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Comparing BV Solutions of Rate Independent Processes*

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Many nonequilibrium rate independent processes arising in elastoplasticity, ferromagnetism and phase transitions are described by an evolution variational inequality with a convex constraint in a Hilbert space. The resulting solution operator is called *play operator* and acts on absolutely continuous functions. For nonregular data two natural notions of BV solutions have been proposed by the authors, giving rise to different extensions of the play operator to BV. We prove that these extensions are equal if and only if the convex constraint is a non-obtuse polyhedron.

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1. Introduction

Mathematical models of material memory are often based on the following evolution variational inequality. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{Z} \subseteq \mathcal{H}$ be a closed convex subset containing 0. Given $T > 0$ and $u : [0,T] \longrightarrow \mathcal{H}$, find $\xi : [0,T] \longrightarrow \mathcal{H}$ such that

$$\langle u(t) - \xi(t) - z, \xi'(t) \rangle \geq 0 \quad \forall z \in \mathcal{Z},\ t \in [0,T], \tag{1}$$

$$u(t) - \xi(t) \in \mathcal{Z} \quad \forall t \in [0,T], \tag{2}$$

with a given initial condition

$$u(0) - \xi(0) = z_0 \in \mathcal{Z}, \tag{3}$$

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where ξ' denotes the time derivative of ξ. Variational inequalities of the form (1)–(3) play an important role in (thermo)-plasticity, ferromagnetism, and phase transitions. The references [6, 7, 8, 13] contain surveys of the physical models described by (1)–(3). The special one-dimensional case $\mathcal{H} = \mathbb{R}$ has been deeply studied by several authors: we refer to the monographs [5, 20, 2].

Inequality (1)–(3) can be solved by using classical tools from the theory of evolution equations governed by maximal monotone operators. In particular it is well known that, keeping the initial condition $z_0 \in \mathcal{Z}$ fixed, for every $u \in \mathrm{AC}([0,T];\mathcal{H})$, the space of $\mathcal{H}$-valued absolutely continuous maps, there exists a unique $\xi \in \mathrm{AC}([0,T];\mathcal{H})$ satisfying (1)–(3) almost everywhere. The resulting solution operator $\mathsf{P} : \mathrm{AC}([0,T];\mathcal{H}) \longrightarrow \mathrm{AC}([0,T];\mathcal{H}) : u \mapsto \xi$ is usually called *(vector) play operator* following [5]. The suggestive terms *input* and *output* are used for u and ξ respectively. A relevant feature of P is *rate independence*, i.e.

$$\mathsf{P}(u \circ \phi) = \mathsf{P}(u) \circ \phi \tag{4}$$

whenever $u \in \mathrm{AC}([0,T];\mathcal{H})$ and $\phi : [0,T] \longrightarrow [0,T]$ is an increasing surjective absolutely continuous reparametrization of time. The play operator is also *causal*, i.e.

$$u = v \quad \text{on } [0,t] \quad \Longrightarrow \quad \mathsf{P}(u)(t) = \mathsf{P}(v)(t)$$

for every $u, v \in \mathrm{AC}([0,T];\mathcal{H})$ and $t \in [0,T]$. Rate independence together with causality characterizes the so called *hysteresis operators* according to the classification of [20].

An important issue to be considered is the extension of the operator P to classes of functions more general than $\mathrm{AC}([0,T];\mathcal{H})$. The first answer to this question can be found in [5] where the play operator is extended to $\mathrm{BV}([0,T];\mathcal{H}) \cap C([0,T];\mathcal{H})$, the space of continuous functions of bounded variation. In [7], it is shown that this operator still admits a variational representation, where the variational inequality (1) is replaced by the integral inequality

$$\int_0^T \big\langle u(t) - \xi(t) - z(t), \mathrm{d}\xi(t) \big\rangle \geq 0 \quad \forall z \in C([0,T];\mathcal{Z}), \tag{5}$$

in the sense of Riemann-Stieltjes integral. The problem is first solved for step functions, then a solution for continuous BV inputs is found by an *a priori estimates-limit procedure*. In [7] it is also proved the continuity of P with respect to the topology of uniform convergence. In the paper [9] the play operator is further extended to the space of possibly discontinuous functions of bounded variation. This extension is achieved by considering the variational integral inequality

$$\int_0^T \big\langle u(t+) - \xi(t+) - z(t), \mathrm{d}\xi(t) \big\rangle \geq 0 \quad \forall z \in \mathrm{Reg}([0,T];\mathcal{Z}), \tag{6}$$

where the integral is understood in the sense of Young, and $\mathrm{Reg}([0,T];\mathcal{Z})$ denotes the space of regulated $\mathcal{Z}$-valued functions. Later, in [10, 11], the Kurzweil integral (cf. [12]) was used instead in the framework of (6). In both Young and Kurzweil setting, the continuity with respect to the topology of uniform convergence is proved, and (6) allows to extend P even to the space of regulated functions provided $\mathcal{Z}$ has nonempty interior. Let us call P_K this extension of P defined by means of the Kurzweil integral. In [11] it is proved that P_K is even locally Lipschitz continuous with respect to strong BV norm

$$\|u\|_{\mathrm{BV}} := \|u\|_\infty + \mathrm{V}(u, [0, T]),$$

where $\|u\|_\infty$ is the supremum norm of u and $\mathrm{V}(u, [0, T])$ is the (total) variation of u, provided $\mathcal{Z}$ is a smooth domain.

In [18] another extension P_R is defined by means of reparametrizations in the following way. For $u \in \mathrm{BV}([0, T]; \mathcal{H})$ let $\ell_u : [0, T] \longrightarrow [0, T]$ be the arc length (normalized by a multiplicative factor) defined by

$$\ell_u(t) := \frac{T}{\mathrm{V}(u, [0, T])} \mathrm{V}(u, [0, t]), \tag{7}$$

and let us consider the unique Lipschitz map $\widetilde{u} : [0, T] \longrightarrow \mathcal{H}$ such that

$$u = \widetilde{u} \circ \ell_u,$$
$$\widetilde{u} \text{ is affine on } [\ell_u(t-), \ell_u(t+)] \text{ for every jump point } t \text{ of } u.$$

In other words we are reparametrizing u by the (normalized) arc-length and we fill in the jumps with segments. Then the extension of P is defined by setting

$$\mathsf{P}_R(u) := \mathsf{P}(\widetilde{u}) \circ \ell_u,$$

i.e. the classical play operator is applied to $\widetilde{u}$, and then the jumps are reinserted by composing with the arc length ℓ_u. We could also say that a generalized form of rate independence is used in order to define P_R. In [18] it is also shown that P_R can be defined as the extension by continuity with respect to the strict metric of BV, which is defined by

$$d_s(u, v) := \|u - v\|_{L^1} + |\,\mathrm{V}(u, [0, T]) - \mathrm{V}(v, [0, T])|. \tag{8}$$

Indeed if $u \in \mathrm{BV}([0, T]; \mathcal{H})$ is approximated by a sequence $u_n \in \mathrm{AC}([0, T]; \mathcal{H})$ converging in the strict metric of BV then it is shown that

$$\mathsf{P}(u_n) \to \mathsf{P}_R(u) \ \text{ in } L^1([0, T]; \mathcal{H})$$

as $n \to \infty$. The continuity of P in $\mathrm{BV}([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H})$ with respect to the strict metric is also proved but in general P is not continuous on the whole space BV: in [18] it is proved that P_R is continuous in BV if and only if $\mathcal{Z}$ is a vector subspace or

$$\mathcal{Z} = \{x \in \mathcal{H} \ : \ -\alpha \le \langle f, x\rangle \le \beta\} \tag{9}$$

for some $f \in \mathcal{H} \smallsetminus \{0\}$ and $\alpha, \beta \in [0, \infty]$. It follows that when $\mathcal{H}$ is one-dimensional then P_R is continuous with respect to the BV-norm. The one-dimensional case is examined in detail in [16, 15, 17]. In particular in [17] it is proved that P_K and P_R coincide when $\dim(\mathcal{H}) = 1$, whereas in [18] an example shows that in general the two extensions P_K and P_R are different for $\dim(\mathcal{H}) > 1$. The aim of this paper is to compare these two extensions by providing a simple condition that describes when P_K and P_R are equal. We limit ourselves to the finite dimensional case and we prove the following characterization:

$$\mathsf{P}_K = \mathsf{P}_R \quad \Longleftrightarrow \quad \mathscr{Z} \text{ is a non-obtuse polyhedron,} \tag{10}$$

where a polyhedron is called *non-obtuse* if the inner angles formed by its faces are less or equal than $\pi/2$.

Now we give a brief plan of the paper. In the next section we recall the main definitions and notations about convex sets and vector valued functions of bounded variation. In Section 3 we recall the precise definitions of play operator and we state the main theorem of the paper. Sections 4 and 5 are devoted to the proof of the two implications of (10). Finally in the last section we provide a more explicit description of non-obtuse polyhedra.

2. Preliminaries

In this section we recall the basic notions on convex analysis and on functions of bounded variation. Throughout the paper $\mathbb{N}$ will denote the set of positive integers, and $\mathbb{R}_+ := [0, \infty[$. The symbol $\mathscr{L}^1$ indicates the one-dimensional Lebesgue measure.

2.1. Convex analysis

In the sequel of the paper we assume that

$$\begin{cases} \mathcal{H} \text{ is a finite dimensional real Hilbert space} \\ \qquad \text{with inner product } (x, y) \longmapsto \langle x, y \rangle, \\ \|x\| := \langle x, x \rangle^{1/2}, \quad x \in \mathcal{H}. \end{cases} \tag{11}$$

We use the notation $B_r(x_0) := \{x \in \mathcal{H} \ : \ \|x - x_0\| < r\}$ for open balls in $\mathcal{H}$, and the general topological notions of interior, closure and boundary of a subset $\mathcal{S} \subseteq \mathcal{H}$, will be respectively denoted by $\mathrm{int}(\mathcal{S})$, $\mathrm{cl}(\mathcal{S})$ and $\partial\mathcal{S}$. We also set $d(x, \mathcal{S}) := \inf\{\|x - y\| \ : \ y \in \mathcal{S}\}$, $\mathcal{S} - x := \{y - x \ : \ y \in \mathcal{S}\}$, and $\lambda\mathcal{S} := \{\lambda y \ : \ y \in \mathcal{S}\}$ for every $x \in \mathcal{H}$ and $\lambda \in \mathbb{R}$.

If $\mathcal{C}$ is a closed convex subset of $\mathcal{H}$, then the projection on $\mathcal{C}$ is denoted by $\mathrm{Proj}_{\mathcal{C}}$, i.e. for $x \in \mathcal{H}$ given, $y = \mathrm{Proj}_{\mathcal{C}}(x)$ is the unique point such that $d(x, \mathcal{C}) = \|x - y\|$, and it is also characterized by the fact that $y \in \mathcal{C}$ and satisfies the variational inequality

$$\langle x - y, v - y \rangle \leq 0 \quad \forall v \in \mathcal{C}.$$

When $\mathcal{C}$ is equal to an affine subspace $\mathcal{A}$, then $y = \mathrm{Proj}_{\mathcal{A}}(x)$ if and only if $y \in \mathcal{A}$ and $\langle x - y, v - y \rangle = 0$ for every $v \in \mathcal{A}$; moreover if $x_0 \in \mathcal{A}$ and $\mathcal{V} := \mathcal{A} - x_0$, then $\mathrm{Proj}_{\mathcal{A}}(y) = x_0 + \mathrm{Proj}_{\mathcal{V}}(y - x_0)$.

We say that $\mathcal{K} \subseteq \mathcal{H}$ is a *cone* if $\lambda\mathcal{K} \subseteq \mathcal{K}$ for every $\lambda > 0$. Here are two important examples. If $\mathcal{C}$ is convex and $x \in \mathcal{C}$, the *normal cone to* $\mathcal{C}$ *at* x is the set $N_{\mathcal{C}}(x)$ defined by

$$N_{\mathcal{C}}(x) := \{y \in \mathcal{H} \ : \ \langle y, v - x\rangle \leq 0 \ \forall v \in \mathcal{C}\}, \tag{12}$$

and the *tangent cone to* $\mathcal{C}$ *at* x is the set $T_{\mathcal{C}}(x)$ defined by

$$T_{\mathcal{C}}(x) := \{y \in \mathcal{H} \ : \ \langle y, n\rangle \leq 0 \ \forall n \in N_{\mathcal{C}}(x)\}. \tag{13}$$

Observe that $N_{\mathcal{C}}(x)$ and $T_{\mathcal{C}}(x)$ are closed convex cones. Moreover, as $\mathcal{H}$ is finite dimensional, $N_{\mathcal{C}}(x) \setminus \{0\} \neq \varnothing$ for every $x \in \partial\mathcal{C}$ (see, e.g., [4, Lemma 4.2.1, p. 124]).

A relevant class of closed convex sets is given by polyhedra:

Definition 2.1. A set $\mathcal{P} \subseteq \mathcal{H}$ is called a *(closed convex) polyhedron* if there exist $n_1, \ldots, n_p \in \partial B_1(0)$ and $c_1, \ldots, c_p \in \mathbb{R}$ such that

$$\mathcal{P} = \{x \in \mathcal{H} \ : \ \langle n_j, x\rangle \leq c_j, \ j = 1, \ldots, p\}. \tag{14}$$

For $x \in \mathcal{P}$ we set $J(x) := \{j \in \{1, \ldots, p\} \ : \ \langle n_j, x\rangle = c_j\}$, the *set of active constraints at* x.

In particular we will deal with a special class of polyhedra which we call "non-obtuse". Here is the precise definition.

Definition 2.2. A polyhedron $\mathcal{P}$ of the form (14) is called *non-obtuse* if $\langle n_i, n_j\rangle \leq 0$ whenever $1 \leq i < j \leq p$.

A polyhedron $\mathcal{P}$ given by (14) is clearly a closed convex set, and $\partial\mathcal{P} = \{x \in \mathcal{P} : J(x) \neq \varnothing\}$. Note that $0 \in \mathcal{P}$ if and only if $c_j \geq 0$ for all $j \in \{1, \ldots, p\}$.

The next proposition describes the normal and the tangent cones to a polyhedron (see [4, p. 138]).

Proposition 2.3. *Assume that* $n_1, \ldots, n_p \in \partial B_1(0)$, $c_1, \ldots, c_p \in \mathbb{R}$ *and* $\mathcal{P}$ *is defined by* (14). *If* $x \in \partial\mathcal{P}$ *then*

$$N_{\mathcal{P}}(x) = \left\{ \sum_{j \in J(x)} \alpha_j n_j \ : \ \alpha_j \geq 0 \right\}$$

and

$$T_{\mathcal{P}}(x) = \big\{w \in \mathcal{H} \ : \ \langle w, n_j\rangle \leq 0 \ \forall j \in J(x)\big\}.$$

2.2. Functions of bounded variation

If $D \subseteq \mathbb{R}$ and $u : D \longrightarrow \mathcal{H}$ are given, then for every interval $I \subseteq D$ the *(pointwise) variation of* u *on* I is the real extended number $\mathrm{V}_p(u, I)$ defined by

$$\mathrm{V}_p(u, I) := \sup\left\{ \sum_{j=1}^{m} \|u(t_j) - u(t_{j-1})\| \ : \ m \in \mathbb{N}, \ t_j \in I, \ t_1 < \ldots < t_m \right\}.$$

We define the *essential variation of* u *on an open interval* by setting

$$\mathrm{V}_e(u, I) := \inf\{\mathrm{V}_p(v, I) \ : \ v = u \ \mathcal{L}^1\text{-a.e. in } I\}, \quad I \text{ open.}$$

Now we set $D = [a, b]$ with $a, b \in \mathbb{R}$, $a < b$. We say that $u : [a, b] \longrightarrow \mathcal{H}$ is *of bounded (pointwise) variation (on* $[a, b]$*)* if $\mathrm{V}_p(u, [a, b]) < \infty$. It is well known that any function u of bounded variation belongs to the space of regulated functions $\mathrm{Reg}([a, b]; \mathcal{H})$, i.e. it admits left and right limits $u(t-), u(t+)$ in $\mathcal{H}$ at any point $t \in [a, b]$, with the convention that $u(a-) = u(a)$ and $u(b+) = u(b)$; moreover its derivative $u'(t)$ exists at $\mathcal{L}^1$-a.e. point $t \in [a, b]$. Since we will primarily deal with left continuous functions we define the vector space of functions

$$\mathrm{BV}_L([a, b]; \mathcal{H}) := \{u : [a, b] \longrightarrow \mathcal{H} \ : \ \mathrm{V}_p(u, [a, b]) < \infty, \ u \text{ is left continuous}\}. \tag{15}$$

Observe that we are not identifying functions which are equal $\mathcal{L}^1$-almost everywhere, therefore for $u : [a, b] \longrightarrow \mathcal{H}$ it can happen that $\mathrm{V}_p(u, [a, b]) = \infty$ and $\mathrm{V}_e(u, [a, b]) < \infty$, where $\mathrm{V}_e(u, [a, b])$, the *essential variation of* u *on* $[a, b]$, is defined by

$$\mathrm{V}_e(u, [a, b]) := \mathrm{V}_e(\overline{u}, \mathbb{R}), \tag{16}$$

the function $\overline{u} : \mathbb{R} \longrightarrow \mathcal{H}$ being the following extension of u:

$$\overline{u}(t) := \begin{cases} u(t) & \text{if } t \in [a, b] \\ u(a) & \text{if } t < a \\ u(b) & \text{if } t > b. \end{cases}$$

Notice that if $\mathrm{V}_p(u, [a, b]) < \infty$ then $\mathrm{V}_e(u, [a, b]) = \mathrm{V}_e(u,]a, b[) + \|u(a) - u(a+)\| + \|u(b) - u(b-)\|$, and we also have that

$$\mathrm{V}_e(u, [a, b]) = \mathrm{V}_e(u,]a, b[) + \|u(a) - u(a+)\| = \mathrm{V}_p(u, [a, b]) \ \ \forall u \in \mathrm{BV}_L([a, b]; \mathcal{H}).$$

Therefore when we consider left continuous functions we are essentially dealing with Lebesgue equivalence class of functions with a special view on the initial point a, allowing us to take into account Dirac masses at a. For the proofs of the previous properties we refer the reader to [1, Appendix], [9, Section 1] and [18, Section 2]. The use of left continuous functions is also justified by the fact that the viscous regularizations of rate independent processes converge to a left continuous function when the viscosity coefficient tends to zero (cf. [10, Theorem 2.4]).

For $u \in \mathrm{AC}([a, b]; \mathcal{H})$ there exists $g \in L^1([a, b]; \mathcal{H})$ such that $u(t) = u(a) + \int_a^t g(s)\, \mathrm{d}s$ for every $t \in [a, b]$, and $u'(t) = g(t)$ for $\mathcal{L}^1$-a.e. $t \in [a, b]$. If $\mathrm{D}u$ denotes the distributional derivative of u then it is well known that

$$\mathrm{AC}([a, b]; \mathcal{H}) = \{u \in \mathrm{BV}_L([a, b]; \mathcal{H}) \, : \, \mathrm{D}u = u' \mathcal{L}^1\}.$$

Notice also that $\mathrm{Lip}([0, T]; \mathcal{H}) = \{u \in \mathrm{AC}([a, b]; \mathcal{H}) \ : \ u' \in L^\infty([0, T]; \mathcal{H})\}$, where $\mathrm{Lip}([a, b]; \mathcal{H})$ is the space of Lipschitz functions on $[a, b]$ (we refer to [1, Appendix] for the proofs).

If $u \in \mathrm{BV}_L([a,b];\mathcal{H})$, we define $\ell_u : [a,b] \longrightarrow [a,b]$ by

$$\ell_u(t) := \begin{cases} a + \dfrac{b-a}{\mathrm{V}_p(u,[a,b])}\mathrm{V}_p(u,[a,t]) & \text{if } \mathrm{V}_p(u,[a,b]) \neq 0, \\ a & \text{if } \mathrm{V}_p(u,[a,b]) = 0, \end{cases} \qquad t \in [a,b]. \tag{17}$$

Indeed, ℓ_u is a left continuous function.

The proof of the following proposition can be found in [18, Proposition 3.1] and [19, Proposition 2.1] and it is based on [3, Section 2.5.16, p. 109].

Proposition 2.4. *Let $u \in \mathrm{BV}_L([a,b];\mathcal{H})$ be given, and let $\ell_u : [a,b] \longrightarrow [a,b]$ be its "normalized" arc length defined by* (17)*. Then there exists a unique $\widetilde{u} \in$ $\mathrm{Lip}([a,b];\mathcal{H})$ such that*

$$u = \widetilde{u} \circ \ell_u, \tag{18}$$

$$\widetilde{u} \textit{ is affine on } [\ell_u(t), \ell_u(t+)] \quad \forall t \in [a,b], \tag{19}$$

$$\mathrm{Lip}(\widetilde{u}) \leq \frac{\mathrm{V}_p(u,[a,b])}{b-a}, \tag{20}$$

$\mathrm{Lip}(\widetilde{u})$ *being the Lipschitz constant of* $\widetilde{u}$.

The function $\widetilde{u}$ in Proposition 2.4 is nothing but the reparametrization of u by the arc length, normalized by a constant factor allowing us to define $\widetilde{u}$ on $[a,b]$. When u jumps at a time $t \in [a,b]$, then $\widetilde{u}$ is obtained by filling in the jump with a segment, i.e. we define $\widetilde{u}$ to be affine on $[\ell_u(t), \ell_u(t+)]$. Thus the variations of $\widetilde{u}$ and u are the same.

We will consider two topologies on $\mathrm{BV}_L([a,b];\mathcal{H})$. The first one is the topology of uniform convergence on bounded sets in BV, the natural topology of BV as a subset of the space of bounded functions. It is induced by the supremum norm

$$\|u\|_\infty := \sup_{t\in[a,b]} \|u(t)\|, \quad u \in \mathrm{BV}_L([a,b];\mathcal{H}).$$

The second is the topology induced by the *strict metric*

$$d_s(u,v) := \|u-v\|_{L^1([a,b];\mathcal{H})} + |\,\mathrm{V}_p(u,[a,b]) - \mathrm{V}_p(v,[a,b])|, \quad u,v \in \mathrm{BV}_L([a,b];\mathcal{H}).$$

The strict metric is very natural because any BV function can be approximated in the strict metric by a sequence in $C^\infty([a,b];\mathcal{H})$ (cf. [18, Proposition A.2]). Since we are dealing with left continuous functions, the two topologies possess the Hausdorff property. It is easy to see that the strict metric induces a topology that is not linear. The following approximation result also holds (cf. [19, Proposition 3.1]).

Proposition 2.5. *For every $u \in \mathrm{BV}_L([a,b];\mathcal{H})$ there exists a sequence of left continuous step functions $u_h : [a,b] \longrightarrow \mathcal{H}$ of the form*

$$u_h(t) := \begin{cases} u_0^h & \textit{if } t = a \\ u_k^h & \textit{if } t \in \left]t_{k-1}^h, t_k^h\right],\ k = 1,\ldots,m \end{cases}$$

with $m \in \mathbb{N}$, $a = t_0^h < \cdots < t_m^h = b$ *and* $\{u_0^h, \ldots, u_m^h\} \subseteq \mathcal{H}$, *such that* $u_h \to u$ *uniformly and* $\mathrm{V}_p(u_h, [a,b]) \to \mathrm{V}_p(u, [a,b])$ *as* $h \to \infty$.

3. Review of the play operator and main result

In this section we recall the definition of the play operator in the classical framework of absolutely continuous functions and its extensions to the space of functions of bounded variation. We will assume that

$$\mathcal{Z} \text{ is a closed convex subset of } \mathcal{H}, \tag{21}$$

$$0 \in \mathcal{Z}. \tag{22}$$

We also fix

$$z_0 \in \mathcal{Z} \tag{23}$$

and

$$T > 0. \tag{24}$$

Let us start with the following result, whose proof can be found, e.g., in [1, Proposition 3.4, Remark 3.7]).

Theorem 3.1. *For every* $u \in \mathrm{AC}([0,T];\mathcal{H})$ *there exists a unique* $\xi \in \mathrm{AC}([0,T];\mathcal{H})$ *such that*

$$u(t) - \xi(t) \in \mathcal{Z} \quad \forall t \in [0,T] \tag{25}$$

$$\langle u(t) - \xi(t) - z, \xi'(t)\rangle \geq 0 \quad \forall z \in \mathcal{Z}, \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \tag{26}$$

$$u(0) - \xi(0) = z_0. \tag{27}$$

If $u \in \mathrm{AC}([0,T];\mathcal{H})$ and ξ is the unique solution of (25)–(27), by setting $\mathsf{P}(u) := \mathsf{P}(u; z_0) := \xi$ we define an operator

$$\mathsf{P} = \mathsf{P}(\cdot\,, z_0) : \mathrm{AC}([0,T];\mathcal{H}) \longrightarrow \mathrm{AC}([0,T];\mathcal{H}).$$

As mentioned in the introduction, P is called the *(vector) play operator*, and the set $\mathcal{Z}$ is called the *characteristic* of the operator P. The notation $\mathsf{P}(u, z_0)$ is used in order to emphasize the dependence on z_0 and will often be omitted in the sequel. Apart from the *rate independence* (4), the play operator also enjoys of the semigroup property: if $u \in \mathrm{AC}([0,T];\mathcal{H})$ then

$$\mathsf{P}(u, z_0)(t_2) = \mathsf{P}(u(\cdot + t_1), \mathsf{P}(u, z_0)(t_1))(t_2 - t_1) \quad \forall t_1, t_2 \in [0,T],\ t_1 < t_2, \tag{28}$$

where the play operator $\mathsf{P}(\cdot, \mathsf{P}(u, z_0)(t_1))$ in the right hand side is acting on the space $\mathrm{AC}([0, T - t_1];\mathcal{H})$. Both rate independence and semigroup property are well known and easy to check.

Now we recall the first extension of P to the space of functions with bounded variation obtained in [9, Theorem 2.3].

Theorem 3.2. *For every* $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ *there exists a unique* $\xi \in \mathrm{BV}_L([0,T];\mathcal{H})$ *such that*

$$u(t) - \xi(t) \in \mathcal{Z} \quad \forall t \in [0,T], \tag{29}$$

$$\int_0^T \langle u(s+) - \xi(s+) - z(s), \mathrm{d}\xi(s)\rangle \geq 0 \quad \forall z \in \mathrm{Reg}([0,T];\mathcal{H}),\ z([0,T]) \subseteq \mathcal{Z}, \tag{30}$$

$$u(0) - \xi(0) = z_0, \tag{31}$$

where the integral in (30) *is meant in the sense of Kurzweil (Young) or in the sense of the differential measure* $\mathrm{D}\xi$ *(see* [12], [10, *Section* 2], [9, *Section* 3] *and* [18, *Section* A4]*). Moreover if* $u \in \mathrm{AC}([0,T];\mathcal{H})$ *then* $\xi = \mathsf{P}(u)$.

We do not go into detail about the Kurzweil integral here because we will not need the inequality (30) in its full form, but only one of its consequences stated in the next Theorem 3.4. The previous Theorem 3.2 allows us to define the "Kurzweil" extension P_K of the play operator

$$\mathsf{P}_K : \mathrm{BV}_L([0,T];\mathcal{H}) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{H}) \tag{32}$$

assigning to every $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ the unique function $\mathsf{P}_K(u) := \xi \in \mathrm{BV}_L([0,T];\mathcal{H})$ satisfying (29)–(31). Finally we describe the extension P_R which is instead based on "Reparametrizations":

Theorem 3.3. *Let us define the operator* $\mathsf{P}_R : \mathrm{BV}_L([0,T];\mathcal{H}) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{H})$ *by setting*

$$\mathsf{P}_R(u) := \mathsf{P}(\widetilde{u}) \circ \ell_u, \quad u \in \mathrm{BV}_L([0,T];\mathcal{H}), \tag{33}$$

where ℓ_u *is the arc length defined in* (17)*,* $\widetilde{u}$ *is reparametrization by the arc length given by Proposition* 2.4 *(with* $a = 0$*,* $b = T$*), and* P *is the classical play operator. Then* $\mathsf{P}_R(u) = \mathsf{P}(u)$ *for* $u \in \mathrm{AC}([0,T];\mathcal{H})$ *and* P_R *is the unique continuous extension of* P *in the following sense: if* $u_h \to u$ *strictly in* BV *then* $\mathsf{P}_R(u_h) \to \mathsf{P}_R(u)$ *pointwise and in* $L^1([0,T];\mathcal{H})$.

The proof of the previous result is given in [18, Proposition 3.3, Theorem 3.7] and [19, Theorem 4.1].

Strictly related to the play operator is the *stop operator* which is defined in any of the previous cases in the following way:

$$\mathsf{S}(u) := u - \mathsf{P}(u), \quad u \in \mathrm{AC}([0,T];\mathcal{H}),$$

$$\mathsf{S}_K(u) := u - \mathsf{P}_K(u), \quad \mathsf{S}_R(u) := u - \mathsf{P}_R(u), \quad u \in \mathrm{BV}([0,T];\mathcal{H}).$$

Observe that in the absolutely continuous case we have

$$\langle \mathsf{P}(u)'(t), \mathsf{S}(u)'(t)\rangle = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a,b], \tag{34}$$

as one easily deduce putting $z = \mathsf{S}(u)(t \pm h)$ in (26) and taking the limit as $h \to 0+$.

The classical play operator P has a useful geometric interpretation (cf. [14, p. 347]): in fact let us rewrite (25)–(27) as

$$\xi'(t) \in N_{\xi(t)+\mathcal{Z}}(u(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \tag{35}$$

$$\xi(0) = \xi_0 := u(0) - z_0, \tag{36}$$

and let us interpret the point $u(t)$ as the motion of a pivot that at the initial time $t = 0$ is inserted in the convex set $y_0 + \mathcal{Z}$. The pivot starts to move according the law of motion $u(t)$. If initially $u(0)$ lies in the interior of $\xi(0) + \mathcal{Z}$ then $N_{\xi_0+\mathcal{Z}}(u(t)) = \{0\}$, thus $\xi(t) \equiv \xi_0$ solves the inclusion as long as $u(t)$ does not touch the boundary of $\xi_0 + \mathcal{Z}$. When $u(t)$ touches the boundary of $\xi_0 + \mathcal{Z}$ then $\xi(t)$ starts to move in such a way that $\xi'(t) \in N_{\xi(t)+\mathcal{Z}}(u(t))$. We interpret this solution by saying that the convex set $\xi_0+\mathcal{Z}$ moves in the direction of the outward normal in $u(t) \in \xi(t) + \mathcal{Z}$. This interpretation is maybe easier to visualize if we assume that $z_0 = u(0) = 0 \in \mathcal{Z}$. Accordingly, the stop operator S can also be interpreted in this way: if $x(t) = \mathsf{S}(u)(t) = u(t) - \xi(t)$ then $x(t) \in \mathcal{Z}$ for every $t \in [0,T]$ and

$$x'(t) - u'(t) \in N_{\mathcal{Z}}(x(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \tag{37}$$

$$x(0) = z_0. \tag{38}$$

Hence $x(t) = u(t) - \xi_0$ as long as $u(t)$ lies in the interior of $\xi_0 + \mathcal{Z}$; when $u(t)$ touches the boundary of $\xi_0 + \mathcal{Z}$ pointing outward of $\mathcal{Z}$, then $x(t)$ moves on $\partial\mathcal{Z}$ in the tangent direction $x'(t) \in T_{\mathcal{Z}}(x(t))$, because of (34).

The geometric interpretation of P_R is clear: we consider the Lipschitz reparametrization $\widetilde{u}$ of $u(t)$ by filling in the jumps, we apply the classical operator obtaining $\mathsf{P}(\widetilde{u})$, and we get $\mathsf{P}_R(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$ by reinserting the jumps.

Instead P_K is based on projections and its geometrical meaning is clear from the action on step functions, as the following Theorem shows (cf. [9, Theorem 2.3, Proposition 4.3]).

Theorem 3.4. *The operator* $\mathsf{P}_K : \mathrm{BV}_L([0,T];\mathcal{H}) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{H})$ *is continuous with respect to uniform convergence. Moreover if* $m \in \mathbb{N}$, $0 = t_0 < \cdots < t_m = T$, $u_0, \ldots, u_m \in \mathcal{H}$ *and* $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ *is the step function defined by*

$$u(t) = \begin{cases} u_0 & \text{if } t = 0, \\ u_k & \text{if } t \in \,]t_{k-1}, t_k], \ k = 1, \ldots, m, \end{cases} \tag{39}$$

then

$$\mathsf{S}_K(u; z_0)(t) = \begin{cases} x_0 := z_0 & \text{if } t = 0 \\ x_k := \mathrm{Proj}_{\mathcal{Z}}(x_{k-1} + u_k - u_{k-1}) & \text{if } t \in \,]t_{k-1}, t_k], \ k = 1, \ldots, m. \end{cases} \tag{40}$$

The previous Theorem 3.4 also shows that the solution $x = \mathsf{S}_K(u)$ (equivalently $\xi = \mathsf{P}_K(u)$) is the time discrete approximation of (37)–(38) obtained by using an implicit Euler scheme. It follows that the solution $\xi = \mathsf{P}_K(u)$ coincides with the solution, as defined in [14], of the sweeping process

$$-\xi'(t) \in N_{\mathcal{C}(t)}(\xi(t))$$

in the particular case $\mathcal{C}(t) = u(t) - \mathcal{Z}$.

Concerning the action of P_R on the step function $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ defined in (39), if $s_k := \ell_u(t_k)$, then we have that

$$\widetilde{u}(s_k) = u_k \quad \forall k = 0, \dots, m, \tag{41}$$

$$\widetilde{u} \text{ is affine on } [s_{k-1}, s_k] \quad \forall k = 1, \dots, m \tag{42}$$

and, using also the semigroup property (28),

$$\mathsf{S}_R(u, z_0) = \begin{cases} \mathsf{S}(\widetilde{u}, z_0)(0) = z_0 & \text{if } t = 0 \\ \mathsf{S}(\widetilde{u}, z_0)(s_k) = \mathsf{S}(\widetilde{u}(\cdot + s_{k-1}), \mathsf{S}(\widetilde{u}, z_0)(s_{k-1}))(s_k - s_{k-1}) & \text{if } t \in \,]t_{k-1}, t_k] \end{cases} \tag{43}$$

for $k = 1, \dots, m$.

Let us also observe that if $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ and $v_0 \in \mathcal{H}$, then

$$v(t) := u(t) - v_0 \qquad \Longrightarrow \qquad \mathsf{P}_K(v) = \mathsf{P}_K(u) - v_0, \quad \mathsf{P}_R(v) = \mathsf{P}_R(u) - v_0, \tag{44}$$

as it is easily seen on step functions and then on any u, thanks to the continuity of the operators.

In [18, Section 5.3] is proved that

$$\mathsf{P}_K(u) = \mathsf{P}_R(u) \quad \forall u \in \mathrm{BV}_L([0,T];\mathcal{H}) \cap C([0,T];\mathcal{H})$$

and it is shown by means of an example that in general, the operators P_K and P_R are different on $\mathrm{BV}_L([0,T];\mathcal{H})$ unless $\mathcal{H}$ is one-dimensional (see also [17]). In the next two sections we prove the following geometric characterization of the cases where the two operators coincide.

Theorem 3.5. *Let* $\dim(\mathcal{H}) < \infty$. *Then* $\mathsf{P}_K = \mathsf{P}_R$ *if and only if* $\mathcal{Z}$ *is a non-obtuse polyhedron.*

4. If $\mathsf{P}_K = \mathsf{P}_R$ then $\mathcal{Z}$ is a non-obtuse polyhedron

In this section we prove that a necessary condition for P_K and P_R to be equal is that $\mathcal{Z}$ is a non-obtuse polyhedron. Let us recall that $\mathcal{H}$ is a finite dimensional real Hilbert space and let us start with the following

Lemma 4.1. *Let* $\mathcal{Z}$ *be a closed convex subset of* $\mathcal{H}$ *and let* $z \in \mathcal{Z}$ *and* $\rho > 0$ *be such that* $B_\rho(z) \subseteq \mathcal{Z}$. *If* $y \in \partial\mathcal{Z} \cap \partial B_\rho(z)$, *then* $N_{\mathcal{Z}}(y) = \mathbb{R}_+(y - z)$.

Proof. It is enough to prove that if $n \in N_{\mathcal{Z}}(y) \cap \partial B_1(0)$ then

$$\langle n, y - z \rangle = \|n\| \|y - z\| = \rho. \tag{45}$$

Observe that $z + \rho n \in \mathcal{Z}$, therefore, by definition of normal cone, $\langle n, y - z - \rho n \rangle \geq 0$, i.e. $\langle n, y - z \rangle \geq \langle n, \rho n \rangle = \|n\| \rho = \rho$. On the other hand $\langle n, y - z \rangle \leq \|n\| \rho = \rho$ by the Cauchy-Schwarz inequality, hence (45) is proved. □

Lemma 4.2. *If $\mathcal{Z}$ and $\mathcal{W}$ are closed convex sets in $\mathcal{H}$ such that* $\mathrm{int}(\mathcal{Z}) \neq \varnothing$*, $\mathcal{Z} \subseteq \mathcal{W}$ and $\partial \mathcal{Z} \subseteq \partial \mathcal{W}$, then $\mathcal{Z} = \mathcal{W}$.*

Proof. Let us start by observing that in a general topological space we have

$$A, B \text{ nonempty connected open sets}, \quad A \subseteq B, \quad \partial A \subseteq \partial B \quad \Longrightarrow \quad A = B. \tag{46}$$

In order to see this fact observe first A is relatively open in B, as $A = A \cap B$; on the other hand $\mathrm{cl}(A) \cap B = (A \cup \partial A) \cap B = (A \cap B) \cup (\partial A \cap B) = A \cup \varnothing$ because $\partial A \subseteq \partial B$, hence $A = \mathrm{cl}(A) \cap B$ and A is relatively closed in B. From the connectedness of B it follows that $A = B$. We can conclude by applying (46) with $A = \mathrm{int}(\mathcal{Z})$, $B = \mathrm{int}(\mathcal{W})$, because $\mathrm{int}(\mathcal{Z}) \subseteq \mathrm{int}(\mathcal{W})$ and for any convex set $\mathcal{C}$ with nonempty interior we have $\partial \mathcal{C} = \partial \, \mathrm{int}(\mathcal{C})$. □

Proposition 4.3. *Assume that $\mathcal{Z}$ is a closed convex subset of $\mathcal{H}$ and that* $\mathrm{int}(\mathcal{Z}) \neq \varnothing$*. Then $\mathcal{Z}$ is a polyhedron if and only if the set*

$$A = \{ n \in \partial B_1(0) \ : \ \exists x \in \partial \mathcal{Z}, \ N_{\mathcal{Z}}(x) = \mathbb{R}_+ n \}$$

is finite.

Proof. If $\mathcal{Z}$ is a polyhedron then the finiteness of A is a straightforward consequence of Proposition 2.3. Now let us assume that $A = \{n_1, \ldots, n_p\}$ for some $p \in \mathbb{N}$, thus there are points $x_1, \ldots, x_p \in \partial \mathcal{Z}$ such that $N_{\mathcal{Z}}(x_j) = \mathbb{R}_+ n_j$ for every $j = 1, \ldots, p$. Let us set $c_j := \langle n_j, x_j \rangle$ for every $j = 1, \ldots, p$ and consider the polyhedron $\mathcal{Z}_A := \{ x \in \mathcal{H} \ : \ \langle n_j, x \rangle \leq c_j \ j = 1, \ldots, p \}$. We have that

$$\mathcal{Z} \subseteq \mathcal{Z}_A, \tag{47}$$

because if $z \in \mathcal{Z}$ then $\langle n_j, z - x_j \rangle \leq 0$ for every $j = 1, \ldots, p$, hence $\langle n_j, z \rangle \leq \langle n_j, x_j \rangle \leq c_j$. Now we prove that actually $\mathcal{Z}_A = \mathcal{Z}$. By virtue of Lemma 4.2 it is enough to show that

$$\partial \mathcal{Z} \subseteq \partial \mathcal{Z}_A. \tag{48}$$

To this aim let us take $z \in \partial \mathcal{Z}$ and let $z_k \in \mathrm{int}(\mathcal{Z})$ be a sequence such that $z_k \to z$. We set $\rho_k := d(z_k, \partial \mathcal{Z})$, thus $B_{\rho_k}(z_k) \subseteq \mathcal{Z}$ and there exists $y_k \in \partial \mathcal{Z} \cap \partial B_{\rho_k}(z_k)$. Thanks to Lemma 4.1 $N_{\mathcal{Z}}(y_k) = \mathbb{R}_+(y_k - z_k)$ hence $N_{\mathcal{Z}}(y_k) = \mathbb{R}_+ n_{j_k}$ for some $j_k \in \{1, \ldots, p\}$, i.e.

$$\langle n_{j_k}, w - y_k \rangle \leq 0 \ \ \forall w \in \mathcal{Z}. \tag{49}$$

On the other hand we also have that

$$\langle n_{j_k}, w - x_{j_k} \rangle \leq 0 \ \ \forall w \in \mathcal{Z}, \tag{50}$$

hence taking $w = x_{j_k}$ in (49) and $w = y_k$ in (50) we obtain

$$\langle n_{j_k}, x_{j_k} - y_k \rangle = 0, \tag{51}$$

i.e.

$$\langle n_{j_k}, y_k \rangle = c_{j_k}. \tag{52}$$

Observe that $y_k \to z$, indeed $\|y_k - z\| \leq \|y_k - z_k\| + \|z_k - z\| = \rho_k + \|z_k - z\| \leq 2\|z_k - z\|$. Moreover, passing to a subsequence of k, which we do not relabel, we can find $h \in \{1, \dots, p\}$ such that $n_{j_k} \to n_h$ and $c_{j_k} \to c_h$ as $k \to \infty$. Therefore taking the limit in (52) we get

$$\langle n_h, x_h - z \rangle = 0, \tag{53}$$

hence $z \in \partial \mathcal{Z}_A$ and (48) is proved. □

Lemma 4.4. *Assume that $\mathcal{Z}$ is a closed convex subset of $\mathcal{H}$ and that $\operatorname{int}(\mathcal{Z}) \neq \varnothing$. If $\mathcal{Z}$ is not a polyhedron then there exist $x_1, x_2 \in \partial\mathcal{Z}$ and $n_1, n_2 \in \partial B_1(0)$ such that $x_1 \neq x_2$, $N_{\mathcal{Z}}(x_1) = \mathbb{R}_+ n_1$, $N_{\mathcal{Z}}(x_2) = \mathbb{R}_+ n_2$, and $0 < \langle n_1, n_2 \rangle < 1$.*

Proof. Since $\mathcal{Z}$ is not a polyhedron, by Proposition 4.3 there exist two infinite sets $\{n^k \in \partial B_1(0) \,:\, k \in \mathbb{N}\}$ and $\{x^k \in \partial\mathcal{Z} \,:\, k \in \mathbb{N}\}$ such that $N_{\mathcal{Z}}(x^k) = \mathbb{R}_+ n^k$ for every $k \in \mathbb{N}$. We have that, at least for a subsequence which we do not relabel, there exists $n^0 \in \partial B_1(0)$ such that $n^k \to n^0$ as $k \to \infty$. Observe that $\lim_{k\to\infty} \langle n^k, n^{k+1} \rangle = \langle n^0, n^0 \rangle = 1$, therefore if $\varepsilon > 0$ is arbitrarily fixed, then $1 - \varepsilon < \langle n^k, n^{k+1} \rangle$ for all k sufficiently large; on the other hand $\langle n^k, n^{k+1} \rangle < 1$, thus the lemma is proved with $n_1 = n^h$, $n_2 = n^{h+1}$ and $x_1 = x^h$, $x_2 = x^{h+1}$ for some $h \in \mathbb{N}$ sufficiently large. Note that $x_1 \neq x_2$ since $x^k \neq x^h$ for $k \neq h$. □

Lemma 4.5. *Assume that $\mathcal{Z}$ is a closed convex subset of $\mathcal{H}$, $\operatorname{int}(\mathcal{Z}) \neq \varnothing$, and $\mathcal{Z}$ is not a non-obtuse polyhedron. Then there exist $x_0, x_1, x_2 \in \partial\mathcal{Z}$, $n_1, n_2 \in \partial B_1(0)$, $r_1, r_2 > 0$, and $\lambda > 0$ such that*

$$x_1 \neq x_2, \qquad 0 < \langle n_1, n_2 \rangle < 1, \tag{54}$$

$$N_{\mathcal{Z}}(x_k) = \mathbb{R}_+ n_k, \quad k = 1, 2, \tag{55}$$

and putting

$$u_1 := x_1 + r_1 n_1, \qquad u_2 := x_2 + r_2 n_2, \tag{56}$$

then

$$u_1 = x_0 + \lambda(u_2 - u_1), \qquad u_2 = x_0 + (\lambda + 1)(u_2 - u_1), \tag{57}$$

$$\langle u_2 - u_1, n_1 \rangle > 0. \tag{58}$$

Proof. The situation is illustrated on Figure 4.1. If $\mathcal{Z}$ is a not a polyhedron then Lemma 4.4 yields the existence of $x_1, x_2 \in \partial\mathcal{Z}$ and $n_1, n_2 \in \partial B_1(0)$ satisfying (54)–(55). This fact is trivial if instead $\mathcal{Z}$ is a polyhedron which is not non-obtuse. Now observe that

$$\langle x_2 - x_1 + r n_2, n_1 \rangle = \langle x_2 - x_1, n_1 \rangle + r \langle n_1, n_2 \rangle \quad \forall r > 0. \tag{59}$$

hence, taking $r_2 > 0$ sufficiently large and putting

$$u_2 := x_2 + r_2 n_2, \tag{60}$$

$$w_0 := u_2 - x_1 = (x_2 + r_2 n_2) - x_1, \tag{61}$$

then

$$\langle w_0, n_1 \rangle > 0. \tag{62}$$

Now we claim that there exists $\delta_0 > 0$ such that

$$y_0 := x_1 - \delta_0 w_0 = (1+\delta_0)x_1 - \delta_0 u_2 \in \mathrm{int}(\mathcal{Z}). \tag{63}$$

In order to prove (63) we assume by contradiction that $x_1 - \delta w_0 \notin \mathrm{int}(\mathcal{Z})$ for every $\delta > 0$, therefore

$$x_\delta := \mathrm{Proj}_{\mathcal{Z}}(x_1 - \delta w_0) \in \partial\mathcal{Z} \quad \forall \delta > 0. \tag{64}$$

Now let us consider a sequence $\delta_k \searrow 0$ and pick a unit vector $n_k \in N_{\mathcal{Z}}(x_1 - \delta_k w_0)$ for every $k \in \mathbb{N}$, thus

$$\langle n_k, z - x_1 + \delta_k w_0 \rangle \leq 0 \quad \forall z \in \mathcal{Z}. \tag{65}$$

Passing, if needed, to a subsequence, we can take the limit as $k \to \infty$ and find $n^* \in \partial B_1(0)$ such that $n_k \to n^*$ as $k \to \infty$ and $\langle n^*, z - x_1 \rangle \leq 0$ for every $z \in \mathcal{Z}$, i.e. $n^* \in N_{\mathcal{Z}}(x_1)$. It follows that $n^* = n_1$, since $N_{\mathcal{Z}}(x_1) = \mathbb{R}_+ n_1$. Now taking $z = x_1$ in (65) we obtain that $\langle n_k, w_0 \rangle \leq 0$ for every k, and letting k go to infinity yields $\langle n_1, w_0 \rangle \leq 0$, contradicting (62). Hence we have proved the existence of $\delta_0 > 0$ satisfying (63). Now we set

$$y_r := y_0 + r n_1, \quad r \in \mathbb{R}, \tag{66}$$

and prove that there exist $r_0 > 0$ such that $y_{r_0} \in \partial\mathcal{Z}$. Indeed, the function $f(r) := \langle n_1, x_1 - y_r \rangle = \delta_0 \langle n_1, u_2 - x_1 \rangle - r$ is continuous, $f(0) > 0$ by (62), $f(r) < 0$ for r large. We have the implication $y_r \in \mathcal{Z} \Rightarrow f(r) \geq 0$. Hence, $y_r \notin \mathcal{Z}$ for $r > \delta_0 \langle n_1, u_2 - x_1 \rangle$, and some

$$0 < r_0 \leq \delta_0 \langle n_1, u_2 - x_1 \rangle \tag{67}$$

with the desired property necessarily exists. We now check that (57) holds provided we choose $x_0 = y_{r_0}$, $\lambda = \delta_0$, $r_1 = r_0/(1+\delta_0)$, $u_1 = x_1 + r_1 n_1$. Indeed, then

$$(1+\lambda)u_1 - \lambda u_2 = (1+\delta_0)x_1 + (1+\delta_0)r_1 n_1 - \delta_0 u_2 = y_0 + r_0 n_1 = x_0\,,$$

and (57) follows. To prove (58), we easily compute

$$\langle n_1, u_2 - u_1 \rangle = \langle n_1, u_2 - x_1 \rangle - \frac{r_0}{1+\delta_0} \geq \frac{1}{1+\delta_0}\langle n_1, u_2 - x_1 \rangle > 0$$

by virtue of (67) and (62), and the proof is complete. □

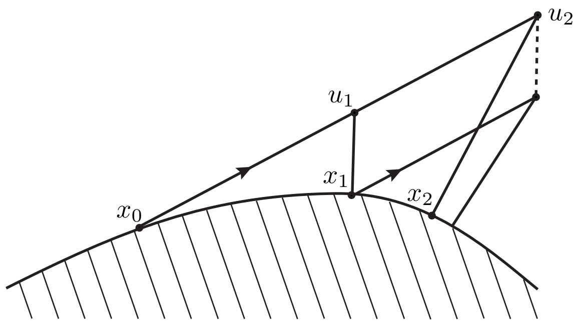

Figure 4.1: Illustration to Lemma 4.5.

Theorem 4.6. *Assume that* (21)–(24) *hold,* $\mathrm{int}(\mathcal{Z}) \neq \varnothing$*, and* $\mathcal{Z}$ *is not a non-obtuse polyhedron. Let* $x_0, x_1, x_2 \in \partial\mathcal{Z}$*,* $n_1, n_2 \in \partial B_1(0)$ *and* u_1, u_2 *be given by Lemma* 4.5*. For* $0 < t_1 < t_2 < T$ *define* $u, v \in \mathrm{BV}_L([0,T];\mathcal{H})$ *by*

$$u(t) := \begin{cases} z_0 & \textit{if } t = 0 \\ x_0 & \textit{if } 0 < t \leq t_1 \\ u_1 & \textit{if } t_1 < t \leq t_2 \\ u_2 & \textit{if } t_2 < t \leq T \end{cases} \tag{68}$$

and

$$v(t) := \begin{cases} z_0 & \textit{if } t = 0 \\ x_0 & \textit{if } 0 < t \leq t_1 \\ u_2 & \textit{if } t_1 < t \leq T. \end{cases} \tag{69}$$

Then $\mathsf{S}_K(u; z_0)(T) \neq \mathsf{S}_K(v; z_0)(T)$*.*

Proof. Here again, Figure 4.1 shows the situation. Observe that $u_1 \neq u_2$, hence $u \neq v$. Since $x_k = \mathrm{Proj}_{\mathcal{Z}}(u_k)$, $k = 1, 2$, thanks to Theorem 3.4 we have

$$\mathsf{S}_K(u; z_0)(t) = \begin{cases} z_0 & \text{if } t = 0 \\ x_0 & \text{if } 0 < t \leq t_1 \\ \mathrm{Proj}_{\mathcal{Z}}(u_1) = x_1 & \text{if } t_1 < t \leq t_2 \\ \mathrm{Proj}_{\mathcal{Z}}(x_1 + u_2 - u_1) & \text{if } t_2 < t \leq T \end{cases} \tag{70}$$

and

$$\mathsf{S}_K(v; z_0)(t) = \begin{cases} z_0 & \text{if } t = 0 \\ x_0 & \text{if } 0 < t \leq t_1 \\ \mathrm{Proj}_{\mathcal{Z}}(u_2) = x_2 & \text{if } t_1 < t \leq T. \end{cases} \tag{71}$$

If we assume by contradiction that $\mathsf{S}_K(u; z_0)(T) = \mathsf{S}_K(v; z_0)(T)$, then

$$x_2 = \mathrm{Proj}_{\mathcal{Z}}(x_1 + u_2 - u_1), \tag{72}$$

i.e.
$$x_1 + u_2 - u_1 - x_2 \in N_{\mathcal{Z}}(x_2).$$
Since $N_{\mathcal{Z}}(x_2) = \mathbb{R}_+ n_2$ we infer that there exists $\mu \geq 0$ such that
$$x_1 + u_2 - u_1 - x_2 = \mu n_2.$$
On the other hand by (56) of Lemma 4.5 there exist $r_k > 0$, $k = 1, 2$, such that $x_1 + u_2 - u_1 - x_2 = r_2 n_2 - r_1 n_1$, hence
$$r_2 n_2 - r_1 n_1 = \mu n_2$$
and we have found a contradiction because $n_1 \neq n_2$. $\square$

The previous result allows us to infer one implication of Theorem 3.5.

Corollary 4.7. *Assume that* (21)–(24) *hold and that* $\mathcal{Z}$ *is not a non-obtuse polyhedron. Then* $\mathsf{P}_K \neq \mathsf{P}_R$.

Proof. Let us start with the case $\mathrm{int}(\mathcal{Z}) \neq \varnothing$. If u and v are defined as in Theorem 4.6 then, thanks to (57), $\widetilde{u} = \widetilde{v}$ so that $\mathsf{S}_R(u; z_0)(T) = \mathsf{S}_R(v; z_0)(T)$ and we are done. When $\mathrm{int}(\mathcal{Z}) = \varnothing$, let $\mathcal{V}$ be the affine hull containing $\mathcal{Z}$. Then $\mathcal{V}$ is a closed vector subspace of $\mathcal{H}$ and it is a Hilbert space endowed with the inner product induced by $\mathcal{H}$. If $\mathsf{P}^{\mathcal{V}} : \mathrm{AC}([0,T];\mathcal{V}) \longrightarrow \mathrm{AC}([0,T];\mathcal{V})$ denotes the play operator in the space $\mathcal{V}$ with characteristic $\mathcal{Z}$, then $\mathsf{P}(u) = \mathsf{P}^{\mathcal{V}}(u)$ for every $u \in \mathrm{AC}([0,T];\mathcal{H})$ such that $u(t) \in \mathcal{V}$ for every $t \in [0,T]$, since $u'(t) \in \mathcal{V}$ for $\mathcal{L}^1$-a.e. $t \in [0,T]$. It follows that $\mathsf{P}_R(u) = \mathsf{P}^{\mathcal{V}}_R(u)$ whenever $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ and $u([0,T]) \subseteq \mathcal{V}$. Moreover Theorem 3.4 yields $\mathsf{P}_K(u) = \mathsf{P}^{\mathcal{V}}_K(u)$ for any $\mathcal{V}$-valued functions u. Therefore, since $\mathcal{Z}$, as a subset of $\mathcal{V}$, has nonempty interior and is not a non-obtuse polyhedron, we can conclude by applying to $\mathsf{P}^{\mathcal{V}}$ the result in the case $\mathrm{int}(\mathcal{Z}) \neq \varnothing$. $\square$

5. If $\mathcal{Z}$ is a non-obtuse polyhedron, then $\mathsf{P}_K = \mathsf{P}_R$

In this section we prove that the two extensions P_K and P_R coincide if $\mathcal{Z}$ is a non-obtuse polyhedron. We fix
$$p \in \mathbb{N}, \qquad n_1, \dots, n_p \in \partial B_1(0), \qquad c_1, \dots, c_p \in \mathbb{R}_+, \tag{73}$$
and we consider the polyhedron
$$\mathcal{Z} = \{x \in \mathcal{H} \ : \ \langle n_j, x \rangle \leq c_j, \ j = 1, \dots, p\}. \tag{74}$$
We assume that $\mathcal{Z}$ is non-obtuse, therefore $n_1, \dots, n_p$ can be chosen in such a way that
$$\langle n_i, n_j \rangle \leq 0 \ \ \forall i, j, \ 0 \leq i < j \leq p. \tag{75}$$
The following lemma allows to reduce to a very particular case and it holds for any $\mathcal{Z}$.

Lemma 5.1. *Assume that* (21)–(24) *hold. If* $\mathsf{P}_K(v, x_0) = \mathsf{P}_R(v, x_0)$ *for any* $x_0 \in \partial\mathcal{Z}$ *and any* $v : [0,T] \longrightarrow \mathcal{H}$ *of the form*

$$v(t) = \begin{cases} x_0 & \text{if } \ t = 0, \\ x_0 + e & \text{if } \ t \in \,]0,T]\,, \end{cases} \qquad e \in \mathcal{H},$$

with

$$x_0 + te \notin \mathcal{Z} \quad \forall t > 0,$$

then $\mathsf{P}_K(u, z_0) = \mathsf{P}_R(u, z_0)$ *for any* $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ *and any* $z_0 \in \mathcal{Z}$.

Proof. By Proposition 2.5 any $u \in \mathrm{BV}_L([0,T];\mathcal{H})$ can be approximated by a sequence of left continuous step functions converging both in the strict topology and in the uniform convergence topology. Therefore from the continuity properties stated in Theorem 3.3 and Theorem 3.4 it follows that $\mathsf{P}_K = \mathsf{P}_R$ if they coincide on the set of left continuous step functions. From (44) we infer that it is not restrictive to prove that they coincide on the set of step functions u such that $u(0) = z_0$. We will prove that $\mathsf{S}_K(u, z_0) = \mathsf{S}_R(u, z_0)$. Let us start by considering the special step function

$$u(t) := \begin{cases} z_0 & \text{if } t = 0 \\ u_1 & \text{if } t \in \,]0,T]. \end{cases} \tag{76}$$

In this case $\widetilde{u}(s) = z_0 + (s/T)(u_1 - z_0)$ for every $s \in [0,T]$ and

$$\mathsf{S}_K(u; z_0)(t) = \begin{cases} z_0 & \text{if } t = 0 \\ \mathrm{Proj}_{\mathcal{Z}}(u_1) & \text{if } t \in \,]0,T], \end{cases}$$

$$\mathsf{S}_R(u, z_0) = \begin{cases} z_0 & \text{if } t = 0 \\ \mathsf{S}(\widetilde{u}, z_0)(T) & \text{if } t \in \,]0,T]. \end{cases}$$

If $u_1 \in \mathcal{Z}$ then $\mathsf{S}(\widetilde{u}, z_0) = \widetilde{u}$, hence $\mathsf{S}_K(u, z_0) = \mathsf{S}_R(u, z_0)$. Thus we assume that $u_1 \notin \mathcal{Z}$. Let $t_0 \in [0,T[$ be such that $x_0 := z_0 + (t_0/T)(u_1 - z_0) \in \partial\mathcal{Z}$ and $z_0 + (t/T)(u_1 - z_0) \notin \mathcal{Z}$ for any $t > t_0$, and define

$$w(t) := \begin{cases} z_0 & \text{if } t = 0 \\ x_0 & \text{if } t \in \,]0,t_0] \\ u_1 & \text{if } t \in \,]t_0,T]. \end{cases}$$

Then $\widetilde{u} = \widetilde{w}$ and

$$\mathsf{S}_K(w, z_0)(t) = \begin{cases} z_0 & \text{if } t = 0 \\ x_0 & \text{if } t \in \,]0,t_0] \\ \mathrm{Proj}_{\mathcal{Z}}(u_1) & \text{if } t \in \,]t_0,T], \end{cases}$$

$$\mathsf{S}_R(w, z_0)(t) = \begin{cases} z_0 & \text{if } t = 0 \\ x_0 & \text{if } t \in \,]0,t_0] \\ \mathsf{S}(\widetilde{u}, z_0)(T) & \text{if } t \in \,]t_0,T]. \end{cases}$$

Therefore $\mathsf{S}_K(u, z_0) = \mathsf{S}_R(u, z_0)$ if and only if $\mathsf{S}_K(w, z_0)(T) = \mathsf{S}_R(w, z_0)(T)$. On the other hand

$$\mathsf{S}_K(w, z_0)(T) = \mathrm{Proj}_{\mathcal{Z}}(u_1) = \mathsf{S}_K(w(\cdot + t_0), x_0)(T - t_0) \tag{77}$$

and, since $\widetilde{u}(\cdot + t_0) = \widetilde{w(\cdot + t_0)}$, by the semigroup property,

$$\begin{aligned} \mathsf{S}_R(w, z_0)(T) &= \mathsf{S}(\widetilde{u}, z_0)(T) = \mathsf{S}(\widetilde{u}(\cdot + t_0), x_0)(T - t_0) \\ &= \mathsf{S}(\widetilde{w(\cdot + t_0)}, x_0)(T - t_0) = \mathsf{S}_R(w(\cdot + t_0), x_0)(T - t_0). \end{aligned} \tag{78}$$

Thanks to the assumption and to a time reparametrization we know that $\mathsf{S}_K(w(\cdot + t_0), x_0)(T - t_0) = \mathsf{S}(\widetilde{w(\cdot + t_0)}, x_0)(T - t_0)$, hence from (77)–(78) we infer that $\mathsf{S}_K(u, z_0) = \mathsf{S}_R(u, z_0)$. The general case of a step function of the form (39) with $u_0 = z_0$ can be reduced to the previous case (76) thanks to formulas (40)–(43). $\square$

Lemma 5.2. *Let $\mathcal{Z}$ be a non-obtuse polyhedron defined by* (73)–(75). *Let us consider $j_0 \in \{1, \dots, p\}$ and assume that*

$$\mathcal{Z}_{j_0} := \{x \in \mathcal{Z} \,:\, \langle n_{j_0}, x\rangle = c_{j_0}\}$$

is nonempty with an element $x_{j_0} \in \mathcal{Z}_{j_0}$. Then $\mathcal{Z}_{j_0} - x_{j_0}$ is a non-obtuse polyhedron in $\mathcal{H}_{j_0} := \{x \in \mathcal{H} \,:\, \langle n_{j_0}, x\rangle = 0\}$.

Proof. For $i \in \{1, \dots, p\}$ set

$$\tilde{n}_i := n_i - \langle n_i, n_{j_0}\rangle n_{j_0} \in \mathcal{H}_{j_0},$$

and denote

$$\widetilde{J} := \{i \in \{1, \dots, p\} \,:\, \tilde{n}_i \neq 0\}.$$

Therefore

$$j_0 \notin \widetilde{J}$$

and, since $n_j = \langle n_j, n_{j_0}\rangle n_{j_0}$ for $j \notin \widetilde{J}$,

$$j \notin \widetilde{J} \quad \Longrightarrow \quad n_j \in \{n_{j_0}, -n_{j_0}\}.$$

Furthermore if $\{1, \dots, p\} \smallsetminus (\widetilde{J} \cup \{j_0\}) \neq \varnothing$, then

$$j \notin \widetilde{J}, \quad j \neq j_0 \quad \Longrightarrow \quad c_j \geq -c_{j_0},$$

indeed $c_{j_0} = \langle n_{j_0}, x_{j_0}\rangle = -\langle -n_{j_0}, x_{j_0}\rangle = -\langle n_j, x_{j_0}\rangle \geq -c_j$. Notice that we have $\langle \widetilde{n}_i, \widetilde{n}_j\rangle \leq 0$ for every $i, j \in \widetilde{J}$ with $i \neq j$. Indeed $j_0 \notin \widetilde{J}$, hence a straightforward calculation provides

$$\langle \widetilde{n}_i, \widetilde{n}_j\rangle = \langle n_i, n_j\rangle - \langle n_i, n_{j_0}\rangle\langle n_j, n_{j_0}\rangle \leq 0.$$

For $i \in \widetilde{J}$ put $c_i^0 := \langle n_i, x_{j_0} \rangle \le c_i$. The proof of Lemma 5.2 will be complete if we check that

$$\mathcal{Z}_{j_0} - x_{j_0} = \widetilde{\mathcal{Z}}_{j_0} := \big\{ x \in \mathcal{H}_{j_0} \,:\, \langle x, \tilde{n}_i/\|\tilde{n}_i\| \rangle \le (c_i - c_i^0)/\|\tilde{n}_i\| \ \forall i \in \widetilde{J} \big\}.$$

The inclusion $\mathcal{Z}_{j_0} - x_{j_0} \subseteq \widetilde{\mathcal{Z}}_{j_0}$ is obvious. Let now $\tilde{z} \in \widetilde{\mathcal{Z}}_{j_0}$ be arbitrary. In particular $\tilde{z} \in \mathcal{H}_{j_0}$, thus

$$\langle \tilde{z} + x_{j_0}, n_{j_0} \rangle = c_{j_0}. \tag{79}$$

Moreover, for $i \in \widetilde{J}$ we have, as $\widetilde{\mathcal{Z}}_{j_0} \subseteq \mathcal{H}_{j_0}$,

$$\begin{aligned} \langle \tilde{z} + x_{j_0}, n_i \rangle &= \langle \tilde{z}, \tilde{n}_i \rangle + \langle n_i, n_{j_0} \rangle \langle \tilde{z}, n_{j_0} \rangle + \langle x_{j_0}, n_i \rangle \\ &= \langle \tilde{z}, \tilde{n}_i \rangle + c_i^0 \le (c_i - c_i^0) + c_i^0 \le c_i \quad \forall i \in \widetilde{J}. \end{aligned} \tag{80}$$

Finally if there exists an index $j \in \{1, \dots, p\}$ with $j \notin (\widetilde{J} \cup \{j_0\})$, then $n_j = -n_{j_0}$ and

$$\langle \tilde{z} + x_{j_0}, n_j \rangle = -\langle \tilde{z} + x_{j_0}, n_{j_0} \rangle = -c_{j_0} \le c_j. \tag{81}$$

From (79)–(81) we infer that $\tilde{z} + x_{j_0} \in \mathcal{Z}_{j_0}$, which we wanted to prove. □

Lemma 5.3. *Let $\mathcal{Z}$ be a non-obtuse polyhedron defined by* (73)–(75)*. Let us consider $J \subseteq \{1, \dots, p\}$ and assume that*

$$\mathcal{Z}_J := \{ x \in \mathcal{Z} \,:\, \langle n_j, x \rangle = c_j \ \forall j \in J \}$$

is nonempty with an element $x_J \in \mathcal{Z}_J$. Then $\mathcal{Z}_J - x_J$ is a non-obtuse polyhedron in $\mathcal{H}_J := \{ x \in \mathcal{H} \,:\, \langle n_j, x \rangle = 0 \ \forall j \in J \}$.

Proof. We proceed by induction. Let # denote the number of elements of a finite set. Lemma 5.3 follows from Lemma 5.2 if $\#J = 1$. For $\#J \ge 2$ assume that the statement is proved for all sets J' such that $\#J' < \#J$, fix an element $j_0 \in J$, and set $J' := J \setminus \{j_0\}$. By induction hypothesis, the set $\mathcal{Z}_{J'} - x_J$ is a non-obtuse polyhedron in $\mathcal{H}_{J'} := \{ x \in \mathcal{H} \,:\, \langle n_j, x \rangle = 0 \ \forall j \in J' \}$, and we apply Lemma 5.2 with $\mathcal{Z}_{J'} - x_J$ instead of $\mathcal{Z}$ and $x_{j_0} = 0$. □

Lemma 5.4. *Assume that $\mathcal{Z}$ is a non-obtuse polyhedron defined by* (73)–(75)*. Let $x_0 \in \partial\mathcal{Z}$, $e \notin T_{\mathcal{Z}}(x_0)$ be given and let us set*

$$J_0 := \{ j \in J(x_0) \ :\ \langle n_j, e \rangle \ge 0 \},$$

$$\mathcal{A}_0 := \{ x \in \mathcal{H} \ :\ \langle n_j, x \rangle = c_j \ \forall j \in J_0 \}.$$

Then $J_0 \neq \varnothing$,

$$\mathrm{Proj}_{\mathcal{Z}}(x_1 + e) \in \mathcal{A}_0 \quad \forall x_1 \in \mathcal{A}_0, \tag{82}$$

and

$$\begin{aligned} \mathrm{Proj}_{\mathcal{Z}}(x_1 + e) &= \mathrm{Proj}_{\mathcal{Z} \cap \mathcal{A}_0}(\mathrm{Proj}_{\mathcal{A}_0}(x_1 + e)) \\ &= \mathrm{Proj}_{(\mathcal{Z} \cap \mathcal{A}_0) - x_0}(\mathrm{Proj}_{\mathcal{A}_0}(x_1 + e) - x_0) + x_0 \quad \forall x_1 \in \mathcal{Z} \cap \mathcal{A}_0. \end{aligned} \tag{83}$$

Proof. It is obvious that $J_0 \neq \varnothing$. Now let us set $x := \mathrm{Proj}_{\mathcal{Z}}(x_1 + e)$ and assume by contradiction that (82) is not true. It follows that there exists $j_0 \in J_0$ such that $\langle n_{j_0}, x\rangle < c_{j_0}$. Hence $n_{j_0} \notin N_{\mathcal{Z}}(x)$ and we can find $\alpha_1, \ldots, \alpha_m \geq 0$ such that

$$x_1 + e - x = \sum_{j \neq j_0} \alpha_j n_j,$$

thus, as $\mathcal{Z}$ is non-obtuse,

$$\langle x_1 + e - x, n_{j_0}\rangle = \sum_{j \neq j_0} \alpha_j \langle n_j, n_{j_0}\rangle \leq 0.$$

On the other hand

$$\begin{aligned} \langle x_1 + e - x, n_{j_0}\rangle &= c_{j_0} + \langle e, n_{j_0}\rangle - \langle x, n_{j_0}\rangle \\ &\geq c_{j_0} - \langle x, n_{j_0}\rangle > 0. \end{aligned}$$

This is a contradiction proving that $J_0 \subseteq J(x)$, i.e. (82). Now we show the first equality in (83). If $v := \mathrm{Proj}_{\mathcal{A}_0}(x_1 + e)$, then $\langle x_1 + e - v, y - v\rangle = 0$ for all $y \in \mathcal{A}_0$, since $\mathcal{A}_0$ is an affine subspace. Hence, for every $z \in \mathcal{Z} \cap \mathcal{A}_0$ we have

$$\begin{aligned} \langle v - x, z - x\rangle &= \ -\langle x_1 + e - v, z - x\rangle + \langle x_1 + e - x, z - x\rangle \\ &\leq \ -\langle x_1 + e - v, z - x\rangle \\ &= \ -\langle x_1 + e - v, z - v\rangle + \langle x_1 + e - v, x - v\rangle = 0. \end{aligned}$$

Therefore the equality is proved since $x \in \mathcal{Z} \cap \mathcal{A}_0$. The second equality in (83) is obtained by translating in x_0. $\square$

Lemma 5.5. *Under the same assumptions of Lemma* 5.4*, set*

$$\mathcal{V}_0 := \mathcal{A}_0 - x_0, \qquad \mathcal{Z}_0 := (\mathcal{Z} \cap \mathcal{A}_0) - x_0.$$

Let

$$\mathsf{S}^0 : \mathrm{AC}([0,T];\mathcal{V}_0) \longrightarrow \mathrm{AC}([0,T];\mathcal{V}_0), \tag{84}$$
$$\mathsf{S}^0_K : \mathrm{BV}_L([0,T];\mathcal{V}_0) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{V}_0), \tag{85}$$
$$\mathsf{S}^0_R : \mathrm{BV}_L([0,T];\mathcal{V}_0) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{V}_0) \tag{86}$$

be the stop operator in the Hilbert space $\mathcal{V}_0$, and its extensions to BV*, with characteristic $\mathcal{Z}_0$ and initial data $z_0 = 0$. If*

$$u(t) := \begin{cases} x_0 & \text{if } t = 0 \\ x_0 + e & \text{if } t \in \,]0,T], \end{cases} \qquad v(t) := \mathrm{Proj}_{\mathcal{V}_0}(u(t) - x_0),$$

then

$$\mathsf{S}_K(u, x_0) = x_0 + \mathsf{S}^0_K(v, 0), \qquad \mathsf{S}_R(u, x_0) = x_0 + \mathsf{S}^0_R(v, 0). \tag{87}$$

Proof. If $m \in \mathbb{N}$, $0 = t_0 < \cdots < t_m = T$, and $u_m \in \mathrm{BV}_L([0,T];\mathcal{H})$ is defined by

$$u_m(t) := \begin{cases} x_0 & \text{if } t = 0 \\ x_0 + t_k e & \text{if } t \in \,]t_{k-1}, t_k],\ k = 1, \ldots, m, \end{cases} \tag{88}$$

then we have

$$\mathsf{S}_K(u_m, x_0)(t) = \begin{cases} x_0 & \text{if } t = 0 \\ x_k := \mathrm{Proj}_{\mathcal{Z}}(x_{k-1} + (t_k - t_{k-1})e) & \text{if } t \in \,]t_{k-1}, t_k] \end{cases}$$

for any $k = 1, \ldots, m$, and from Lemma 5.4 it follows that

$$x_k = \mathsf{S}_K(u_m, x_0)(t_k) \in \mathcal{Z} \cap \mathcal{A}_0 \quad \forall k = 1, \ldots, m. \tag{89}$$

Define $v_m \in \mathrm{BV}_L([0,T];\mathcal{V}_0)$ by

$$\begin{aligned} v_m(t) :=\ & \mathrm{Proj}_{\mathcal{V}_0}(u_m(t) - x_0) = \mathrm{Proj}_{\mathcal{A}_0}(u_m(t)) - x_0 \\ =\ & \begin{cases} 0 & \text{if } t = 0 \\ t_k \,\mathrm{Proj}_{\mathcal{V}_0}(e) & \text{if } t \in \,]t_{k-1}, t_k],\ k = 1, \ldots, m. \end{cases} \end{aligned} \tag{90}$$

For any $k = 1, \ldots, m$ we have that

$$\mathsf{S}^0_K(v_m, 0)(t) = \begin{cases} \eta_0 := 0 & \text{if } t = 0 \\ \eta_k := \mathrm{Proj}_{\mathcal{Z}_0}(\eta_{k-1} + (t_k - t_{k-1})\,\mathrm{Proj}_{\mathcal{V}_0}(e)) & \text{if } t \in \,]t_{k-1}, t_k]. \end{cases}$$

We prove by an iterative argument that $\mathsf{S}^0_K(u_m, x_0) = x_0 + \mathsf{S}^0_K(v_m, 0)$, i.e. $x_k = x_0 + \eta_k$ for every $k = 1, \ldots, m$. For $k = 0$ this is obvious. For $k > 1$ we assume that $x_{k-1} = x_0 + \eta_{k-1}$ and from (89) we get

$$\begin{aligned} \mathrm{Proj}_{\mathcal{A}_0}(x_{k-1} + (t_k - t_{k-1})e) =\ & \mathrm{Proj}_{\mathcal{A}_0}(x_{k-1}) + (t_k - t_{k-1})\,\mathrm{Proj}_{\mathcal{V}_0}(e) \\ =\ & x_{k-1} + (t_k - t_{k-1})\,\mathrm{Proj}_{\mathcal{V}_0}(e), \end{aligned}$$

hence, thanks to (83) we obtain

$$\begin{aligned} \eta_k =\ & \mathrm{Proj}_{\mathcal{Z}_0}(\eta_{k-1} + (t_k - t_{k-1})\,\mathrm{Proj}_{\mathcal{V}_0}(e)) \\ =\ & \mathrm{Proj}_{(\mathcal{A}_0 \cap \mathcal{Z}) - x_0}(x_{k-1} - x_0 + (t_k - t_{k-1})\,\mathrm{Proj}_{\mathcal{V}_0}(e)) \\ =\ & \mathrm{Proj}_{(\mathcal{A}_0 \cap \mathcal{Z}) - x_0}(\mathrm{Proj}_{\mathcal{A}_0}(x_{k-1} + (t_k - t_{k-1})e) - x_0) \\ =\ & -x_0 + \mathrm{Proj}_{\mathcal{Z}}(x_{k-1} + (t_k - t_{k-1})e) = -x_0 + x_k. \end{aligned}$$

Thus we have proved that $\mathsf{S}^0_K(u_m, x_0) = x_0 + \mathsf{S}^0_K(v_m, 0)$. The case $m = 1$ yields the first equality in (87). Now observe that for the reparametrizations of u and v we have the formulas $\widetilde{u}(t) = x_0 + (t/T)e$ and $\widetilde{v}(t) = (t/T)\,\mathrm{Proj}_{\mathcal{V}_0}(e)$, $t \in [0,T]$, thus $\widetilde{u}$ and $\widetilde{v}$ can be uniformly approximated by sequences u_m and v_m of the form

(88) and (90). Since $\mathsf{S}_K^0(u_m, x_0) = x_0 + \mathsf{S}_K^0(v_m, 0)$ for every m, by the continuity of the stop operators we get $\mathsf{S}(\widetilde{u}, x_0) = x_0 + \mathsf{S}^0(\widetilde{v}, 0)$. It follows that

$$\begin{aligned}\mathsf{S}_R^0(v,0)(t) &= \begin{cases} 0 & \text{if } t = 0 \\ \mathsf{S}^0(\widetilde{v},0)(T) = \mathsf{S}(\widetilde{u},0)(T) - x_0 & \text{if } t \in \,]0,T] \end{cases} \\ &= \mathsf{S}_R(u,x_0)(t)\end{aligned}$$

for any $t \in [0,T]$ and (87) is completely proved. □

We are now in position to prove the missing implication of our main theorem.

Theorem 5.6. *If $\mathcal{Z}$ is a non-obtuse polyhedron then $\mathsf{P}_K = \mathsf{P}_R$.*

Proof. We argue by induction on $d = \dim(\mathcal{H}) \in \mathbb{N}$, the dimension of $\mathcal{H}$. If $d = 1$ we know that the two operators are the same. If $d > 1$ we assume that the theorem is true for any $k \leq d-1$ and we prove it for d. Thanks to Lemma 5.1 it is enough to prove that $\mathsf{S}_K(u, x_0) = \mathsf{S}_R(u, x_0)$ with $x_0 \in \partial\mathcal{Z}$ and $u : [0,T] \longrightarrow \mathcal{H}$ of the form

$$u(t) = \begin{cases} x_0 & \text{if } t = 0 \\ x_0 + e & \text{if } t \in \,]0,T], \end{cases} \qquad e \in \mathcal{H},$$

with $x_0 + te \notin \mathcal{H}$ for every $t > 0$. Since $\mathcal{Z}$ is a polyhedron this is equivalent to saying that $e \notin T_{\mathcal{Z}}(x_0)$. Hence if we set

$$J_0 := \{j \in J(x_0) \ : \ \langle n_j, e\rangle \geq 0\}, \qquad \mathcal{A}_0 := \{x \in \mathcal{H} \ : \ \langle n_j, x\rangle = c_j \ \forall j \in J_0\},$$
$$\mathcal{V}_0 := \mathcal{A}_0 - x_0, \qquad \mathcal{Z}_0 := (\mathcal{Z} \cap \mathcal{A}_0) - x_0,$$

if $\mathsf{S}_K^0 : \mathrm{BV}_L([0,T];\mathcal{V}_0) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{V}_0)$ and $\mathsf{S}_R^0 : \mathrm{BV}_L([0,T];\mathcal{V}_0) \longrightarrow \mathrm{BV}_L([0,T];\mathcal{V}_0)$ are the stop operators defined in Lemma 5.5, and if $v \in \mathrm{BV}_L([0,T];\mathcal{V}_0)$ is defined by

$$v(t) := \mathrm{Proj}_{\mathcal{V}_0}(u(t) - x_0), \quad t \in [0,T],$$

from Lemma 5.5 we get

$$\mathsf{S}_K(u, x_0) = x_0 + \mathsf{S}_K^0(v, 0), \qquad \mathsf{S}_R(u, x_0) = x_0 + \mathsf{S}_R^0(v, 0). \tag{91}$$

By Lemma 5.3 $\mathcal{Z}_0$ is a non-obtuse polyhedron in $\mathcal{V}_0$. As $\dim(\mathcal{V}_0) < d$, by the induction hypothesis we have that $\mathsf{S}_K^0(v,0) = \mathsf{S}_R^0(v,0)$, hence $\mathsf{S}_K(u,x_0) = \mathsf{S}_R(u,x_0)$ and the theorem is completely proved. □

6. Description of non-obtuse polyhedra

In this section we provide a description of non-obtuse polyhedra in finite dimensional spaces. Let $\mathcal{Z}$ be a non-obtuse polyhedron defined by (73)–(75).

For a subset $J \subseteq \{1, \dots, p\}$ we introduce the subspace $L(J)$ and the positive cone $C(J)$ generated by vectors n_j with $j \in J$, as well as the normal cone $N(J)$ to $C(J)$, by the formulas

$$L(J) = \Big\{x \in \mathcal{H} : x = \sum_{i \in J} a_i n_i\,,\ a_i \in \mathbb{R}\ \forall i \in J\Big\}\,, \tag{92}$$

$$C(J) = \Big\{x \in \mathcal{H} : x = \sum_{i \in J} a_i n_i\,,\ a_i \geq 0\ \forall i \in J\Big\}\,, \tag{93}$$

$$N(J) = \big\{y \in L(J) : \langle y, n_i\rangle \leq 0\ \forall i \in J\big\}\,. \tag{94}$$

We can assume that $\mathcal{H}_p := L(\{1, \dots, p\}) = \mathcal{H}$. Indeed, if this is not the case, then $\mathcal{H}$ can be decomposed into the direct sum $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_p^\perp$ of $\mathcal{H}_p$ and its orthogonal complement $\mathcal{H}_p^\perp$. The set $\mathcal{Z}$ now turns out to be the "cylinder" $\mathcal{Z} = (\mathcal{Z} \cap \mathcal{H}_p) + \mathcal{H}_p^\perp$, and the analysis can be reduced to $\mathcal{Z} \cap \mathcal{H}_p$.

The case $p = d = \dim(\mathcal{H})$ is easy. We find z_0 such that $\langle z_0, n_i\rangle = c_i$ for $i = 1, \dots, d$. Then $\mathcal{Z}$ is the cone

$$\mathcal{Z} = z_0 + N(\{1, \dots, d\}). \tag{95}$$

For $p > d$ we assume, possibly after a permutation of indices, that the first d vectors are linearly independent. For subspaces $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{H}$ we denote

$$\mathcal{V}_1 \perp \mathcal{V}_2 \iff \langle v_1, v_2\rangle = 0 \quad \forall v_1 \in \mathcal{V}_1, \forall v_2 \in \mathcal{V}_2.$$

Proposition 6.1. *Let $p = d + r$, $r \geq 1$, let $\dim L(\{1, \dots, d\}) = d$, and let (73)–(75) hold. Set $\tilde{J} = \{d{+}1, \dots, d{+}r\}$. Then there exist nonempty subsets $J_1, \dots, J_r$ of $\{1, \dots, d\}$, $J_i \cap J_k = \varnothing$ for $i \neq k$, $\bigcup_{i=1}^r J_i = J_0 \subseteq \{1, \dots, d\}$, and numbers $a_{ij} > 0$ for $j \in J_i$ such that*

$$n_{d+i} + \sum_{j \in J_i} a_{ij} n_j = 0\,, \quad \text{for } i = 1, \dots, r\,, \tag{96}$$

$$L(\{d+i\} \cup J_i) \perp L(\{d+l\} \cup J_l) \quad \text{for } i \neq l\,, \tag{97}$$

$$L(\tilde{J} \cup J_0) \perp L(\{1, \dots, d\} \setminus J_0)\,. \tag{98}$$

Proof. For every $i = 1, \dots, r$ there exist $\gamma_{ij} \in \mathbb{R}$, $j = 1, \dots, d$, such that

$$n_{d+i} = \sum_{j=1}^{d} \gamma_{ij} n_j\,. \tag{99}$$

Set $J_i = \{j \in \{1, \dots, d\} : \gamma_{ij} \neq 0\}$, $J_i^- = \{j \in J_i : \gamma_{ij} < 0\}$, $J_i^+ = J_i \setminus J_i^-$, and

$$x := n_{d+i} + \sum_{j \in J_i^-} |\gamma_{ij}| n_j = \sum_{k \in J_i^+} \gamma_{ik} n_k =: y. \tag{100}$$

The scalar product $\langle x, y\rangle$ is non-positive by (75), hence $x = y = 0$. Since all $n_1, \dots, n_d$ are linearly independent, we necessarily have $J_i^+ = \varnothing$, and (96) follows with $a_{ij} = |\gamma_{ij}|$. To prove that $J_i \cap J_k = \varnothing$, we proceed by contradiction. Assume that the set $\hat{J}_{ik} = J_i \cap J_k$ is nonempty. For an arbitrary $\beta > 0$ we have

$$n_{d+i} + \sum_{j \in J_i \setminus \hat{J}_{ik}} a_{ij} n_j = \beta n_{d+k} + \sum_{j \in J_k \setminus \hat{J}_{ik}} \beta a_{kj} n_j + \sum_{j \in \hat{J}_{ik}} (\beta a_{kj} - a_{ij}) n_j. \tag{101}$$

We first choose β large such that $\beta a_{kj} - a_{ij} > 0$ for all $j \in \hat{J}_{ik}$. We are in the situation of (100) of two vectors $x = y$, $\langle x, y\rangle \le 0$, hence $x = y = 0$. Hence

$$n_{d+i} + \sum_{j \in J_i \setminus \hat{J}_{ik}} a_{ij} n_j = 0,$$

thus from (96) it follows that

$$\sum_{j \in \hat{J}_{ik}} a_{ij} n_j = 0$$

and we conclude from the linear independence of $\{n_1, \dots, n_d\}$ that $\hat{J}_{ik} = \varnothing$. A similar argument yields (97). Indeed, we have

$$0 = \left\langle n_{d+i} + \sum_{j \in J_i} a_{ij} n_j \, , \, n_{d+l} + \sum_{k \in J_l} a_{lk} n_k \right\rangle \le 0,$$

hence $\langle n_j, n_k\rangle = 0$ for all $j \in \{d+i\} \cup J_i$, $k \in \{d+l\} \cup J_l$. Statement (98) is trivial. Indeed, for $j \in J_0$, we find i such that $j \in J_i$, and take the scalar product of (96) with any n_s, $s \in \{1, \dots, d\} \setminus J_0$. We obtain as before

$$0 = \left\langle n_{d+i} + \sum_{j \in J_i} a_{ij} n_j \, , \, n_s \right\rangle \le 0,$$

hence $\langle n_{d+i}, n_s\rangle = 0$, $\langle n_j, n_s\rangle = 0$ for $j \in J_i$, and the proof is complete. □

The interpretation of Proposition 6.1 is the following. As above, let # denote the number of elements of a finite set. A polyhedron of the form (73)–(75) with normal vectors satisfying (96) is a *non-obtuse simplex* in a space of dimension $\#J_i$, that is, an interval if $\#J_i = 1$, a triangle if $\#J_i = 2$, a tetrahedron if $\#J_i = 3$, etc. The convex set $\mathcal{Z}$ with normal vectors from $\{1, \dots, d\} \setminus J_0$ is a cone as in (95).

From the condition

$$\sum_{i=1}^{r} \#J_i \le d$$

we conclude that necessarily $r \le d$. The case $r = d$ is indeed special. Then $\#J_i = 1$ for all $i = 1, \dots, d$ and, after a possible permutation, we obtain from

(96) that $n_{d+i} = -n_i$ for all $i = 1, \dots, d$. In other words, the only non-obtuse polyhedron with $p = 2d$ is a *parallelepiped*.

The general case considered in Proposition 6.1 can be summarized as follows: Every non-obtuse polyhedron can be decomposed into an orthogonal combination of simplices and at most one cone. In the three dimensional case, for example, we have a parallelepiped for $p = 6$, a bounded "cylinder" with triangular basis or a semi-infinite parallelepiped for $p = 5$, and a tetrahedron or a semi-infinite "cake segment" for $p = 4$. If $\dim(\mathcal{H}) = 4$ there are many more alternatives, the most special one being an orthogonal decomposition into two triangles for $p = 6$.

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