementary halfspaces of W_m such that $W_m^+ \cap W_m^- = W_{m-1}$. Since ℓ is a lower semicontinous function (it is obtained by adding a suitable affine functional on X to a function $g = f + \overline{\alpha}K$, with $\overline{\alpha} > 0$), such that $\lim_{\|x\|\to+\infty} \ell(x) = +\infty$ (owing to condition (13′)), and since W_m^+ is a closed set, there exists a point $x^+ \in W_m^+$ such that

$$\ell(x^+) = \inf_{W_m^+} \ell = \inf_X \ell.$$

Likewise, there is a point point $x^- \in W_m^-$ such that

$$\ell(x^-) = \inf_{W_m^-} \ell = \inf_X \ell.$$

Moreover, since $\inf_X \ell < \inf_{W_{m-1}} \ell$, neither of points x^+ and x^- can stay in $W_{m-1} = W_m^+ \cap W_m^-$, so that $x^+ \neq x^-$, hence ℓ has two distinct global minimum points, but this is a contradiction, because $0 \in \Sigma_K(\ell)$.

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