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# On a Shape Derivative Formula with Respect to Convex Domains

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We extend to the case of $W^{1,1}_{\rm loc}$ functions a formula, known for $C^1$ functions, for the computation of the shape derivative of an integral cost functional with respect to a class of admissible convex domains. The convexity context allows one to use the support functions of the domains to express the shape derivative. We shall also illustrate the formula by applying it to the computation of the shape derivative for a shape optimization problem and by giving an algorithm.

## 1. Introduction

This paper is devoted essentially to the proof of a formula for the computation of the shape derivative for an integral cost functional with respect to a class of admissible convex domains, using what is called *support functions* in convex analysis to express such a derivative. Originally, this was motivated by numerical considerations. Indeed, we needed such a formula when we considered some free boundary problems of Bernoulli type in [8] from a numerical point of view. However, this work is still in preparation. Here, to illustrate the result, we shall only apply the formula to the computation of the shape derivative of a simple shape optimization problem and we shall give an algorithm.

In order to be more precise, let us define the set of admissible domains by

$$\mathcal{U} = \left\{\Omega \subset \mathbb{R}^n\,;\ \Omega \text{ is non empty, open, bounded, convex and of class } C^2\right\},$$

and let us consider the functional $J$ defined on $\mathcal{U}$ by

$$J(\Omega) = \int_\Omega f\,dx,$$

where $f$ is a fixed function defined in $\mathbb{R}^n$. The starting point of our work is the reference [1] where A. Niftiyev and Y. Gasimov proved a formula giving the

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derivative of $J$ with respect to $\Omega \in \mathcal{U}$ by means of the support functions of the domains. Their result says the following:

**Theorem 1.1 (A. Niftiyev, Y. Gasimov).** *If $\Omega_0, \Omega \in \mathcal{U}$ and the function $f$ is of class $C^1$, then, the limit*

$$\lim_{\varepsilon \to 0^+} \frac{J((1-\varepsilon)\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon}$$

*exists and is equal to*

$$\int_{\partial\Omega_0} f(x)\ \left(P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x))\right)\, d\sigma(x), \tag{1}$$

*where $\nu_0(x)$ denotes the exterior unit normal vector to $\partial\Omega_0$ at $x$, and $P_{\Omega_0}$, $P_\Omega$ are the support functions of the domains $\Omega_0$, $\Omega$, respectively.*

Recall that the support function of a bounded convex set $A$ in $\mathbb{R}^n$ is given by

$$P_A(x) = \sup_{y \in A} \sum_{k=1}^{n} x_k y_k.$$

See Section 2 for more details.

Note in the above statement the unusual deformation $(1-\varepsilon)\Omega_0 + \varepsilon\Omega$ of $\Omega_0$. This is suggested by the convexity context and the applications considered in [1]. Note also that this is only a semi-derivative. See the remarks in Section 5 for more details.

We also refer to the papers [11] and [12] where the above formula is used without proof and applied to some problems of partial differential equations.

Our objective here is to generalize such a result to the case where $f$ is in the Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, that is, $f$ is a locally integrable function whose distributional (or weak) partial derivatives are also locally integrable in $\mathbb{R}^n$. It goes without saying that such a generalization is natural and can be of interest elsewhere in applications, the Sobolev spaces being nowadays widely used.

As said above, our interest in such a formula came first from a numerical study undertaken in [8]. In fact, we believe that, in the context of convexity and numerical implementation, the use of support functions is more advantageous than that of vectors fields. We refer, for example, to [9], page 155, for explanations about the difficulties that arise when implementing numerically the minimization of domain integral functionals, via some gradient method, by using the usual expression of the shape derivative by vector fields. Very briefly, the reason is that, when using vector fields, at each iteration we have to extend the vector field (obtained only on the boundary) to all the domain or to re-mesh, and both approaches are expensive, while when using support functions, at each iteration we get not only a set of boundary points but also a support function which, by taking its subdifferential at the origin, gives the next domain.

We also discovered afterwards that a formula similar to that in the above theorem was known in the classical theory of convex sets since a long time for $f = 1$ and when one uses the usual deformation $\Omega_0 + \varepsilon\Omega$ of $\Omega_0$. It says that, if $f = 1$,

$$\lim_{\varepsilon \to 0^+} \frac{J(\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} P_\Omega \circ \nu_0 \, d\sigma.$$

See [7], [4] or [10], for example. Hence, the result of Niftiyev-Gasimov and our result below (Theorem 1.1) look like extensions of such a classical formula. See the remarks of Section 5.

Even if we have followed the same plan as [1], the proof of such a generalization turned out to be not straightforward and required a number of geometrical and analytical lemmas. In a first step, one assumes that $\Omega$ is strongly convex. The strong convexity and the $C^2$ regularity of the domains allow one to do some geometric constructions which show that the variable domain $(1-\varepsilon)\Omega_0 + \varepsilon\Omega$ can be viewed as the image of $\Omega_0$ by some diffeomorphism. The general case is then treated by approximating $\Omega$ by a sequence of strongly convex domains via some arguments using compactness in $L^1$ and the Brunn-Minkowski theory.

The outline of the paper is as follows. In Section 2, we recall few facts about convex sets. In Section 3, we state the main result of the paper and prove a corollary which is also a generalization. Section 4 is devoted to the proof of the main result via a number of lemmas. In Section 5, we make remarks and add complements to the main result. In Section 6, in order to illustrate the result of this paper, we give a simple application of it to a problem of shape optimization via some partial differential equation.

The main result of this paper was announced in [13] where a sketch of its proof was given.

## 2. Preliminaries

Let us recall briefly some notions and facts about convex sets that we shall use in the sequel. For more details or proofs, we refer for example to [7], [2], [4], or [6].

We shall denote by $\mathcal{C}$ the set of closed convex bounded subsets of $\mathbb{R}^n$.

Let us start by recalling the operations of addition and multiplication by non-negative numbers in $\mathcal{C}$. If $A$ and $B$ are in $\mathcal{C}$ and $\lambda \geq 0$, then,

$$A + B = \{x + y;\ x \in A,\ y \in B\}$$
$$\text{and} \quad \lambda A = \{\lambda x;\ x \in A\}$$

are also in $\mathcal{C}$. However, $\mathcal{C}$ is not a linear space. Note that it can be realized as a subset of some linear space, but we shall not use this fact here.

If $A \in \mathcal{C}$, we shall mainly use its *support function* $P_A$ defined in $\mathbb{R}^n$ by

$$P_A(x) = \sup_{y \in A} x.y\,,$$

where $x.y = xy$ denotes the standard scalar product in $\mathbb{R}^n$. One can easily see that the support function $P_A$ is continuous, convex and positively homogeneous, that is, $P_A(\lambda x) = \lambda P_A(x)$, $\lambda > 0$. Conversely, for each continuous convex positively homogeneous function $P$ on $\mathbb{R}^n$, there exists a unique closed convex bounded set $A$, such that $P = P_A$. In fact, the set $A \in \mathcal{C}$ is reconstructed as a subdifferential of the function $P$ at the origin $x = 0$, that is,

$$A = \partial P(0) = \{y \in \mathbb{R}^n : P(x) \geq x.y, \forall x \in \mathbb{R}^n\}.$$

Thus, we have a one-one correspondance between $\mathcal{C}$ and the set of all continuous convex positively homogeneous functions $P$ on $\mathbb{R}^n$. The interest in using the support functions is that instead of working with a set of subsets of $\mathbb{R}^n$, one works with functions defined on $\mathbb{R}^n$, which allows one of course to use all the means of functional analysis.

The support functions have other remarkable properties. For all $A, B \in \mathcal{C}$ and $\lambda \geq 0$, one can easily prove that

$$P_{A+B} = P_A + P_B \quad \text{and} \quad P_{\lambda A} = \lambda P_A. \tag{2}$$

## 3. Main results

Recall that $\mathcal{U}$ is the set of non empty bounded convex open subsets of $\mathbb{R}^n$ which are of class $C^2$. If $\Omega \in \mathcal{U}$, its closure $\overline{\Omega}$ is in $\mathcal{C}$ and its support function $P_{\overline{\Omega}}$ is well-defined. Since $\sup_{y\in\Omega} xy = \sup_{y\in\overline{\Omega}} xy$, we shall use the notation $P_\Omega$ instead of $P_{\overline{\Omega}}$. We can now state the main result of this paper. It concerns the functional

$$J(\Omega) = \int_\Omega f\,dx \tag{3}$$

and reads as follows.

**Theorem 3.1.** *For all $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega_0, \Omega \in \mathcal{U}$, we have the limit*

$$\lim_{\varepsilon\to 0^+} \frac{J((1-\varepsilon)\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x)\,(P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x)))\,d\sigma(x), \tag{4}$$

*where $\nu_0(x)$ denotes the exterior unit normal vector to $\partial\Omega_0$ at $x$.*

The expression (4) will be called *the shape derivative of $J$ at $\Omega_0$ in the direction $\Omega$.*

The proof of this statement will be given in the next section. Before, we give and prove a corollary of this result, which is at the same time an extension of it. Indeed, it concerns the case where the function $f$ depends on the parameter $\varepsilon$, a situation which is frequent in the applications.

**Corollary 3.2.** *Let $\Omega_0, \Omega \in \mathcal{U}$ and let $\Omega_\varepsilon$ stands for $(1-\varepsilon)\Omega_0 + \varepsilon\Omega$ where $0 \leq \varepsilon \leq 1$. Let $f_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0 \leq 1$, be a family of functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

*$f_0 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and assume that the limit*

$$g = \lim_{\varepsilon \to 0^+} \frac{f_\varepsilon - f_0}{\varepsilon}$$

*exists in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ or, at least, in $L^1(D)$, where $D$ is a bounded open set in $\mathbb{R}^n$ which contains all the $\Omega_\varepsilon$. Then, the (right) derivative at $0$ of*

$$I(\varepsilon) = \int_{\Omega_\varepsilon} f_\varepsilon \, dx$$

*exists and*

$$I'(0) = \int_{\Omega_0} g(x)\, dx + \int_{\partial\Omega_0} f_0(x) \left( P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x)) \right) d\sigma(x),$$

*where $\nu_0(x)$ denotes the exterior unit normal vector to $\partial\Omega_0$ at $x$.*

**Proof of Corollary 3.2.** We can write

$$\frac{1}{\varepsilon}(I(\varepsilon) - I(0)) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)$$

where

$$I_1(\varepsilon) = \frac{1}{\varepsilon}\left( \int_{\Omega_\varepsilon} f_0\, dx - \int_{\Omega_0} f_0\, dx \right),$$

$$I_2(\varepsilon) = \int_{\Omega_\varepsilon} \left( \frac{f_\varepsilon - f_0}{\varepsilon} - g \right) dx \quad \text{and} \quad I_3(\varepsilon) = \int_{\Omega_\varepsilon} g\, dx.$$

It follows from Theorem 1.1 that

$$\lim_{\varepsilon \to 0^+} I_1(\varepsilon) = \int_{\partial\Omega_0} f_0(x) \left( P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x)) \right) d\sigma(x),$$

and from the assumption, since

$$|I_2(\varepsilon)| \le \int_D \left| \frac{f_\varepsilon - f_0}{\varepsilon} - g \right| dx,$$

that $I_2(\varepsilon) \to 0$ when $\varepsilon \to 0$. At last, since the characteristic functions $\mathbf{1}_{\Omega_\varepsilon}$ converge almost everywhere to the characteristic function $\mathbf{1}_{\Omega_0}$ as one can check easily, it follows from the Lebesgue convergence theorem that $I_3(\varepsilon) \to \int_{\Omega_0} g(x)\, dx$ when $\varepsilon \to 0$, which achieves the proof of the corollary. □

## 4. Proof of Theorem 3.1

Let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $\Omega_0, \Omega \in \mathcal{U}$. The proof is fairly long and, in a first stage, we shall prove the formula assuming that $\Omega$ is *strongly convex*, that is, near each point of its boundary, the open set $\Omega$ is defined by $\{\varphi < 0\}$ and its boundary $\partial\Omega$ by $\{\varphi = 0\}$, with some $C^2$ function $\varphi$ whose *Hessian matrix is strictly positive.* Recall that such an $\Omega$ is then strictly convex, in the sense that it contains the (relative) interior of all intervals whose endpoints are on the boundary. Note also that this strict convexity is not equivalent to the strong convexity.

### 4.1. The case where $\Omega$ is strongly convex

The proof is based on some geometric construction which will show that the set $(1-\varepsilon)\Omega_0+\varepsilon\Omega$ is the deformation of $\Omega_0$ via some diffeomorphism (in fact, a Lipschitz homeomorphism), which reduces the problem to a known situation. We shall need several lemmas.

We start by constructing a map between $\overline{\Omega}_0$ and $\overline{\Omega}$. In fact, the construction begins with the boundaries $\Gamma_0=\partial\Omega_0$ and $\Gamma=\partial\Omega$.

**Lemma 4.1.** *Let $\Omega_0,\Omega\in\mathcal{U}$ and assume that $\Omega$ is strongly convex. Then, there exists a map $a_0:\Gamma_0\to\Gamma$, such that*

(*i*) *For all $x\in\Gamma_0$, $P_\Omega(\nu_0(x))=\nu_0(x).a_0(x)$.*
(*ii*) *For all $x\in\Gamma_0$, $\nu(a_0(x))=\nu_0(x)$ where $\nu(y)$ denotes the exterior unit normal vector to $\Gamma$ at $y$.*
(*iii*) *The map $a_0:\Gamma_0\to\Gamma$ is of class $C^1$.*

**Proof.** (*i*) The continuous function $L_x:y\mapsto\nu_0(x).y$ reaches its maximum on the compact set $\overline{\Omega}$ at least at one point $y_1\in\overline{\Omega}$. It follows from the linearity of this function that $y_1\in\partial\overline{\Omega}=\Gamma$, and from the strong convexity of $\overline{\Omega}$ that $y_1$ is unique. By setting $a_0(x)=y_1$, we have, of course, $P_\Omega(\nu_0(x))=\nu_0(x).a_0(x)$.

(*ii*) Since $L_x(a_0(x))=\max_\Gamma L_x$, $a_0(x)$ is a critical point for $L_x|_\Gamma$. Hence,

$$d(L_x|_\Gamma)(a_0(x))=0.$$

Using the linearity of $L_x$ and the formula $d(L_x|_\Gamma)(y)=dL_x(y)|_{T_y\Gamma}$, we obtain that

$$L_x(v)=0,\quad\forall v\in T_{a_0(x)}\Gamma,$$

and consequently that

$$\nu(a_0(x))=\lambda(x)\,\nu_0(x),$$

where $\lambda(x)=\pm1$. To show that $\lambda(x)=1$, write

$$\nu(a_0(x)).a_0(x)=\lambda(x)\,\nu_0(x).a_0(x)=\lambda(x)\,P_\Omega(\nu_0(x)).$$

Assume first that $0\in\Omega$. So, $P_\Omega(\nu_0(x))\geq 0$, and since $\Omega$ is open and convex, we have $\nu(y).y>0$, for all $y\in\Gamma$. Hence, $\lambda(x)=1$. If $0\notin\Omega$, we perform a translation. Let $c\in\Omega$ and denote by $\widetilde{\Omega}=\Omega-c$, by $\widetilde{\nu}$ the exterior unit normal vector field to $\partial\widetilde{\Omega}$ and by $\widetilde{a}_0:\Gamma_0\to\partial\widetilde{\Omega}$ the map one obtains by the first part when $\Omega$ is replaced by $\widetilde{\Omega}$. Clearly, we have

$$P_\Omega(\nu_0(x))=P_{\widetilde{\Omega}}(\nu_0(x))+c.\nu_0(x)=\nu_0(x).\widetilde{a}_0(x)+c.\nu_0(x),$$

so that, by uniqueness, $\widetilde{a}_0(x)=a_0(x)-c$. Since $\nu(a_0(x))=\widetilde{\nu}(\widetilde{a}_0(x))$, we have the same $\lambda(x)$ which is equal to 1 by the first case, since $0\in\widetilde{\Omega}$.

(*iii*) Let $x_0$ be an arbitrary point in $\Gamma_0$ and let us show that $a_0$ is of class $C^1$ near $x_0$, that is, in a neighbourhood of $x_0$ on $\Gamma_0$. We can assume that $\Gamma$ is defined by the equation $\varphi = 0$ near $y_0 = a_0(x_0)$, where $\varphi$ is a $C^2$ function in a neighbourhood (in $\mathbb{R}^n$) of $y_0$ and such that $\nu = \nabla\varphi/|\nabla\varphi|$ and, since $\Omega$ is strongly convex, $\varphi''$ is a strictly positive matrix at $y_0$. Now, the differential of $\nu$ (as a map on $\mathbb{R}^n$) at $y_0$ can be written as

$$d\nu(y_0) = \psi(y_0)\,\varphi''(y_0) + \nabla\varphi(y_0)\,{}^t\nabla\psi(y_0) \quad \text{where } \psi(y) = \frac{1}{|\nabla\varphi(y)|}.$$

Let $v \in \mathbb{R}^n$ be tangent to $\Gamma$ at $y_0$ and such that $d\nu(y_0)v = 0$. Thus, $(d\nu(y_0)v).v = 0$, that is,

$$\psi(y_0)\,(\varphi''(y_0)v).v + (\nabla\varphi(y_0).v)\,({}^t\nabla\psi(y_0)\,v) = 0,$$

which is equivalent to $(\varphi''(y_0)v).v = 0$ since $\nabla\varphi(y_0).v = 0$, and to $v = 0$ since $\varphi''(y_0)$ is a positive definite matrix. Hence, we have proved that $d\nu(y_0)|_{T_{y_0}\Gamma}$ is injective, or, more precisely, that $d(\nu|_\Gamma)(y_0) : T_{y_0}\Gamma \to T_{\nu(y_0)}S$ is bijective, where $S$ is the unit sphere in $\mathbb{R}^n$. It follows then from the local inverse function theorem that $\nu$ is a $C^1$ diffeomorphism from a neighborhood of $y_0$ in $\Gamma$ onto a neighbourhood of $\nu(y_0)$ in $S$. Since $\nu(a_0(x)) = \nu_0(x)$, we can write $a_0(x) = \nu^{-1}(\nu_0(x))$ for all $x$ near of $x_0$, which proves that $a_0$ is of class $C^1$ near $x_0$. $\square$

Now, we would like to extend $a_0$ to a map from $\overline{\Omega}_0$ to $\overline{\Omega}$ and even from $\mathbb{R}^n$ to $\mathbb{R}^n$. To do this, we shall use what is called the distance function or gauge function. This is another type of function which is associated to a convex set and characterizes it.

Let $U$ be an open convex bounded subset of $\mathbb{R}^n$ that contains 0. Then, its *distance function* is defined by

$$F_U(x) = \inf\left\{t > 0\,;\ \frac{x}{t} \in U\right\} = 1/\sup\{\lambda > 0\,;\ \lambda\, x \in U\}.$$

The following lemma explains why $F_U$ is called the distance function associated to $U$.

**Lemma 4.2.** *If $U$ is an open convex bounded neighbourhood of 0, then, $F_U$ is a non negative continuous convex positively homogeneous function of degree 1. More precisely, we have the following properties:*

(*i*) $F_U(0) = 0,\ F_U(x) > 0,\ \forall x \neq 0.$
(*ii*) $F_U(\lambda\, x) = \lambda\, F_U(x),\ \forall\lambda > 0,\ \forall x \in \mathbb{R}^n.$
(*iii*) $F_U(x + y) \leq F_U(x) + F_U(y),\ \forall x, y \in \mathbb{R}^n.$
(*iv*) $U = \{x \in \mathbb{R}^n;\ F_U(x) < 1\}.$

When $U$ is symmetric with respect to 0, the distance function $F_U$ is just the norm whose open unit ball is $U$. For a proof of this lemma, we refer for example to [4].

We also need the following

**Lemma 4.3.** *If $U$ is an open convex bounded neighbourhood of 0 which is of class $C^k$, $k \geq 1$, then, the distance function $F_U$ is of class $C^k$ in $\mathbb{R}^n \setminus \{0\}$.*

**Proof.** Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ and set $t_0 = \frac{1}{F_U(x_0)}$. Clearly, we have $\partial U = \{x \in \mathbb{R}^n;\ F_U(x) = 1\}$, so that, $t_0 x_0 \in \partial U$. Since, we can locally write $\partial U = \{\varphi = 0\}$ with a function $\varphi$ which is $C^k$ near $t_0 x_0$, we have $\varphi(t_0 x_0) = 0$, and it follows from the assumptions on $U$ that $\nabla\varphi(t_0 x_0).t_0 x_0 \neq 0$. Consider now the $C^k$ function $\Phi$ defined by $\Phi(t, x) = \varphi(tx)$ near $(t_0, x_0)$. We have

$$\frac{\partial \Phi}{\partial t}(t_0, x_0) = \nabla\varphi(t_0 x_0) x_0 \neq 0.$$

Hence, it follows from the implicit function theorem that the equation $\Phi(t, x) = 0$ can be solved uniquely in $t$ near $(t_0, x_0)$ by a $C^k$ function $\psi(x)$, that is, $\Phi(t, x) = 0 \iff t = \psi(x)$ near $(t_0, x_0)$. Since $\psi(x) = \frac{1}{F_U(x)}$ near $x_0$, we conclude that $F_U$ is of class $C^k$ near $x_0$. □

Now, let us extend the map $a_0$ to $\mathbb{R}^n$. We state the result as

**Lemma 4.4.** *If $\Omega_0, \Omega \in \mathcal{U}$ and $\Omega$ is strongly convex, then, there exists a map $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following properties:*

(*i*) $a = a_0$ *on* $\Gamma_0$.

(*ii*) $a(\Omega_0) \subset \Omega$ *and* $a\big(\mathbb{R}^n \setminus \overline{\Omega}_0\big) \subset \mathbb{R}^n \setminus \overline{\Omega}$.

(*iii*) *$a$ is of class $C^1$ away from some point of $\mathbb{R}^n$ and is everywhere Lipschitz continuous.*

**Proof.** Let $c_0 \in \Omega_0$ and $c \in \Omega$ be fixed points. Since the translated sets $\Omega_0 - c_0$ and $\Omega - c$ are open convex neighbourhoods of 0, we can consider their distance functions which we shall denote simply by $F_0$ and $F$ respectively. If $x \in \mathbb{R}^n \setminus \{c_0\}$, then, $\frac{x-c_0}{F_0(x-c_0)} + c_0$ is a point on $\Gamma_0 = \partial\Omega_0$, so that $a_0\left(\frac{x-c_0}{F_0(x-c_0)} + c_0\right)$ is well defined and belongs to $\Gamma = \partial\Omega$. This suggests to set $a(x) = c + b(x - c_0)$ where $b : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$b(x) = \begin{cases} 0 & \text{if } x = 0, \\ F_0(x)\left(a_0\Big(\frac{x}{F_0(x)} + c_0\Big) - c\right) & \text{if } x \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{5}$$

Clearly, $a = a_0$ on $\Gamma_0$ and if $x \neq c_0$, since $F\left(a(x) - c\right) = F_0(x - c_0)$, we have $a(x) \in \Omega$ if $x \in \Omega_0$, and $a(x) \in \mathbb{R}^n \setminus \overline{\Omega}$ if $x \in \mathbb{R}^n \setminus \overline{\Omega}_0$. Finally, it follows from the preceding lemmas that $b$ is of class $C^1$ in $\mathbb{R}^n \setminus \{0\}$ and positively homogeneous of degree 1. Hence, $a$ is of class $C^1$ in $\mathbb{R}^n \setminus \{c_0\}$ and is everywhere Lipschitz continuous. □

In the next lemma, we shall use basically the map $a$ to parametrize the set $(1-\varepsilon)\Omega_0 + \varepsilon\Omega$ by another map which is almost a diffeomorphism.

**Lemma 4.5.** *Let $\Omega_0, \Omega \in \mathcal{U}$, $\Omega$ being strongly convex, and let $a : \mathbb{R}^n \to \mathbb{R}^n$ be the map constructed in the preceding lemma and set*

$$\Phi_\varepsilon(x) = (1-\varepsilon)x + \varepsilon a(x), \quad x \in \mathbb{R}^n,\ \varepsilon \in [0, 1].$$

*Then, for sufficiently small $\varepsilon$,* (*i*) *$\Phi_\varepsilon$ is a Lipschitz homeomorphism from $\mathbb{R}^n$ onto itself.*

(*ii*) *$\Phi_\varepsilon$ is a $C^1$ diffeomorphism from the complement of a point (of $\Omega_0$) in $\mathbb{R}^n$ onto the complement of a point in $\mathbb{R}^n$.*

(*iii*) $\Phi_\varepsilon(\Omega_0) = (1-\varepsilon)\Omega_0 + \varepsilon\Omega$.

**Proof.** (*i*) Since $\Phi_\varepsilon(x) = x+\varepsilon(a(x)-x)$, this part follows from a classical lemma which says that we obtain a Lipschitz homeomorphism from $\mathbb{R}^n$ onto itself if we perturb the identity map by a Lipschitz map whose Lipschitz constant is less than 1.

(*ii*) We already know that $a$ is of class $C^1$ in $\mathbb{R}^n \setminus \{c_0\}$. Since $a' \in L^\infty(\mathbb{R}^n)$, $\Phi'_\varepsilon(x)$ is invertible for all $x \in \mathbb{R}^n \setminus \{c_0\}$ for sufficiently small $\varepsilon$. Thus, the result follows from the inverse function theorem and the first part.

(*iii*) Let us denote by $\Omega_\varepsilon$ the set $(1-\varepsilon)\Omega_0 + \varepsilon\Omega$. Clearly, we have the inclusion $\Phi_\varepsilon(\Omega_0) \subset \Omega_\varepsilon$. Since $\Omega_\varepsilon$ is connected (because it is convex), we shall obtain the equality if we can show that $\Phi_\varepsilon(\Omega_0)$ is open and closed in $\Omega_\varepsilon$. The fact that it is open being obvious, let us show that it is also closed. Let $(y_k) = (\Phi_\varepsilon(x_k))$ be a sequence in $\Phi_\varepsilon(\Omega_0)$ such that $y_k \to y$ in $\Omega_\varepsilon$. It suffices to prove that $y \in \Phi_\varepsilon(\Omega_0)$. Of course, $(x_k)$ is a sequence in $\Omega_0$, and, since $\Omega_0$ is bounded, we can assume that $x_k \to x$ with $x \in \overline{\Omega}_0$. If $x \in \Omega_0$, the proof is finished. Assume that $x \in \Gamma_0 = \partial\Omega_0$. It follows from Lemma 4.1 that $a(x) \in \Gamma = \partial\Omega$ and that $\nu(a(x)) = \nu_0(x)$. By homothety (with positive ratio), we have the same situation with the points $(1-\varepsilon)x$ and $\varepsilon a(x)$: They are boundary points of $(1-\varepsilon)\Omega_0$ and $\varepsilon\Omega$ respectively, with normals equal to $\nu_0(x)$. This implies that the convex sets $(1-\varepsilon)\Omega_0$ and $\varepsilon\Omega$ have the same tangent space at these boundary points and, moreover, they are both on the same side with respect to this tangent space. It follows then from the lemma below that $(1-\varepsilon)x + \varepsilon a(x)$ is a boundary point of $\Omega_\varepsilon$, that is, that $y = \Phi_\varepsilon(x) \in \partial\Omega_\varepsilon$, which is a contradiction. Thus, the case $x \in \Gamma_0$ is impossible. □

**Lemma 4.6.** *Let $A$, $B$ be subsets of $\mathbb{R}^n$, $a \in \partial A$ and $b \in \partial B$. Assume that there exists a hyperplane $H$ such that both $A-a$ and $B-b$ are in the same closed half space defined by $H$. Then, $a+b \in \partial(A+B)$.*

**Proof.** Let $\varphi$ be a linear form on $\mathbb{R}^n$ such that $H = \mathrm{Ker}(\varphi)$,

$$\varphi(x) \geq \varphi(a), \quad \forall x \in A \quad \text{and} \quad \varphi(y) \geq \varphi(b), \quad \forall y \in B.$$

Then, by summing the two inequalities, we obtain

$$\varphi(z) \geq \varphi(a+b), \quad \forall z \in A+B.$$

Take $v \in \mathbb{R}^n$ such that $\varphi(v) = 1$ and set $z_k = a+b-\frac{1}{k}v$, $k \geq 1$. Clearly, $(z_k)$ is a sequence such that $\varphi(z_k) < \varphi(a+b)$, for all $k$, that is, $z_k \notin \overline{A+B}$, for all $k$. However, $z_k \to a+b$ as $k \to \infty$. Hence, $a+b \in \partial(A+B)$. □

We can now finish the proof of Theorem 3.1 in the case where $\Omega$ is strongly convex. In fact, since $\Phi_\varepsilon(\Omega_0) = (1-\varepsilon)\Omega_0 + \varepsilon\Omega$ by Lemma 4.5, we have reduced the problem to the case of a deformation of $\Omega_0$ by a diffeomorphism or, more precisely, a Lipschitz homeomorphism. Even if $\Phi_\varepsilon$ is not a $C^1$ diffeomorphism in $\mathbb{R}^n$, it is possible to use the known argument which combines a change of variables (see for example [5] for Lipschitz changes of variables) and an approximation by smooth functions (or vice-versa) to obtain that

$$\lim_{\varepsilon\to 0^+} \frac{J(\Phi_\varepsilon(\Omega_0)) - J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x)\,\nu_0(x).(a(x)-x)\,d\sigma(x).$$

The details can be found, for example, in [3], Chapter 5. It remains to remark that it follows from Lemma 4.1 that $P_\Omega(\nu_0(x)) = \nu_0(x).a(x)$ and it is easy to check that $a(x) = x$ when $\Omega = \Omega_0$. Hence, $\nu_0(x).(a(x)-x) = P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x))$.

### 4.2. The general case

The convex domain $\Omega$ being now arbitrary in $\mathcal{U}$, a natural idea is to approximate it by a sequence of strongly convex ones. This is indeed what we shall do. So let us recall the following approximation lemma. Even if it is a classical result, we shall give a proof since we need a particular approximation in the proof of Theorem 3.1.

**Lemma 4.7.** *Let $\Omega \in \mathcal{U}$. Then, there exists a sequence $(\Omega^k)_{k\geq 1}$ of strongly convex open subsets of $\Omega$ such that*

(*i*) $\overline{\Omega}^k \longrightarrow \overline{\Omega}$ *for the Hausdorff topology as $k \longrightarrow \infty$.*
(*ii*) $\operatorname{meas}(\Omega^k) \longrightarrow \operatorname{meas}(\Omega)$ *as $k \longrightarrow \infty$.*

**Proof.** We can describe $\Omega$ and its boundary as

$$\Omega = \{x \in \mathbb{R}^n \ ; \ \varphi(x) < 0\} \quad \text{and} \quad \partial\Omega = \{x \in \mathbb{R}^n \ ; \ \varphi(x) = 0\},$$

where $\varphi$ is a continuous convex function in $\mathbb{R}^n$ which is of class $C^2$ in a neighbourhood $V$ of $\partial\Omega$. Let us define, for $k \geq 1$,

$$\varphi_k(x) = \varphi(x) + \frac{1}{k}(x-c)^2 \quad \text{and} \quad \Omega^k = \{x \in \mathbb{R}^n \ ; \ \varphi_k(x) < 0\}, \tag{6}$$

where $c \in \Omega$ is fixed. Clearly, $\Omega^k$ is non empty, open, convex and $\Omega^k \subset \Omega$, for all $k$. Moreover, since $\varphi_k$ is convex, we also have $\partial\Omega^k = \{x \in \mathbb{R}^n \ ; \ \varphi_k(x) = 0\}$; hence, $V$ is also a neighbourhood of $\partial\Omega^k$ for large enough $k$, say for $k \geq k_0$, and, at least for such $k$, $\varphi_k$ is of class $C^2$ in a neighbourhood of $\partial\Omega^k$ with $\varphi_k'' = \varphi'' + \frac{2}{k}I > 0$ there. Thus, $\Omega^k$ is strongly convex for $k \geq k_0$. Let us now check the convergences stated in the lemma for the sequence $(\Omega^k)$.

(*i*) Let $d^H$ denote the Hausdorff distance for compact sets in $\mathbb{R}^n$. Let us also denote by $\nu(x)$ (resp. $\nu_k(x)$) the exterior unit normal vector to $\partial\Omega$ (resp. $\partial\Omega_k$) at $x$. Since $\Omega_k \subset \Omega$, we have $d^H(\overline{\Omega}_k, \overline{\Omega}) = \sup_{x\in\overline{\Omega}} d(x, \overline{\Omega}_k)$. Let us start by taking $x_k \in \overline{\Omega}$ such that $d^H(\overline{\Omega}_k, \overline{\Omega}) = d(x_k, \overline{\Omega}_k)$. In fact, since $\overline{\Omega}_k \subset \Omega$ and $\overline{\Omega}_k$ is convex, we have more precisely $x_k \in \partial\Omega$. We can also write $d(x_k, \overline{\Omega}_k) = |x_k - y_k|$ with some $y_k \in \partial\Omega_k$. In fact, $y_k$ is the normal projection of $x_k$ on the convex domain $\Omega_k$, that is,

$$x_k - y_k = \lambda_k \, \nu_k(y_k) \tag{7}$$

with $\lambda_k = |x_k - y_k|$. Now, $(x_k)$ and $(y_k)$ being bounded sequences, we can extract convergent subsequences $(x_{k_j})$ and $(y_{k_j})$ with $a = \lim x_{k_j} \in \partial\Omega$ and $b = \lim y_{k_j} \in \overline{\Omega}$. In fact, we have $0 = \varphi_{k_j}(y_{k_j}) = \varphi(y_{k_j}) + \frac{1}{k_j}\,(y_{k_j} - c)^2 \longrightarrow \varphi(b) = 0$. Hence, $b \in \partial\Omega$. Using the expression

$$\nu_k(x) = \frac{\nabla\varphi_k(x)}{|\nabla\varphi_k(x)|} = \frac{\nabla\varphi(x) + \frac{2}{k}\,(x-c)}{|\nabla\varphi(x) + \frac{2}{k}\,(x-c)|},$$

which holds at least in $V$, we can pass to the limit in (7) with $k = k_j$ to obtain

$$a - b = |a - b| \, \nu(b),$$

which implies that $a = b$. Therefore, $d^H(\overline{\Omega}_{k_j}, \overline{\Omega}) = |x_{k_j} - y_{k_j}| \to 0$. Since we have at the same time proved that 0 is the unique accumulation value of the numerical bounded sequence $(d^H(\overline{\Omega}_k, \overline{\Omega}))$, we can conclude that $d^H(\overline{\Omega}_k, \overline{\Omega}) \to 0$ as $k \to \infty$.

$(ii)$ This follows from $(i)$ and the continuity of the Lebesgue measure with respect to the Hausdorff distance. See for example [7]. □

The next and last lemma contains the crucial argument in the proof of the theorem.

**Lemma 4.8.** *Let $\Omega_0, \Omega \in \mathcal{U}$. Then, there exists a sequence $(\Omega^k)_{k\geq 1}$ of strongly convex open subsets of $\Omega$ satisfying the conclusions of Lemma 4.7 and such that:*

$(i)$ *For all $\varepsilon \in [0,1]$, $\lim_{k\to\infty} \operatorname{meas}(\Omega^k_\varepsilon) = \operatorname{meas}(\Omega_\varepsilon)$, where $\Omega^k_\varepsilon = (1-\varepsilon)\Omega_0 + \varepsilon\Omega^k$ and $\Omega_\varepsilon = (1-\varepsilon)\Omega_0 + \varepsilon\Omega$.*

$(ii)$ *More precisely, there exists a non negative sequence $(\gamma_k)$ such that $\gamma_k \to 0$ and, for all $k \geq 1$ and all $\varepsilon \in [0,1]$,*

$$\operatorname{meas}(\Omega_\varepsilon) - \operatorname{meas}(\Omega^k_\varepsilon) \leq \varepsilon\,\gamma_k.$$

$(iii)$ *For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and all $\delta > 0$, there exists $k_1 \in \mathbb{N}$ such that, for all $k \geq k_1$ and all $\varepsilon \in [0,1]$,*

$$\left| \int_{\Omega^k_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon} f(x)\,dx \right| \leq \varepsilon\,\delta.$$

**Proof.** Let $(\Omega^k)$ be as given by Lemma 4.7.

$(i)$ Let us use the formula

$$\sup_{|x|=1} |P_K(x) - P_L(x)| = d^H(K, L), \quad K, L \in \mathcal{C}, \tag{8}$$

where $d^H$ denotes the Hausdorff distance for compact sets in $\mathbb{R}^n$. See, for example, [2], [7] or [6]. Using (8), we can write

$$\begin{aligned} d^H(\overline{\Omega}_\varepsilon, \overline{\Omega}_\varepsilon^k) &= \sup_{|x|=1} |P_{\Omega_\varepsilon}(x) - P_{\Omega_\varepsilon^k}(x)| \\ &= \varepsilon \sup_{|x|=1} |P_{\Omega}(x) - P_{\Omega^k}(x)| = \varepsilon\, d^H(\overline{\Omega}, \overline{\Omega}^k) \longrightarrow 0, \end{aligned}$$

as $k \longrightarrow \infty$, as it follows from Lemma 4.7. The result is then the consequence of the continuity of the Lebesgue measure with respect to the Hausdorff distance.

$(ii)$ According to the Brunn-Minkowski theory on mixed volumes, we can write

$$\operatorname{meas}(\Omega_\varepsilon) = \operatorname{meas}(\overline{\Omega}_\varepsilon) = \operatorname{meas}[(1-\varepsilon)\overline{\Omega}_0 + \varepsilon\overline{\Omega}] = \sum_{j=0}^{n} C_j(\Omega_0, \Omega)\,(1-\varepsilon)^{n-j}\,\varepsilon^j,$$

where the non negative constants $C_j(\Omega_0, \Omega)$, the so called mixed volumes, depend only on $\Omega_0$ and $\Omega$. In particular, $C_0(\Omega_0, \Omega) = \operatorname{meas}(\Omega_0)$. We refer for this to [4], [6] or [7]. Similarly,

$$\operatorname{meas}(\Omega_\varepsilon^k) = \operatorname{meas}(\overline{\Omega}_\varepsilon^k) = \sum_{j=0}^{n} C_j(\Omega_0, \Omega^k)\,(1-\varepsilon)^{n-j}\,\varepsilon^j.$$

Therefore, $\operatorname{meas}(\Omega_\varepsilon \backslash \Omega_\varepsilon^k) = \sum_{j=0}^{n} \alpha_j^k\,(1-\varepsilon)^{n-j}\,\varepsilon^j$ with $\alpha_j^k = C_j(\Omega_0, \Omega) - C_j(\Omega_0, \Omega^k)$. An obvious and, at the same time, important remark is that $\alpha_0^k = \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_0) = 0$. Another remark which is not less important is that $\alpha_j^k \geq 0$ for all $j$ and $k$, thanks to the fact that the mixed volumes are monotone with respect to each factor. We deduce from $(i)$ and these remarks that, for all $j$, $\alpha_j^k \to 0$ as $k \to \infty$ and, consequently, that

$$\operatorname{meas}(\Omega_\varepsilon \setminus \Omega_\varepsilon^k) = \varepsilon \sum_{j=1}^{n} \alpha_j^k (1-\varepsilon)^{n-j}\,\varepsilon^{j-1} \leq \varepsilon \sum_{j=1}^{n} \alpha_j^k = \varepsilon\gamma_k,$$

where $\gamma_k = \sum_{j=1}^{n} \alpha_j^k$ is independent of $\varepsilon$ and $\gamma_k \to 0$ as $k \to \infty$, which proves the second part of the lemma.

$(iii)$ Let $\delta > 0$. We can write

$$\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon^k} f(x)\,dx\right) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon \backslash \Omega_\varepsilon^k} f(x)\,dx = \int_{A_\varepsilon^k} \widetilde{f}(x,t)\,dx\,dt,$$

where $A_\varepsilon^k = (\Omega_\varepsilon \setminus \Omega_\varepsilon^k) \times [0, \frac{1}{\varepsilon}]$ and $\widetilde{f}(x,t) = f(x)$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Since $\widetilde{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$, it follows from the absolute continuity property with respect to the Lebesgue measure that there exists $\eta > 0$ such that, for all measurable subsets $A$ of $\mathbb{R}^{n+1}$,

$$\operatorname{meas}(A) \leq \eta \quad \Longrightarrow \quad \int_A |\widetilde{f}| dx dt \leq \delta.$$

Now, we have

$$\text{meas}(A_\varepsilon^k) = \frac{1}{\varepsilon}\,\text{meas}(\Omega_\varepsilon \setminus \Omega_\varepsilon^k) \leq \frac{1}{\varepsilon}\,\varepsilon\,\gamma_k = \gamma_k \longrightarrow 0,$$

as it follows from $(ii)$. Hence, since the sequence $(\gamma_k)$ does not depend on $\varepsilon$, there exists $k_1 \in \mathbb{N}$ such that, for all $\varepsilon \in [0,1]$ and all $k \geq k_1$, $\text{meas}(A_\varepsilon^k) \leq \eta$, and therefore,

$$\left|\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon^k} f(x)\,dx\right| \leq \varepsilon \int_{A_\varepsilon^k} |\widetilde{f}(x,t)|\,dx\,dt \leq \varepsilon\,\delta,$$

for all $k \geq k_1$ and all $\varepsilon \in [0,1]$, which achieves the proof. $\square$

**End of proof of Theorem 3.1.** Using the notations of the preceding lemma, we can write

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_0} f(x)\,dx\right) - \int_{\partial\Omega_0} f(x)\,(P_\Omega - P_{\Omega_0})\,(\nu_0(x))\,d\sigma \\
&= \frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon^k} f(x)\,dx\right) \\
&\qquad + \frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon^k} f(x)\,dx - \int_{\Omega_0} f(x)\,dx\right) - \int_{\partial\Omega_0} f(x)\,(P_{\Omega^k} - P_{\Omega_0})\,(\nu_0(x))\,d\sigma \\
&\qquad + \int_{\partial\Omega_0} f(x)\,(P_{\Omega^k} - P_\Omega)\,(\nu_0(x))\,d\sigma.
\end{aligned}$$

Let $\delta > 0$ be arbitrary. It follows from Lemma 4.8 that there exists $k_1 \in \mathbb{N}$ such that, for all $k \geq k_1$ and all $\varepsilon \in ]0,1]$,

$$\frac{1}{\varepsilon}\left|\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon^k} f(x)\,dx\right| \leq \delta.$$

Furthermore, we have

$$\begin{aligned}
\left|\int_{\partial\Omega_0} f\,(P_{\Omega^k} - P_\Omega) \circ \nu_0\,d\sigma\right| &\leq \int_{\partial\Omega_0} |f|\,d\sigma \sup_{|y|=1} |P_{\Omega^k}(y) - P_\Omega(y)| \\
&= \int_{\partial\Omega_0} |f|\,d\sigma\; d^H(\overline{\Omega}^k, \overline{\Omega})
\end{aligned}$$

which tends to 0 as $k \to \infty$ by Lemma 4.7. Hence, there exists $k_2 \in \mathbb{N}$ such that, for all $k \geq k_2$,

$$\left|\int_{\partial\Omega_0} f\,(P_{\Omega^k} - P_\Omega) \circ \nu_0\,d\sigma\right| \leq \delta.$$

Now, let $k_0$ stand for $\max\{k_1, k_2\}$. The set $\Omega^{k_0}$ being strongly convex, it follows from the study of this case done above that there exists $\varepsilon_\delta > 0$ such that, for all $\varepsilon \leq \varepsilon_\delta$,

$$\left|\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon^{k_0}} f(x)\,dx - \int_{\Omega_0} f(x)\,dx\right) - \int_{\partial\Omega_0} f(x)\left(P_{\Omega^{k_0}} - P_{\Omega_0}\right)(\nu_0(x))\,d\sigma\right| \leq \delta.$$

Summing up, we have

$$\left|\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_0} f(x)\,dx\right) - \int_{\partial\Omega_0} f(x)\left(P_{\Omega} - P_{\Omega_0}\right)(\nu_0(x))\,d\sigma\right| \leq 3\delta,$$

for all $\varepsilon \leq \varepsilon_\delta$, and this achieves the proof of Theorem 1.1. □

## 5. Remarks and complements

**Remark 5.1.** One can wonder why we do not use the classical and simpler deformation $\widetilde{\Omega}_\varepsilon = \Omega_0 + \varepsilon\Omega$ of $\Omega_0$. As it is said before, we have followed [1], and for the applications given there, the deformation $\Omega_\varepsilon = (1-\varepsilon)\Omega_0 + \varepsilon\Omega$ seems to be more appropriate in the context of convexity. Essentially, it allows sets of admissible domains like

$$\begin{gathered}\{\Omega; \Omega \subset D\}, \quad D \text{ being a fixed convex subset of } \mathbb{R}^n, \\ \text{or} \quad \{\Omega; E \subset \Omega\}, \quad E \text{ being a fixed subset of } \mathbb{R}^n, \\ \text{or} \quad \{\Omega; E \subset \Omega \subset D\},\end{gathered}$$

to be stable under such a deformation for all $\varepsilon \in [0,1]$.

Anyhow, one can show that the two deformations give rise to equivalent results. Assume for example that the limit

$$\lim_{\varepsilon\to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega_0)}{\varepsilon}$$

exists and is equal to

$$\int_{\partial\Omega_0} f\left(P_\Omega - P_{\Omega_0}\right) \circ \nu_0\,d\sigma,$$

and let us consider the limit

$$\lim_{\varepsilon\to 0^+} \frac{J(\widetilde{\Omega}_\varepsilon) - J(\Omega_0)}{\varepsilon}.$$

We can write $\varepsilon = \delta/(1-\delta)$ with $\delta = \varepsilon/(1+\varepsilon)$, so that $\widetilde{\Omega}_\varepsilon = \frac{1}{1-\delta}\Omega_\delta$, and

$$\frac{J(\widetilde{\Omega}_\varepsilon) - J(\Omega_0)}{\varepsilon} = \frac{1-\delta}{\delta}\left(\int_{\Omega_\delta} f_\delta\,dx - \int_{\Omega_0} f\,dx\right)$$
$$\text{where } f_\delta(x) = f\left(\frac{x}{1-\delta}\right)\left(\frac{1}{1-\delta}\right)^n.$$

Now, we have

$$\frac{f_\delta - f}{\delta} \longrightarrow nf(x) + f'(x)x \quad \text{in } L^1_{\mathrm{loc}},$$

as $\delta \to 0$. Indeed, this is easy if $f$ is of class $C^1$ with a compact support, and the general case is obtained by using the density of the space of test functions in $W^{1,1}_{\mathrm{loc}}$. Then, applying Corollary 3.2 (or rather its proof) yields the formula

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{J(\widetilde{\Omega}_\varepsilon) - J(\Omega_0)}{\varepsilon} &= \int_{\Omega_0} (nf(x) + f'(x)x)\, dx + \int_{\partial\Omega_0} f\,(P_\Omega - P_{\Omega_0}) \circ \nu_0 \, d\sigma \\ &= \int_{\partial\Omega_0} f\, P_\Omega \circ \nu_0 \, d\sigma,\end{aligned}$$

where the last equality is obtained by integration by parts.

Conversely, using almost the same argument by writing $\Omega_\varepsilon = (1-\varepsilon)\widetilde{\Omega}_\delta$ with $\delta = \varepsilon/(1-\varepsilon)$, one can easily show that the last formula implies that of Theorem 3.1. Hence the following

**Theorem 5.2.** *Under the assumptions of Theorem* 3.1*, we have*

$$\lim_{\varepsilon\to 0^+} \frac{J(\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f\, P_\Omega \circ \nu_0 \, d\sigma. \tag{9}$$

**Remark 5.3.** The formula of Theorem 5.2 is an extension of a well known formula in the Brunn-Minkowski theory which says that, if $K$ and $L$ are convex bodies in $\mathbb{R}^n$,

$$\lim_{\varepsilon\to 0^+} \frac{\mathrm{meas}(K + \varepsilon L) - \mathrm{meas}(K)}{\varepsilon} = \int_{\partial K} P_L(\nu(x))\, d\sigma(x), \tag{10}$$

where $\nu(x)$ denotes the exterior unit normal vector to $\partial K$ at $x$. See [7], [4] or [10]. Thus, this is the case $f = 1$ of Theorem 5.2. However, (10) holds without any regularity on the convex sets. Of course, convex bodies have Lipschitz boundaries and this makes the boundary integral in (10) meaningful. Note that the boundary integrals (4) and (9) are also well defined if $\Omega_0$ and $\Omega$ are only convex open bounded subsets of $\mathbb{R}^n$. This is why we believe that, even if the $C^2$ regularity of $\Omega_0$ and $\Omega$ is crucially needed in the geometric construction used in the proof, Theorems 1.1 and 3.1 should hold without any regularity assumption on the domains. Unfortunately, we have not been able to produce a proof of Theorem 1.1 (or Theorem 3.1) based on the formula (10) or its (classical) proof. However, using arguments similar to that used in the proof of Theorem 3.1, we can at least treat the case where $\Omega$ is an arbitrary convex open bounded subset of $\mathbb{R}^n$. The case of an arbitrary convex open bounded $\Omega_0$ seems to be more delicate. Anyhow, we shall return to this problem in a future work. Thus, we have

**Theorem 5.4.** *Let $\Omega$ be an arbitrary convex open bounded subset of $\mathbb{R}^n$. For all $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega_0 \in \mathcal{U}$, we have*

$$\lim_{\varepsilon\to 0^+} \frac{J((1-\varepsilon)\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f\,(P_\Omega - P_{\Omega_0}) \circ \nu_0 \, d\sigma. \tag{11}$$

*and*

$$\lim_{\varepsilon\to 0^+} \frac{J(\Omega_0+\varepsilon\Omega)-J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f\, P_\Omega \circ \nu_0 \, d\sigma. \tag{12}$$

**Proof.** We give here only the proof of (11), that of (12) being similar.

We follow the same idea as that in the second part of the proof of Theorem 1.1, that is, we use the approximation technique, and in fact we have just to precise Lemma 4.7 by

**Lemma 5.5.** *Let $\Omega$ be a convex bounded open subset of $\mathbb{R}^n$. Then, there exists a sequence $(\Omega^k)_{k\geq 1}$ of $C^\infty$ convex bounded open subsets of $\Omega$ such that*

(*i*) $\overline{\Omega}^k \longrightarrow \overline{\Omega}$ *for the Hausdorff topology as* $k \longrightarrow \infty$.
(*ii*) $\mathrm{meas}(\Omega^k) \longrightarrow \mathrm{meas}(\Omega)$ *as* $k \longrightarrow \infty$.

**Proof.** The result on approximation of a convex body by regular ones using the Hausdorff metric is classical. Here, we just indicate how to obtain a sequence of such smooth $\Omega^k$ *which are subsets of* $\Omega$, which is needed further. One can assume that $0 \in \Omega$, that is, the closure $A = \overline{\Omega}$ is a neighbourhood of 0.

We first apply, for example, the regularization process of [7], Theorem 3.3.1, to the polar (or dual) set $A^\star$ of $A$. We obtain a sequence $(A_k^\star)$ of convex bodies whose support functions are $C^\infty$ (away from 0) and such that $d^H(A_k^\star, A^\star) \to 0$ as $k \to \infty$, where $d^H$ denotes the Hausdorff distance for compact sets in $\mathbb{R}^n$. In fact, we even have the precise estimate

$$d^H(A_k^\star, A^\star) \leq \frac{R}{k}, \quad k \geq 1,$$

if $A^\star \subset \overline{B}(0,R)$. Now, it follows from the definition of the Hausdorff distance that, if we set

$$C_k = A_k^\star + \overline{B}\left(0, \frac{R}{k}\right), \quad k \geq 1,$$

we have the inclusion $A^\star \subset C_k$, for all $k \geq 1$, and we remark here that one can even obtain the fact that $C_k$ is a neighbourhood of $A^\star$ just by replacing the $R$ by some $R' > R$. Clearly, the support functions of the $C_k$ are also $C^\infty$ and it follows from the classical formula

$$\sup_{|x|=1} |P_K(x) - P_L(x)| = d^H(K,L), \quad K, L \in \mathcal{C}, \tag{13}$$

that $d^H(C_k, A^\star) \to 0$ as $k \to \infty$ also.

Finally, if we set $A_k = C_k^\star$, that is, $A_k$ is the polar set of $C_k$, one can easily check that:

- $A_k \subset (A^\star)^\star = A, \ \forall k \geq 1.$
- The $A_k$ are $C^\infty$ since their distance functions are the support functions of the $C_k$.

- $d^H(A_k, A) \to 0$ as $k \to \infty$, since $A^\star$ is a neighbourhood of 0.

It remains to take $\Omega^k$ to be the interior of $A_k$ to get the first part of the lemma. Note that we even have $\overline{\Omega}^k = A_k \subset \mathring{A} = \Omega$, $k \geq 1$, since one can take $C_k$ to be a neighbourhood of $A^\star$ as remarked above.

The second part follows from $(i)$ and the continuity of the Lebesgue measure with respect to the Hausdorff distance. See for example [7]. □

Now, the lemma which is similar to Lemma 4.8 also holds with exactly the same proof (using of course the approximating sequence obtained in Lemma 5.5):

**Lemma 5.6.** *Let $\Omega$ be a convex bounded open subset of $\mathbb{R}^n$ and $\Omega_0 \in \mathcal{U}$. Then, there exists a sequence $(\Omega^k)_{k\geq 1}$ of smooth convex open subsets of $\Omega$ satisfying the conclusions of Lemma* 5.5 *and such that:*

$(i)$ *For all $\varepsilon \in [0,1]$, $\lim_{k\to\infty} \operatorname{meas}(\Omega^k_\varepsilon) = \operatorname{meas}(\Omega_\varepsilon)$, where $\Omega^k_\varepsilon = (1-\varepsilon)\Omega_0 + \varepsilon\Omega^k$ and $\Omega_\varepsilon = (1-\varepsilon)\Omega_0 + \varepsilon\Omega$.*

$(ii)$ *More precisely, there exists a non negative sequence $(\gamma_k)$ such that $\gamma_k \to 0$ and, for all $k \geq 1$ and all $\varepsilon \in [0,1]$,*

$$\operatorname{meas}(\Omega_\varepsilon) - \operatorname{meas}(\Omega^k_\varepsilon) \leq \varepsilon\, \gamma_k.$$

$(iii)$ *For all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and all $\delta > 0$, there exists $k_1 \in \mathbb{N}$ such that, for all $k \geq k_1$ and all $\varepsilon \in [0,1]$,*

$$\left| \int_{\Omega^k_\varepsilon} f(x)\, dx - \int_{\Omega_\varepsilon} f(x)\, dx \right| \leq \varepsilon\, \delta.$$

Finally, one can finish the proof of Theorem 5.2 exactly as that of Theorem 3.1 by applying Lemma 5.6 and, of course, the result of Theorem 3.1. □

Of course, one can also extend Corollary 3.2 to the case of an arbitrary $\Omega$, the proof being exactly the same.

**Remark 5.7.** Another relevant question is: What happens when $\varepsilon \to 0^-$ ? Let us consider the following limit

$$\lim_{\varepsilon\to 0^+} \frac{J((1+\varepsilon)\Omega_0 - \varepsilon\Omega) - J(\Omega_0)}{-\varepsilon}. \tag{14}$$

As before, we can write $(1+\varepsilon)\Omega_0 - \varepsilon\Omega = \frac{1}{1-2\delta}((1-\delta)\Omega_0 + \delta(-\Omega))$ with $\delta = \varepsilon/(1+2\varepsilon)$. Then, by arguing as in Remark 5.1, one can show that, under the assumptions of Theorem 3.1 (or that of Theorem 5.2), the limit (14) exists and is equal to

$$\int_{\partial\Omega_0} f\,(-P_{-\Omega} - P_{\Omega_0}) \circ \nu_0\, d\sigma,$$

since $-\Omega = \{-x; x \in \Omega\}$ is also in $\mathcal{U}$. Obviously, this limit is equal to the limit from the right (4) if and only if

$$-\int_{\partial\Omega_0} f\, P_{-\Omega} \circ \nu_0\, d\sigma = \int_{\partial\Omega_0} f\, P_\Omega \circ \nu_0\, d\sigma,$$

that is, if and only if

$$\int_{\partial\Omega_0} f\, P_{\Omega-\Omega} \circ \nu_0 \, d\sigma = 0. \tag{15}$$

This condition is obviously satisfied if $f = 0$ on $\partial\Omega_0$. But, in general, given $\Omega_0$ and if $\int_{\partial\Omega_0} f\, d\sigma \neq 0$, it is not satisfied for all $\Omega$, for if $\Omega$ is the unit ball, $P_{\Omega-\Omega}(x) = 2|x|$. Moreover, if $f = 1$ (or a function that do not change sign on $\partial\Omega_0$), it is not satisfied at all. Indeed, since $\Omega-\Omega$ is a symmetric neighbourhood of 0, $P_{\Omega-\Omega}$ is a norm and so $P_{\Omega-\Omega} \circ \nu_0 > 0$ on $\partial\Omega_0$. This is to say that, even in natural cases, condition (15) is rarely satisfied, that is, the right and left derivatives are not equal.

What can we conclude from this? In fact, this remark highlights some phenomenon: The deformation $\Omega_0 \mapsto \Omega_\varepsilon = (1-\varepsilon)\Omega_0 + \varepsilon\Omega$ (or $\widetilde{\Omega}_\varepsilon = \Omega_0 + \varepsilon\Omega$) for $\varepsilon > 0$ and that for $\varepsilon < 0$ are not of the same geometric nature. This can be understood by going back to the argument used in the first part of the proof of Theorem 3.1: When $\Omega$ is strongly convex and of class $C^2$, the deformation $\Omega_0 \mapsto \Omega_\varepsilon$ is equivalent for $\varepsilon > 0$ to that obtained by using the $C^1$ vector field $V(x) = a(x) - x$, that is, $\Omega_0 \mapsto (Id + \varepsilon V)(\Omega_0)$. With respect to the latter deformation, it is well known that the functional $J$ is left and right differentiable, with *equal* derivatives, at any $\Omega_0 \in \mathcal{U}$ and for all $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, whereas, with respect to the former deformation, $J$ is left and right differentiable at any $\Omega_0 \in \mathcal{U}$, with *distinct* derivatives for $f = 1$ (for example).

## 6. Application

To illustrate the formula of Theorem 3.1, we give an example of application to a shape optimization problem and compute the shape derivative of some functional related to the solution of a partial differential equation.

Let $D$ be an open bounded (convex) non empty subset of $\mathbb{R}^n$ and $\mathcal{U}_D = \{\Omega \in \mathcal{U};\ \Omega \subset D\}$. If $\Omega \in \mathcal{U}_D$, $u_d \in H^1(D)$ and $f \in L^2(D)$, we consider the problem

$$\begin{cases} \text{Minimize } J(\Omega) = J_1(\Omega, u_\Omega) = \displaystyle\int_\Omega (u_\Omega(x) - u_d(x))^2 dx \\ \text{where } u_\Omega \text{ is the solution of} \\ (\mathcal{P}) \begin{cases} -\Delta\, u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \end{cases} \tag{16}$$

Let $u_0$ be the solution of $(\mathcal{P})$ on $\Omega_0 \in \mathcal{U}_D$ and $u_\varepsilon$ be the solution of $(\mathcal{P})$ on $\Omega_\varepsilon = (1-\varepsilon)\,\Omega_0 + \varepsilon\,\Omega$, $0 \le \varepsilon \le 1$. Assuming $\Omega$ strongly convex, we know from Lemma 4.5 that $\Omega_\varepsilon$ can be considered as a deformation of the domain $\Omega_0$ by the vector field $V(x) = a(x) - x$, that is $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)$. Therefore, at least when $f \in H^1(D)$, we can write

$$\widetilde{u}_\varepsilon = \widetilde{u}_0 + \varepsilon\, u_0' + \varepsilon\, v_\varepsilon \quad \text{where } v_\varepsilon \to 0 \ \text{ in } L^2(D),$$

and $u_0'$ is the shape derivative of $\widetilde{u}_0$ with respect to the vector field $V$. See, for example, [3], Chapter 5. It follows from that result that

$$\frac{1}{\varepsilon}[(\widetilde{u}_\varepsilon - u_d)^2 - (\widetilde{u}_0 - u_d)^2] - 2\, u_0'(\widetilde{u}_0 - u_d) \longrightarrow 0$$

in $L^1(D)$ as $\varepsilon \to 0$, which allows one to apply Corollary 3.2 to obtain

$$\lim_{\varepsilon \to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega_0)}{\varepsilon} = \int_{\Omega_0} 2\, u_0'(\widetilde{u}_0 - u_d)\, dx + \int_{\partial\Omega_0} (\widetilde{u}_0 - u_d)^2\, (P_\Omega - P_{\Omega_0}) \circ \nu_0\, d\sigma.$$

Now, in this expression of the shape derivative of $J$, even the domain integral can be written as a boundary integral. Indeed, by following, for example, the same method as [3], one can show that $u_0'$ satisfies the boundary value problem

$$\begin{cases} -\Delta\, u_0' = 0 & \text{in } \Omega_0, \\ u_0' = -\frac{\partial u_0}{\partial \nu_0}\, V \cdot \nu_0 & \text{on } \partial\Omega_0. \end{cases} \tag{17}$$

Note here that, since $\Omega_0$ is of class $C^2$, $u_0$ is in fact in $H^2(\Omega_0)$, so that the derivative $\frac{\partial u_0}{\partial \nu_0}$ is well defined on $\partial\Omega_0$. Since $\Omega$ is strongly convex, it follows from Lemma 4.1 that $V \cdot \nu_0 = (P_\Omega - P_{\Omega_0}) \circ \nu_0$, so that

$$\begin{cases} -\Delta\, u_0' = 0 & \text{in } \Omega_0, \\ u_0' = -\frac{\partial u_0}{\partial \nu_0}\, (P_\Omega - P_{\Omega_0}) \circ \nu_0 & \text{on } \partial\Omega_0. \end{cases} \tag{18}$$

Now, using the solution of the following adjoint state boundary value problem

$$\begin{cases} -\Delta\, \psi = 2\,(u_0 - u_d) & \text{in } \Omega_0, \\ \psi = 0 & \text{on } \partial\Omega_0, \end{cases} \tag{19}$$

and the Green's formula, we can write

$$\begin{aligned} \int_{\Omega_0} u_0'\, 2\,(u_0 - u_d)\, dx &= \int_{\Omega_0} u_0'\, (-\Delta\, \psi)\, dx \\ &= \int_{\partial\Omega_0} \frac{\partial \psi}{\partial \nu_0} \frac{\partial u_0}{\partial \nu_0}\, (P_\Omega - P_{\Omega_0}) \circ \nu_0\, d\sigma. \end{aligned}$$

Finally, we obtain the following expression of the shape derivative

$$\lim_{\varepsilon \to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} \left( \frac{\partial \psi}{\partial \nu_0} \frac{\partial u_0}{\partial \nu_0} + (u_0 - u_d)^2 \right) (P_\Omega - P_{\Omega_0}) \circ \nu_0\, d\sigma.$$

The last thing we propose is an algorithm to solve the shape optimization problem (16):

1. Choose $\Omega_0$, $\rho \in ]0, 1[$ and a precision Eps.

2. Compute $p_0 = P_{\Omega_0}$.
3. Solve the state equation:

$$\begin{cases} -\Delta\, u_0 = f & \text{in } \Omega_0, \\ u_0 = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{20}$$

4. Solve the adjoint state equation:

$$\begin{cases} -\Delta\, \psi_0 = 2\,(u_0 - u_d) & \text{in } \Omega_0, \\ \psi_0 = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{21}$$

5. Compute

$$j_0(p) = \int_{\partial\Omega_0} \left( \frac{\partial\psi_0}{\partial\nu_0} \frac{\partial u_0}{\partial\nu_0} + (u_0 - u_d)^2 \right) (p - p_0) \circ \nu_0 \, d\sigma$$

6. Compute $\bar{p}_0$ the solution of

$$\arg\min_{p \in \mathcal{E}} j(p) \tag{22}$$

   where $\mathcal{E} = \{\Psi \in C(\mathbb{R}^2);\ \Psi \text{ is convex and homogeneous of degree 1 and } \Psi \leq P_D\}$.
7. At step $k$,

$$\text{if } \|u_k - u_d\|_{L^2(\Omega_k)} \leq \text{Eps}, \qquad \textbf{Go to 9}$$

   where $u_k$ is the solution of the state equation in $\Omega_k$ (the domain at step $k$).
8. Compute

$$p_{k+1} = p_k + \rho\,(\bar{p}_k - p_k) \quad \text{and}$$
$$\overline{\Omega}_{k+1} = \partial p_{k+1}(0) = \{y \in \mathbb{R}^n;\ p_{k+1}\,(x) \geq x.y,\ \forall x \in \mathbb{R}^n\}$$

   and **Go to 3**.
9. **End**.

**Remark 6.1.** In the above algorithm, at the $k$th iteration, we take $(\bar{p}_k - p_k)$ as a direction of deformation, a choice justified by the fact that this quantity defined a descent direction.

We also have to say that currently we are successfully implementing such an algorithm for some free boundary problems using the same idea, that is, a shape optimization approach. Of course, the shape functional and its derivative are different from the above ones. This is done in [8].

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## References


[1] A. A. Niftiyev, Y. S. Gasimov: Control by Boundaries and Eigenvalue Problems with Variable Domains, Publishing House of Baku State University, Baku (2004) (in Russian).

[2] V. P. Demyanov, A. M. Rubinov: Bases of Non-Smooth Analysis and Quasi-Differential Calculus, Nauka, Moscow (1990) (in Russian).

[3] A. Henrot, M. Pierre: Variation et Optimisation de Formes. Une Analyse Géométrique, Mathématiques & Applications 48, Springer, Berlin (2005).

[4] L. Hörmander: Notions of Convexity, Progress in Mathematics 127, Birkhäuser, Basel (1994).

[5] L. Evans, R. Gariepy: Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton (1992).

[6] L. Webster: Convexity, Oxford Science Publications, Oxford University Press, Oxford (1994).

[7] R. Schneider: Convex Bodies: The Brunn-Minkowski Theory, Cambridge University Press, Cambridge (1993).

[8] A. Boulkhemair, A. Chakib, A. Nachaoui, A. A. Niftiyev: A new numerical optimal shape design approach for a class of free boundary problems, in preparation.

[9] G. Allaire: Conception Optimale de Structures, Mathématiques et Applications 58, Springer, Berlin (2007) (in French).

[10] Y. D. Burago, V. A. Zalgaller: Geometric Inequalities, Springer, Berlin (1988).

[11] A. A. Niftiyev: On an optimal control problem with unknown boundaries, Trans. Acad. Sci. Azerb., Ser. Phys.-Tech. Math. Sci. 18(3–4) Math. Mech. (1998) 246–251.

[12] Y. S. Gasimov, A. A. Niftiyev: Minimization of eigenvalues of the Schrödinger operator with respect to the domain, Differ. Uravn. 38(6) (2002) 842–844, 863–864 (in Russian); Differ. Equ. 38(6) (2002) 892–896 (in English).

[13] A. Boulkhemair, A. Chakib: On a shape derivative formula in shape optimization, Reports of the National Academy of Sciences of Azerbaidjan 68 CILD (1) (2012) 1–8.