Journal of Convex Analysis
Volume 21 (2014), No. 1, 53–66

Constant Sign and Sign-Changing Solutions for Quasilinear Elliptic Equations with Neumann Boundary Condition

Giuseppina Barletta*
Università degli Studi Mediterranea di Reggio Calabria,
MECMAT-Dipartimento di Meccanica e Materiali, Via Graziella,
Località Feo di Vito, 89100 Reggio Calabria, Italy
giuseppina.barletta@unirc.it

Pasquale Candito*
Università degli Studi Mediterranea di Reggio Calabria,
MECMAT-Dipartimento di Meccanica e Materiali, Via Graziella,
Località Feo di Vito, 89100 Reggio Calabria, Italy
pasquale.candito@unirc.it

Dumitru Motreanu
Département de Mathématiques, Université de Perpignan,
Avenue Paul Alduy 52, 66860 Perpignan, France
motreanu@univ-perp.fr

Received: July 16, 2012
Revised manuscript received: October 24, 2012
Accepted: February 6, 2013

Through variational methods, sub-supersolution and truncation techniques we prove the existence of three nontrivial solutions for a quasilinear elliptic equation with Neumann boundary condition. We provide sign information for each of these solutions: two of them are of opposite constant sign and the third one is sign changing.

Keywords: Quasilinear elliptic equations, sub-supersolution, critical point, Neumann problem, constant sign solutions, sign-changing solutions

2000 Mathematics Subject Classification: 35J62, 49J45

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following quasilinear elliptic equation with Neumann boundary condition

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

*The first and second authors were partially supported by G.N.A.M.P.A.

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

where λ is a positive parameter, $n(x)$ denotes the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$, $\frac{\partial u}{\partial n}$ stands for the normal derivative of u on $\partial\Omega$, $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < \infty$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., measurable in $x \in \Omega$ for all $s \in \mathbb{R}$ and continuous in $s \in \mathbb{R}$ for a.e. $x \in \Omega$). Specifically, we provide verifiable hypotheses on the nonlinearity $f(x, u)$ ensuring the existence of at least three nontrivial solutions of problem (1) provided the parameter λ is sufficiently large, namely $\lambda > \lambda_1$, where λ_1 denotes the second eigenvalue of $-\Delta_p$ on $W^{1,p}(\Omega)$. Moreover, we are able to give precise information on the sign of these solutions: a positive solution, a negative solution, and a sign-changing (or nodal) solution between the opposite constant sign solutions. In addition, we show that a smallest positive solution and a greatest negative solution for problem (1) exist.

Such results are known for quasilinear elliptic equations with Dirichlet boundary condition for which we refer to [4], [5], [7], [14]. In the case of nonlinear Neumann problems, there are many results ensuring the existence of multiple solutions (see [1], [2], [3], [12], [11]), and few results involving the study of the sign of multiple solutions, for instance, we refer to [13] and [19] for the special case of jumping nonlinearities, to [16] for a non parametric problem, to [17] for the case of boundary conditions involving the eigenvalues of the Steklov problem and to [18] for set-valued nonlinearities. The existence of multiple solutions with a complete study of their sign for problem (1) is performed here for the first time. Our approach relies on variational methods combined with sub-supersolution and truncation techniques. We emphasize that we do not impose any growth condition for the nonlinearity $f(x, u)$ in problem (1). The reason for this is that we are dealing in our reasonings only with truncated functionals for which the subcritical growth condition is automatically valid. The treatment parallels the study already done in the case of nonlinear Dirichlet problems taking advantage of various ideas and techniques developed therein, but to adapt them to the Neumann case is not at all a routine matter. There are essential differences between the results and tools used for the Dirichlet and Neumann boundary value problems related to the functional setting, spectral properties of the negative p-Laplacian, sub-supersolution method, variational approach. We have been able to overcome the technical difficulties arisen with the new aspects encountered in our Neumann problem (1) with respect to the corresponding Dirichlet problem thanks to very recent contributions on Neumann problems involving the p-Laplacian (see [1], [3], [12], [11])).

The rest of the paper is organized as follows. Section 2 is devoted to the needed mathematical background. Section 3 presents our results on the existence of the smallest positive solution and the greatest negative solution. Section 4 contains our main result establishing the existence of a third nontrivial solution which is shown to be sign-changing.

2. Mathematical Background

In the analysis of problem (1), we will use the classical Banach spaces $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$ as well as the ordered cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : \ u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}$$

whose interior is nonempty and given by

$$\operatorname{int} C_+ = \{u \in C_+ : \ u(x) > 0 \text{ for all } x \in \overline{\Omega}\}.$$

We denote by $\|\cdot\|_p$ the usual norm on $L^p(\Omega)$ and by $\|\cdot\|$ that on $W^{1,p}(\Omega)$. A weak solution of (1) is any function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega |u|^{p-2}uv dx - \int_\Omega f(x,u)v dx, \quad \forall v \in W^{1,p}(\Omega). \tag{2}$$

Definition 2.1. We say that $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution for problem (1) and $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution for problem (1) if

$$\int_\Omega \left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla v - \lambda |\underline{u}|^{p-2}\underline{u} v + f(x,\underline{u})v\right) dx \leq 0$$

and

$$\int_\Omega \left(|\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla v - \lambda |\overline{u}|^{p-2}\overline{u} v + f(x,\overline{u})v\right) dx \geq 0$$

for all $v \in W^{1,p}(\Omega)$ with $v \geq 0$ a.e. in Ω.

For a later use, we recall some basic facts about the spectrum of the negative p-Laplacian with Neumann boundary condition. So, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The first eigenvalue in (3) is $\lambda_0 = 0$ whose corresponding eigenspace is $\mathbb{R}$. By $\widehat{u}_0$ we denote its positive L^p-normalized eigenfunction, i.e. $\widehat{u}_0(x) = |\Omega|_N^{-\frac{1}{p}}$ where $|\cdot|_N$ stands for the Lebesgue measure on $\mathbb{R}^N$. Since $\lambda_0 = 0$ is isolated and the spectrum $\sigma(p)$ of $(-\Delta_p, W^{1,p}(\Omega))$ is closed, then the second eigenvalue λ_1 satisfies

$$\lambda_1 = \inf [\lambda : \ \lambda \in \sigma(p), \ \lambda > 0] > 0.$$

We will use the following minimax characterization of λ_1 given in [1].

Proposition 2.2. *Let $\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$ and let $S = W^{1,p}(\Omega) \cap \partial B_1^{L^p}$ be endowed with the topology induced by $W^{1,p}(\Omega)$. Then there holds*

$$\lambda_1 = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\le t\le 1} \|\nabla\widehat{\gamma}(t)\|_p^p,$$

where

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], S) : \widehat{\gamma}(-1) = -\widehat{u}_0,\ \widehat{\gamma}(1) = \widehat{u}_0\}.$$

We will also need some facts related to pseudomonotone operators that we now recall. Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets of the pair (X^*, X), by $\rightharpoonup$ the weak convergence, while $\to$ stands for the strong convergence. A map $A : X \to X^*$ is said to be pseudomonotone if for every sequence $\{x_n\}_{n\ge1} \subset X$ such that $x_n \rightharpoonup x$ in X and $\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \le 0$, one has

$$\langle A(x), x - y\rangle \le \liminf_{n\to+\infty}\langle A(x_n), x_n - y\rangle \quad \text{for all } y \in X.$$

A map $A : X \to X^*$ is said to be of type $(S)_+$ if for every sequence $\{x_n\}_{n\ge1} \subset X$ such that $x_n \rightharpoonup x$ in X and

$$\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \le 0,$$

one has $x_n \to x$ in X. The fundamental theorem for pseudomonotone operators (see [6, Theorem 2.99]) reads as follows.

Theorem 2.3. *Let X be a reflexive Banach space, $A : X \to X^*$ a pseudomonotone, bounded, and coercive operator, and $b \in X^*$. Then a solution of the equation $Au = b$ exists.*

Finally, we cite from [3, Proposition 4.1] and [12, Proposition 3.5] the following result.

Proposition 2.4. *The nonlinear map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by*

$$\langle A(u), v\rangle := \int_\Omega (|\nabla u|^{p-2}(\nabla u, \nabla v)_{\mathbb{R}^N} + |u|^{p-2}uv)dx, \quad \forall u, v \in W^{1,p}(\Omega), \tag{4}$$

is bounded, continuous, coercive, monotone (so, maximal monotone and pseudomonotone) and of type $(S)_+$.

Throughout the rest of the paper, the notation $r^\pm$ with $r \in \mathbb{R}$ means the positive and negative parts of r, respectively, that is $r^\pm = \max\{\pm r, 0\}$.

3. Two constant sign solutions

In this section, we prove the existence of two opposite constant sign solutions $u_+ \in \operatorname{int} C_+$ and $u_- \in -\operatorname{int} C_+$ for problem (1). Let $\lambda > 0$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We assume:

$H(f)$:

(i) *there is a positive constant* k *such that for almost all* $x \in \Omega$, *one has* $\min\{f(x,k), -f(x,-k)\} \geq \lambda k^{p-1}$;

(ii) *there are two real numbers* c, α, *with* $c \leq \alpha < \lambda$ *such that the following inequalities hold uniformly for a.e.* $x \in \Omega$:

$$c \leq \liminf_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} \leq \alpha;$$

(iii) f *is bounded on bounded sets.*

Remark 3.1. From (ii) it follows that $f(x,0) = 0$ a.e. in Ω, so $u \equiv 0$ is solution to (1).

Theorem 3.2. *Assume that hypothesis* $H(f)$ *hold. Then problem* (1) *admits two solutions* $u_+ \in \operatorname{int} C_+$ *and* $u_- \in -\operatorname{int} C_+$.

Proof. From $H(f)(i)$ we derive that

$$-\Delta_p(k) - \lambda k^{p-1} + f(x,k) \geq 0\,. \tag{5}$$

Hypothesis $H(f)(ii)$ guarantees that there exists a $\delta \in]0,k[$ such that

$$f(x,s) < \lambda s^{p-1} \tag{6}$$

for almost all $x \in \Omega$ and for all $s \in]0,\delta[$. Then for any $\varepsilon \in]0,\delta[$, we obtain from (6) that

$$-\Delta_p(\varepsilon) - \lambda\varepsilon^{p-1} + f(x,\varepsilon) < 0 \tag{7}$$

for a.a. $x \in \Omega$. Therefore, by (5) and (7), we have that $\underline{u} = \varepsilon$ and $\overline{u} = k$ are a sub-solution and a super-solution for problem (1), respectively.

Now, we introduce a truncation of the identity on $\mathbb{R}$ as follows

$$\tau(s) = \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } \varepsilon < s < k, \\ \varepsilon & \text{if } s \leq \varepsilon. \end{cases}$$

We define the operator $H : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ by

$$\begin{aligned}\langle H(u), v\rangle = & \int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx \\ & - (\lambda+1)\int_\Omega \tau^{p-1}(u)v dx + \int_\Omega f(x,\tau(u))v dx\end{aligned}$$

for all $u, v \in W^{1,p}(\Omega)$. Let us observe that

$$\left|\int_\Omega \tau^{p-1}(u)v dx\right| \leq k^{p-1}\int_\Omega |v| dx, \tag{8}$$

while $H(f)(iii)$ yields a number $L \geq 0$ such that

$$\sup_{(x,s)\in\Omega\times\mathbb{R}} |f(x,\tau(s))| = \sup_{(x,t)\in\Omega\times[0,k]} |f(x,t)| \leq L. \tag{9}$$

Through (8) and (9), an easy computation leads to the boundedness and coercivity of H. In order to show that H is also pseudomonotone operator, we take a sequence $\{u_n\}_{n\geq 1} \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle H(u_n), u_n - u\rangle \leq 0$. Then, along a relabeled subsequence, $u_n \to u$ and $\tau(u_n) \to \tau(u)$ in $L^p(\Omega)$ and a.e. in Ω. So, owing to the boundedness of τ, by the Lebesgue dominated convergence theorem, for any $w \in W^{1,p}(\Omega)$, we obtain

$$\int_\Omega \tau^{p-1}(u)(u-w)dx = \lim_{n\to\infty}\int_\Omega \tau^{p-1}(u_n)(u_n - w)dx, \quad \text{and}$$
$$\int_\Omega f(x,\tau(u))(u-w)dx = \lim_{n\to\infty}\int_\Omega f(x,\tau(u_n))(u-w)dx.$$

We infer that $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$. At this point, Proposition 2.4 enables us to conclude that H is a pseudomonotone operator. Theorem 2.3 yields a point $u \in W^{1,p}(\Omega)$ such that $H(u) = 0$, which reads as

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv\right)dx - (\lambda+1)\int_\Omega \tau^{p-1}(u)vdx \\ + \int_\Omega f(x,\tau(u))vdx = 0, \quad \forall\, v \in W^{1,p}(\Omega).\end{aligned} \tag{10}$$

We claim that $\varepsilon \leq u(x) \leq k$ for a.e. $x \in \Omega$. To this end, taking as test function in (10) $v = (u-k)^+$, we have

$$\begin{aligned}\int_{\{u\geq k\}} \left(|\nabla u|^p + u^{p-1}(u-k)\right)dx - (\lambda+1)\int_{\{u>k\}} k^{p-1}(u-k)dx \\ + \int_{\{u>k\}} f(x,k)(u-k)dx = 0\,.\end{aligned}$$

From $H(f)(i)$ we know that $f(x,k) - \lambda k^{p-1} \geq 0$ a.e. in Ω, thus it follows that

$$0 \leq \int_{\{u>k\}} \left(u^{p-1} - k^{p-1}\right)(u-k)dx \leq 0\,,$$

which results in $u \leq k$ a.e. in Ω. Next, acting with $v = (\varepsilon - u)^+$ in (10), we obtain

$$\begin{aligned}-\int_{\{u\leq\varepsilon\}} |\nabla u|^p dx + \int_{\{u<\varepsilon\}} |u|^{p-2}u(\varepsilon - u)dx \\ -(\lambda+1)\int_{\{u<\varepsilon\}} \varepsilon^{p-1}(\varepsilon-u)dx + \int_{\{u<\varepsilon\}} f(x,\varepsilon)(\varepsilon-u)dx = 0\,.\end{aligned}$$

Using (6), we arrive at

$$0 \leq \int_{\{u<\varepsilon\}} \left(|u|^{p-2}u - \varepsilon^{p-1}\right)(\varepsilon - u)dx \leq 0$$

that implies $|\{u < \varepsilon\}|_N = 0$, which proves the claim.

Since, $\varepsilon \leq u(x) \leq k$ a.e. in Ω, (10) becomes (2) ensuring that u is a weak solution for (1). Moreover, noting that $u \in L^\infty(\Omega)$, $H(f)(iii)$ guarantees that the function $g(x) = f(x, u(x))$ belongs to $L^\infty(\Omega)$, and [10, Theorem 4.9] ensures that $u \in C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$ (see also [8]), so [15, Theorem 5] yields $u \in \operatorname{int} C_+$. Consequently, the desired conclusion holds true with $u = u_+$. Arguing in the same way, we can prove the existence of a (negative) solution $u_- \in -\operatorname{int} C_+$. □

We now focus on the existence of extremal constant sign solutions of problem (1).

Lemma 3.3. *If hypotheses $H(f)$ hold, then problem* (1) *admits a smallest (positive) solution $v_{\varepsilon,+}$ in $[\varepsilon, k]$ and a greatest (negative) solution $v_{\varepsilon,-}$ in $[-k, -\varepsilon]$.*

Proof. Let $\mathcal{S}_\varepsilon$ be the set of all solutions of (1) belonging to $[\varepsilon, k]$, which is nonempty according to Theorem 3.2. As in the case of Dirichlet problems, it can be shown that $\mathcal{S}_\varepsilon$ is directed and compact in $W^{1,p}(\Omega)$ (see [6, Corollary 3.7, Theorem 3.10]. Since $W^{1,p}(\Omega)$ is separable, there exists a sequence $\{v_n\}_{n\geq 1} \subset \mathcal{S}_\varepsilon$ which is dense in $\mathcal{S}_\varepsilon$. We construct a decreasing sequence $\{u_n\}_{n\geq 1} \subset \mathcal{S}_\varepsilon$ as follows. Set $u_1 = v_1$ and, exploiting that $\mathcal{S}_\varepsilon$ is downward directed, we choose $u_{n+1} \in \mathcal{S}_\varepsilon$ such that $u_{n+1} \leq \min\{u_n, v_n\}$. The compactness of $\mathcal{S}_\varepsilon$ and the monotonicity of $\{u_n\}_{n\geq 1}$ render that $u_n \to v_{\varepsilon,+} = \inf_n u_n \in \mathcal{S}_\varepsilon$ in $W^{1,p}(\Omega)$. We observe that $v_{\varepsilon,+} \leq v_n$ for all $n \geq 1$. In view of the density of $\{v_n\}_{n\geq 1}$ in $\mathcal{S}_\varepsilon$, we infer that $v_{\varepsilon,+}$ is the smallest solution in $[\varepsilon, k]$. The existence of the greatest solution $v_{\varepsilon,-}$ in $[-k, -\varepsilon]$ can be done in a similar way. □

Theorem 3.4. *If hypotheses $H(f)$ hold, then problem* (1) *admits a smallest positive solution $v_+ \in \operatorname{int} C_+$ and a greatest negative solution $v_- \in -\operatorname{int} C_+$.*

Proof. We only show the existence of the smallest positive solution, because for the existence of the greatest negative solution we can proceed along the same lines. From Lemma 3.3, we know that for any integer $n \geq 1$ there is a smallest solution $v_{n,+} \in [\frac{1}{n}, k]$. As $\mathcal{S}_{\varepsilon_1} \subseteq \mathcal{S}_{\varepsilon_2}$ whenever $\varepsilon_2 < \varepsilon_1$, one has $v_{\varepsilon_2,+} \leq v_{\varepsilon_1,+}$, so

$$v_{n,+} \rightharpoonup v_+,\ v_{n,+} \to v_+ \text{ in } L^p(\Omega) \quad \text{and} \quad v_{n,+} \downarrow v_+ \text{ a.e.} \in \Omega. \tag{11}$$

Writing (2) for $u = v_{n,+}$ and $v = v_{n,+} - v_+$, we get from (11) that

$$\lim_{n\to+\infty} \int_\Omega |\nabla v_{n,+}|^{p-2} \nabla v_{n,+} \nabla(v_{n,+} - v_+)dx = 0.$$

Then through Proposition 2.4 and (11) we see that $v_{n,+} \to v_+$ in $W^{1,p}(\Omega)$, which enables us to derive that v_+ is a solution for (1).

The next step in the proof is to show that $v_+ \neq 0$. Arguing by contradiction, we suppose that $v_+ = 0$. Setting

$$\tilde{v}_n = \frac{v_{n,+}}{\|v_{n,+}\|} \quad \text{and} \quad h_n = \frac{f(x, v_{n,+})}{(v_{n,+})^{p-1}} \tilde{v}_n^{p-1}, \quad n \geq 1.$$

and bearing in mind $H(f)(ii)$ and (11), up to a subsequence we may assume that

$$\tilde{v}_n \rightharpoonup \tilde{v} \ \text{ in } W^{1,p}(\Omega), \qquad \tilde{v}_n \to \tilde{v} \ \text{ in } L^p(\Omega),$$
$$\tilde{v}_n \to \tilde{v} \ \text{ a.e. on } \Omega, \qquad h_n \rightharpoonup h \ \text{ in } L^{\frac{p}{p-1}}(\Omega).$$

Now acting on (2) (written for $u = v_{n,+}$) with $\tilde{v}_n - \tilde{v}$ and dividing by $\|v_{n,+}\|^{p-1}$ we obtain

$$\int_\Omega |\nabla \tilde{v}_n|^{p-2} \nabla \tilde{v}_n \nabla(\tilde{v}_n - \tilde{v}) dx = \int_\Omega (\lambda \tilde{v}_n^{p-1} - h_n)(\tilde{v}_n - \tilde{v}) dx.$$

Then combining with $H(f)(ii)$, (iii) and the Lebesgue dominated convergence theorem leads to

$$\lim_{n \to +\infty} \int_\Omega |\nabla \tilde{v}_n|^{p-2} \nabla \tilde{v}_n \nabla(\tilde{v}_n - \tilde{v}) dx = 0.$$

From the (S_+) property of $-\Delta_p$ (see Proposition 2.4), it turns out that $\tilde{v}_n \to \tilde{v}$. In particular, we have that $\|\tilde{v}\| = 1$, so $\tilde{v} \neq 0$.

Exploiting $H(f)(ii)$, we note that $h = g\tilde{v}^{k-1}$, with $g \in L^\infty(\Omega)$, $\lambda - g(x) > 0$ for a.e. $x \in \Omega$. We find that $\tilde{v}$ is a solution of

$$\int_\Omega |\nabla \tilde{v}|^{p-2} \nabla \tilde{v} \nabla v dx = \int_\Omega (\lambda - g(x))\, \tilde{v}^{p-1} v dx \quad \forall\, v \in W^{1,p}(\Omega),$$

that is $\tilde{v}$ is an eigenfunction associated with the eigenvalue 1 for the weighted problem

$$\begin{cases} -\Delta_p \tilde{v} = (\lambda - g(x))|\tilde{v}|^{p-2}\tilde{v} & \text{in}\, \Omega \\ \dfrac{\partial \tilde{v}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\lambda - g(x) > 0$ for a.e. $x \in \Omega$, necessarily $\tilde{v}$ must change sign (see [9, Section 6.2]). This contradicts $\tilde{v} \geq 0$, $\tilde{v} \neq 0$. We have thus shown that $v_+ \neq 0$. As before, [10, Theorem 4.9] guarantees that $v_+ \in C^{1,\alpha}(\overline{\Omega})$, while [15, Theorem 5] entails that $v_+ > 0$ in $\overline{\Omega}$, so $v_+ \in \operatorname{int} C_+$.

In order to verify that v_+ is the smallest positive solution of problem (1), let u be a positive solution of (1). By the strong maximum principle (cf. [15, Theorem 5]) we know that $u \in \operatorname{int} C_+$, so there holds $\frac{1}{n} \leq u$ on $\overline{\Omega}$ whenever n is sufficiently large, say $n \geq n_0$. Without loss of generality we may suppose that $u \leq k$ because otherwise we can argue in place of u with a solution of (1) in the ordered interval $[\frac{1}{n_0}, \min\{u, k\}]$, which exists since $\min\{u, k\}$ is a super-solution and $1/n_0$ is a sub-solution of (1). Recalling that $v_{n,+}$ is the smallest solution of (1) in $[\frac{1}{n}, k]$, it follows that $v_{n,+} \leq u$. Letting $n \to \infty$ enables us to obtain that $v_+ \leq u$, which completes the proof. $\square$

4. Sign-changing solutions

In this section we prove the existence of a third nontrivial solution to (1) which has in addition the property to be nodal (sign-changing). We strengthen our assumptions on λ requiring $\lambda > \lambda_1$. In order to develop a variational approach, we set

$$F(x,s) = \int_0^s f(x,t)dt \quad \text{for all } (x,s) \in \Omega \times \mathbb{R} \to \mathbb{R}.$$

Under hypotheses $H(f)$, F is a Carathéodory function such that $F(\cdot,0) = 0$, the partial derivative F_s exists and is a Carathéodory function with $F_s(x,s) = f(x,s)$ for almost all $x \in \Omega$. Using the extremal constant sign solutions u_+ and u_- constructed in Theorem 3.4, we introduce the following functions on $\Omega \times \mathbb{R}$:

$$\tau_+(x,s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } 0 < s < v_+(x), \\ v_+(x) & \text{if } s \ge v_+(x), \end{cases}$$

$$f_+(x,s) = \begin{cases} 0 & \text{if } s \le 0, \\ f(x,s) & \text{if } 0 < s < v_+(x), \\ f(x,v_+(x)) & \text{if } s \ge v_+(x), \end{cases}$$

$$\tau_-(x,s) = \begin{cases} v_-(x) & \text{if } s \le v_-(x), \\ s & \text{if } v_-(x) < s < 0, \\ 0 & \text{if } s \ge 0, \end{cases}$$

$$f_-(x,s) = \begin{cases} f(x,v_-(x)) & \text{if } s \le v_-(x), \\ f(x,s) & \text{if } v_-(x) < s < 0, \\ 0 & \text{if } s \ge 0, \end{cases}$$

$$\tau_0(x,s) = \begin{cases} v_-(x) & \text{if } s \le v_-(x), \\ s & \text{if } v_-(x) < s < v_+(x), \\ v_+(x) & \text{if } s \ge v_+(x), \end{cases}$$

$$f_0(x,s) = \begin{cases} f(x,v_-(x)) & \text{if } s \le v_-(x), \\ f(x,s) & \text{if } v_-(x) < s < v_+(x), \\ f(x,v_+(x)) & \text{if } s \ge v_+(x). \end{cases}$$

Corresponding to these functions, we define

$$F_+(x,s) = \int_0^s f_+(x,t)dt, \qquad F_-(x,s) = \int_0^s f_-(x,t)dt,$$

$$F_0(x,s) = \int_0^s f_0(x,t)dt.$$

Just as an example, we explicitly observe that

$$F_+(x,s)=\begin{cases}0 & \text{if } s\le 0,\\ F(x,s) & \text{if } 0<s<v_+(x),\\ F(x,v_+(x))+(s-v_+(x))f(x,v_+(x)) & \text{if } s\ge v_+(x).\end{cases}$$

The energy functionals related to the three truncated problems that we obtain from (1) corresponding to the above truncations are:

$$E_+(u):=\frac{1}{p}\|u\|^p-(\lambda+1)\int_\Omega\int_0^{u(x)}\tau_+^{p-1}(x,s)dsdx+\int_\Omega F_+(x,u(x))dx,$$

$$E_-(u)=\frac{1}{p}\|u\|^p-(\lambda+1)\int_\Omega\int_0^{u(x)}\tau_-^{p-1}(x,s)dsdx+\int_\Omega F_-(x,u(x))dx,$$

$$E_0(u)=\frac{1}{p}\|u\|^p-(\lambda+1)\int_\Omega\int_0^{u(x)}\tau_0^{p-1}(x,s)dsdx+\int_\Omega F_0(x,u))dx.$$

Clearly, $E_\pm, E_0$ are C^1 functionals on $W^{1,p}(\Omega)$. In this respect, we highlight for instance the differential

$$\begin{aligned}E_+'(v)(w)=&\int_\Omega\left(|\nabla v|^{p-2}\nabla v\nabla w+|v|^{p-2}vw\right)dx\\ &-(\lambda+1)\int_\Omega\tau_+^{p-1}(x,v(x))w(x)dx+\int_\Omega f_+(x,v(x))w(x)dx.\end{aligned}\tag{12}$$

Our main result is the following.

Theorem 4.1. *Assume $\lambda>\lambda_1$ and hypothesis $H(f)$ with $\alpha<\lambda-\lambda_1$. Then problem (1) has at least three nontrivial solutions: a smallest positive solution $v_+\in\operatorname{int}C_+$, a greatest negative solution $v_-\in-\operatorname{int}C_+$ and a nodal solution $u_0\in C^1(\overline{\Omega})$, with $v_-\le u_0\le v_+$.*

Proof. Theorem 3.4 provides the smallest positive solution $v_+\in\operatorname{int}C_+$ and the greatest negative solution $v_-\in-\operatorname{int}C_+$ of problem (1). The proof of the existence of the third nontrivial solution will be done in two steps: first, we show that v_+ and v_- are global minimizers for E_+ and E_- respectively, as well as local minimizers of E_0; afterwards, we apply the mountain pass theorem to E_0 obtaining a nontrivial critical point u_0 with $v_-\le u_0\le v_+$ and $u_0\ne v_-$, $u_0\ne v_+$, which turns out to be a nodal solution for problem (1).

We note that any critical point $v\in W^{1,p}(\Omega)$ of E_+ satisfies

$$0\le v(x)\le v_+(x),\quad \text{for a.a. } x\in\Omega.\tag{13}$$

In order to show (13), by acting on $E_+'(v)=0$ (see (12)) with $w=(v-v_+)^+$, we obtain

$$\begin{aligned}&\int_{\{v\ge v_+\}}\left(|\nabla v|^{p-2}\nabla v\nabla(v-v_+)+v^{p-1}(v-v_+)\right)dx\\ &\quad-(\lambda+1)\int_{\{v>v_+\}}(v_+)^{p-1}(v-v_+)dx+\int_{\{v>v_+\}}f(x,v_+)(v-v_+)dx=0.\end{aligned}$$

Bearing in mind that v_+ is a solution for (1) we can write

$$\int_{\{v\geq v_+\}} |\nabla v_+|^{p-2}\nabla v_+ \nabla(v-v_+)dx \\ -\lambda\int_{\{v>v_+\}}(v_+)^{p-1}(v-v_+)dx + \int_{\{v>v_+\}} f(x,v_+)(v-v_+)dx = 0.$$

Subtracting the second equality from the first one, we have

$$\int_{\{v\geq v_+\}} \left(|\nabla v|^{p-2}\nabla v - \nabla v_+|^{p-2}\nabla v_+\right)\nabla(v-v_+)dx \\ + \int_{\{v\geq v_+\}} \left(v^{p-1}-(v_+)^{p-1}\right)(v-v_+)dx = 0,$$

which implies that $v(x) \leq v_+(x)$ for a.a. $x \in \Omega$. On the other hand, by testing $E'_+(v) = 0$ (see (12)) with $w = -v^-$, we get

$$\int_{\{v<0\}} \left(|\nabla v|^p + |v|^{p-1}\right)dx - \int_{\{v<0\}} f_+(x,v)v^- dx = 0,$$

and because $f_+(x,s) = 0$ whenever $s < 0$, we infer that $v^- = 0$ and (13) is proven.

By $H(f)(ii)$, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(x,s) \leq \frac{\alpha+\varepsilon}{p}s^p \text{ for a.a. } x\in\Omega, \text{ all } s \in]0,\delta[.$$

Taking $0 < \delta < \min v_+$, one has that $F_+(x,s) = F(x,s)$ for all $s \in]0,\delta[$. Then considering a constant function $t \in]0,\delta[$, we see that

$$E_+(t) \leq (-\lambda+\alpha+\varepsilon)\frac{t^p}{p}|\Omega|_N < (-\lambda_1+\varepsilon)\frac{t^p}{p}|\Omega|_N,$$

so for $\varepsilon < \lambda_1$ we get $E_+(t) < 0$. Clearly, E_+ is coercive and weakly lower semicontinuous, thus it has a global minimizer $z \in W^{1,p}(\Omega)$ satisfying $E_+(z) < 0$. This ensures that $z \neq 0$ and owing to (13) it follows that $0 \leq z(x) \leq v_+(x)$ for a.e. $x \in \Omega$. Therefore z is a solution for (1) and the nonlinear regularity theory and strong maximum principle show that $z \in \operatorname{int} C_+$. Due to the minimality property of v_+, we obtain that $z = v_+$. Taking into account that E_+ and E_0 coincide on a $C^1(\overline{\Omega})$-neighborhood of v_+, we see that v_+ is a local $C^1(\overline{\Omega})$-minimizer of E_0, hence it is also a local minimizer of E_0 in $W^{1,p}(\Omega)$ (see [3, Proposition 4.1] and [11, Proposition 24]).

Proceeding in a similar way with E_- in place of E_+ we obtain the corresponding result for v_-. Arguing exactly as for showing (13) we can establish that any critical point v of E_0 satisfies

$$v_-(x) \leq v(x) \leq v_+(x), \quad \text{for a.a. } x\in\Omega. \tag{14}$$

We may assume that v_+ and v_- are strict local minimizers of E_0 because otherwise handling (14) and the extremal properties of v_+ and v_- we find infinitely many sign-changing solutions for (1). Since E_0 satisfies the Palais-Smale condition, we can apply the mountain pass theorem to E_0 to obtain a critical point u_0 of E_0 satisfying

$$\max\{E_0(v_+), E_0(v_-)\} < E_0(u_0) = c := \inf_{\gamma\in\Gamma} \max_{-1\leq t\leq 1} E_0(\gamma(t)), \tag{15}$$

with

$$\Gamma = \left\{\gamma \in C([-1,1], W^{1,p}(\Omega)) :\ \gamma(-1) = v_-,\ \gamma(1) = v_+\right\}.$$

We claim that

$$E_0(u_0) < 0 \tag{16}$$

To this end, we construct a path $\gamma \in \Gamma$ along which E_0 turns out negative. Towards this, we make use of the characterization of the second eigenvalue of the negative $p-$Laplacian given in Proposition 2.2. In addition to $S = W^{1,p}(\Omega)\cap\partial B_1^{L^p}$ endowed with the topology induced by $W^{1,p}(\Omega)$, we consider $S_C = S \cap C^1(\overline{\Omega})$ endowed with the topology induced by $C^1(\overline{\Omega})$. Then, besides

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], S) :\ \widehat{\gamma}(-1) = -\widehat{u}_0,\ \widehat{\gamma}(1) = \widehat{u}_0\},$$

we also take

$$\widehat{\Gamma}_C = \{\widehat{\gamma} \in C([-1,1], S_C) : \widehat{\gamma}(-1) = -\widehat{u}_0,\ \widehat{\gamma}(1) = \widehat{u}_0\}.$$

It is worthwhile to point out that $\widehat{\Gamma}_C$ is dense in $\widehat{\Gamma}$. Given any $\mu \in (0,\ \lambda-\lambda_1-\alpha)$, hypothesis $H(f)(ii)$ yields a $\delta > 0$ such that

$$F(x,s) < \frac{\alpha+\mu}{p}|s|^p \ \text{ for a.a. } x \in \Omega,\ \text{all } |s| < \delta. \tag{17}$$

Then Proposition 2.2 and the density of $\widehat{\Gamma}_C$ in $\widehat{\Gamma}$ allow that for every $\nu \in (0,\ \lambda - \lambda_1 - \alpha - \mu)$ there exists a path $\gamma_0 \in \Gamma_C$ satisfying

$$\max_{-1\leq t\leq 1} \|\nabla\gamma_0(t)\|_p^p < \lambda_1 + \nu. \tag{18}$$

Using the compactness of $\gamma_0([-1,1])$ in $C^1(\overline{\Omega})$, in conjunction with $v_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$, we find that corresponding to $\delta > 0$ defined above there is $\varepsilon > 0$ such that, for all $x \in \Omega$,

$$v_-(x) \leq \varepsilon u(x) \leq v_+(x) \quad \text{and} \quad \varepsilon|u(x)| < \delta,\ \ \forall u = \gamma_0(t) \in \gamma_0([-1,1]). \tag{19}$$

On the basis of (17), (18), (19), we infer that

$$\begin{aligned} E_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}(\|\nabla u\|_p^p - \lambda) + \int_\Omega F(x,\varepsilon u(x))dx \\ &\leq \frac{\varepsilon^p}{p}(\lambda_1 + \nu - \lambda + \alpha + \mu) < 0 \ \text{ for all } t \in [-1,1]. \end{aligned} \tag{20}$$

Notice that $E_+(v_+) = m < c = E_+(u_0)$. Due to the minimality property of v_+ and (13), there are no critical values of E_+ in the interval $]m, c]$, so we can apply the second deformation lemma (see, e.g., [9, p. 366]) to the functional E_+. We obtain a continuous mapping $\eta : [0,1] \times E_+^c \to E_+^c$, where $E_+^c = E_+^{-1}(]-\infty, c])$, such that

$$\eta(0,u) = u, \qquad \eta(1,u) = v_+, \qquad E_+(\eta(t,u)) \le E_+(u), \ \ \forall t \in [0,1], \ \forall u \in E_+^c.$$

The continuous path

$$\gamma_+(t) = \eta(t, \varepsilon\widehat{u}_0)^+ \ \text{ for all } t \in \ [0,1]$$

joins $\varepsilon\widehat{u}_0$ and v_+, and for all $t \in \ [0,1]$ there holds

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) = E_+(\eta(t,\varepsilon\widehat{u}_0)) - \frac{\|\eta(t,\varepsilon\widehat{u}_0)^-\|^p}{p} \le E_+(\varepsilon\widehat{u}_0) < 0. \quad (21)$$

Using the functional E_-, we can similarly construct a continuous path γ_- joining v_- and $-\varepsilon\widehat{u}_0$ such that

$$E_0(\gamma_-(t)) < 0 \ \text{ for all } t \in [0,1]. \quad (22)$$

Concatenating γ_-, $\varepsilon\gamma_0$ and γ_+ we obtain a path γ belonging to Γ. Taking into account (20), (21) and (22), we see that $E_0(\gamma(t)) < 0$ for all $t \in \ [0,1]$, hence (16) holds true. It is clear that u_0 is a nontrivial critical point of E_0 and by (14) we derive that u_0 solves (1). The L^∞-boundedness of u_0 together with $H(f)(iii)$ enables us to apply [10, Theorem 4.9] to deduce that $u_0 \in C^1(\overline{\Omega})$. It remains only to show that u_0 must change sign. Indeed, if for example $u_0 \ge 0$, then by (14) we have $u_0 = v_+$, and this is in contradiction with the fact that $E_+(v_+) < E_+(u_0)$ as known from (15). Arguing in the same way with E_- we can show that the situation $u_0 \le 0$ cannot happen. This amounts to saying that u_0 changes sign, which completes the proof. □

Acknowledgements. The authors are very grateful to the anonymous referee for his/her knowledgeable report, which helped to improve our manuscript.

References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu: The spectrum and an index formula for the Neumann p-Laplacian and multiple solutions for problems with a crossing nonlinearity, Discrete Contin. Dyn. Syst. 25 (2009) 431–456.

[2] G. Bonanno, P. Candito: Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian, Arch. Math. 80 (2003) 424–429.

[3] G. Barletta, N. S. Papageorgiou: A multiplicity theorem for the Neumann p-Laplacian with an asymmetric nonsmooth potential, J. Glob. Optim. 39 (2007) 365–392.

[4] S. Carl, D. Motreanu: Constant-sign and sign-changing solutions for nonlinear eigenvalue problems, Nonlinear Anal. 68 (2008) 2668–2676.

[5] S. Carl, D. Motreanu: Multiple and sign-changing solutions for the multivalued p-Laplacian equation, Math. Nachr. 283 (2010) 965–981.

[6] S. Carl, V. K. Le, D. Motreanu: Nonsmooth Variational Problems and their Inequalities. Comparison Principles and Applications, Springer Monogr. Math., Springer, New York (2007).

[7] S. Carl, K. Perera: Sign-changing and multiple solutions for the p-Laplacian, Abstr. Appl. Anal. 7 (2002) 613–625.

[8] E. DiBenedetto: $C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983) 827–850.

[9] L. Gasiński, N. S. Papageorgiou: Nonlinear Analysis, Chapman & Hall / CRC Press, Boca Raton (2006).

[10] A. Lê: Eigenvalue problems for the p-Laplacian, Nonlinear Anal. 64 (2006) 1057–1099.

[11] S. Miyajima, D. Motreanu, M. Tanaka: Multiple existence results of solutions for the Neumann problems via super- and sub-solutions, J. Funct. Anal. 262 (2012) 1921–1953.

[12] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou: Multiple constant sign and nodal solutions for nonlinear Neumann eigenvalue problems, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 10 (2011) 729–755.

[13] D. Motreanu, M. Tanaka: Sign-changing and constant-sign solutions for $p-$Laplacian problems with jumping nonlinearities, J. Differ. Equ. 249 (2010) 3352–3376.

[14] D. Motreanu, Z. Zhang: Constant and sign changing solutions for systems of quasilinear elliptic equations, Set-Valued Var. Anal. 19 (2011) 255–269.

[15] J. L. Vázquez: A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optimization 12 (1984) 191–202.

[16] P. Winkert: Sign-changing and extremal constant-sign solutions of nonlinear elliptic Neumann boundary value problems, Bound. Value Probl. 2010 (2010), Art. ID 139126.

[17] P. Winkert: Constant-sign and sign-changing solutions for nonlinear elliptic equations with Neumann boundary values, Adv. Differ. Equ. 15(5–6) (2010) 561–599.

[18] P. Winkert: Multiple solution results for elliptic Neumann problems involving set-valued nonlinearities, J. Math. Anal. Appl. 377(1) (2011) 121–134.

[19] J. Zhang, S. Li, Y. Wang, X. Xue: Multiple solutions for semilinear elliptic equations with Neumann boundary condition and jumping nonlinearities, J. Math. Anal. Appl. 371(2) (2010) 682–690.