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λ-Points in Orlicz Spaces

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A criterion for λ-points of the unit ball in Orlicz spaces generated by arbitrary Orlicz functions (that is Orlicz functions which vanish outside zero and which attain infinite values to the right of some point $u > 0$ are not excluded) and equipped with the Orlicz norm is given. Moreover, Orlicz spaces with the λ-property are characterized. In contrast to results in [19], Orlicz spaces considered by us need not always have the λ-property.

Keywords: Orlicz space, λ-point, λ-property, extreme point, Orlicz norm

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1. Introduction

Let $(X, \|\cdot\|_X)$ be a real Banach space and $B(X)$ $(S(X))$ be the closed unit ball (the unit sphere) of X, respectively. Denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of reals. For any $A \subset \mathbb{R}$ denote $A_+ = \{u \in A : u \geq 0\}$.

Before starting with our results, we need to recall some notions.

A point $x \in S(X)$ is said to be an *extreme point of* $B(X)$ if x cannot be written as the arithmetic mean $\frac{1}{2}(y+z)$ of two distinct points $y, z \in S(X)$. The set of all extreme points of $B(X)$ is denoted by $\operatorname{ext} B(X)$.

A Banach space X is said to be *rotund* (**R** for short) if every point of $S(X)$ is an extreme point of $B(X)$.

For any $x \in B(X)$ we define the number

$$\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y,\ e \in \operatorname{ext} B(X),\ y \in B(X)\}.$$

In the case when $\operatorname{ext} B(X) = \varnothing$, we assume that $\lambda(x) = 0$ for any $x \in B(X)$.

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If $\lambda(x) > 0$, then x is called a *λ-point of* $B(X)$. The set of all λ-points of $B(X)$ is denoted by $\lambda[B(X)]$. If every point of $B(X)$ is a λ-point, then X is said to have the *λ-property.* Moreover, if

$$\lambda_X = \inf\{\lambda(x) : x \in S(X)\} > 0,$$

then X is said to have the *uniform λ-property.*

The following result gives a lower estimation of the function λ.

Lemma 1.1 (Proposition 2.12 in [4]). *If X is a Banach space such that* $\operatorname{ext} B(X) \neq \varnothing$, *then*

$$\lambda(x) \geq \max\left\{\frac{1}{2}\left(1 - \|x\|_X\right),\ \lambda\left(\frac{x}{\|x\|_X}\right)\|x\|_X\right\}.$$

Remark 1.2. By Lemma 1.1, any interior point of $B(X)$ is a λ-point whenever $\operatorname{ext} B(X) \neq \varnothing$. Therefore it is sensible to consider only λ-points lying on the unit sphere $S(X)$. Hence we can say that X has the λ-property if and only if $\operatorname{ext} B(X) \neq \varnothing$ and $\lambda(x) > 0$ for any $x \in S(X)$.

The λ-property was introduced by Aron and Lohman [1]. The λ-property is important because for Banach spaces with the λ-property we have that $\overline{\mathrm{co}}\,(\operatorname{ext} B(X)) = B(X)$. Moreover, Aron, Lohman and Granero proved in [2] that a Banach space X has the λ-property if and only if it has the convex series representation property, i.e. for each $x \in B(X)$, there is a sequence (e_k) of extreme points of $B(X)$ and a sequence of non-negative real numbers (λ_k) such that $\sum_{k=1}^{\infty} \lambda_k = 1$ and $x = \sum_{k=1}^{\infty} \lambda_k e_k$. It has been also shown in [2] that the uniform λ-property for a Banach space X is equivalent to the uniform convex series representation property for X, i.e. convex series representation property in which the sequence (λ_k) does not depend on x.

Let (T, Σ, μ) be a measure space with a σ-finite non-atomic and complete measure μ and $L_0 = L_0(T, \Sigma, \mu)$ be the set of all μ-equivalence classes of real and Σ-measurable functions defined on T.

By a *Köthe space* $(E, \|\cdot\|_E)$ we mean a subspace of $L_0(T, \Sigma, \mu)$ which is an ideal in $L_0(T, \Sigma, \mu)$, that is, if $x \in L_0(T, \Sigma, \mu)$, $y \in E$ and $|x(t)| \leq |y(t)|$ μ-a.e. T, then $x \in E$ and $\|x\|_E \leq \|y\|_E$ as well as there exists $x \in E$ such that $x(t) > 0$ for any $t \in T$.

The set $E_+ = \{x \in E : x \geq 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$.

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an *Orlicz function* if it is even, convex, left continuous on the whole $\mathbb{R}_+$, $\Phi(0) = 0$ and Φ is not identically equal to zero. Since Φ is even, without loss of generality we often consider Φ with the domain restricted to the interval $[0, \infty)$. It follows that every Orlicz function Φ is non-decreasing on $\mathbb{R}_+$.

We say a point w is a *point of strict convexity of* Φ (we write $w \in \mathrm{SC}(\Phi)$) if

$$\Phi(w) < \frac{1}{2}(\Phi(u) + \Phi(v))$$

for every $u, v \in \mathbb{R}$ such that $u \neq v$ and $w = \frac{1}{2}(u+v)$. The set $AF(\Phi) := \mathbb{R} \setminus \mathrm{SC}(\Phi)$ is called the *set of all structural affine intervals of* Φ.

For any Orlicz function Φ we let

$$a_\Phi := \sup\{u \geq 0 : \Phi(u) = 0\},$$
$$b_\Phi := \sup\{u > 0 : \Phi(u) < \infty\}$$

and

$$u_\Phi := \sup\{u \geq a_\Phi : u \in \mathrm{SC}(\Phi)\}.$$

Any Orlicz function Φ is strictly increasing on the interval $[a_\Phi, b_\Phi]$.

Denote by p the right hand side derivative of Φ with the domain restricted to the interval $[0, \infty)$. The function Ψ defined by the formula

$$\Psi(u) = \sup\{|u| v - \Phi(v) : v \geq 0\}$$

is called *complementary to* Φ *in the sense of Young.* Denoting by q the right hand side derivative of Ψ.

Given any Orlicz function Φ we define on L_0 the modular I_Φ by

$$I_\Phi(x) = \int_T \Phi(x(t))\, d\mu.$$

By the *Orlicz function space* L_Φ we mean the set of all $x \in L_0$ such that $I_\Phi(cx) < \infty$ for some $c > 0$. The space L_Φ is equipped with the so called *Orlicz norm* that can be expressed by the following Amemiya formula

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

(see [7] or [14]). The set of all $k > 0$ at which the infimum in the Amemiya formula for $\|x\|_\Phi$ is attained (for fixed $x \in L_\Phi$) will be denoted by $K(x)$. Moreover, for any $x \in L_\Phi \setminus \{0\}$, we define

$$k^*(x) = \inf\{k > 0 : I_\Psi(p \circ k|x|) \geq 1\},$$
$$k^{**}(x) = \sup\{k > 0 : I_\Psi(p \circ k|x|) \leq 1\}.$$

Obviously, $k^*(x) \leq k^{**}(x)$ for any $x \in L_\Phi \setminus \{0\}$. It is well known that

$$K(x) = \begin{cases} \varnothing & \text{if } k^*(x) = \infty \\ [k^*(x), k^{**}(x)] & \text{if } k^{**}(x) < \infty \\ [k^*(x), \infty) & \text{if } k^*(x) < \infty \text{ and } k^{**}(x) = \infty \end{cases}$$

(see [4]). Orlicz spaces are important examples of Köthe spaces. For more details on Orlicz spaces we refer to [4, 17, 18].

If the Orlicz space is generated by finitely valued and vanishing only at zero N-functions (i.e. Orlicz functions satisfying $\lim_{u\to 0}\Phi(u)/u = 0$ and $\lim_{u\to\infty}\Phi(u)/u = \infty$) in the case of both (the Luxemburg and the Orlicz) norms, then it has the λ-property [6, 11, 19]. A characterization of λ-points in such kind of Orlicz spaces does not make sense, because every point of $B(L_\Phi)$ is a λ-point. The situation becomes more interesting, when Orlicz spaces are generated by Orlicz functions. Then the Orlicz space need not have the λ-property. In the case of the Luxemburg norm, the λ-property in Orlicz spaces generated by finitely valued Orlicz functions has been characterized by Granero [11]. The Orlicz spaces with the λ-property equipped with the Orlicz norm and generated by arbitrary Orlicz functions, were not considered till now. In this note we give a natural characterization of λ-points of $B(L_\Phi)$, where L_Φ is an Orlicz space equipped with the Orlicz norm and it is generated by an arbitrary Orlicz function. Moreover, we describe Orlicz spaces with the λ-property.

We will use the following criterion for extreme points of the unit ball in Orlicz spaces equipped with the Orlicz norm.

Proposition 1.3 (see [9], Theorem 1). *Let (T,Σ,μ) be a non-atomic and complete measure space and Φ be an arbitrary Orlicz function. Then $x\in S(L_\Phi)$ is an extreme point of $B(L_\Phi)$ if and only if:*

(a) *the set $K(x)$ is a singleton in $(0,\infty)$,*

(b) *$kx(t)\in \mathrm{SC}(\Phi)$ for μ-a.e. $t\in T$, where $\{k\}=K(x)$.*

Moreover, we will use the following characterization of Orlicz function spaces L_Φ without extreme points on its unit sphere.

Proposition 1.4 (see [10], Theorem 1). *The set $\operatorname{ext} B(L_\Phi)$ is empty if and only if $I_\Psi(p\circ u_\Phi\chi_T)\le 1$.*

The above characterization works under convention $\infty\cdot 0 = 0$.

2. λ-points

There are several elementary facts that we record for further use. The first one concerns the fuction λ defined on a Köthe space.

Proposition 2.1. *Let E be a Köthe space. Then*

(a) *$\lambda(x)=\lambda(|x|)$ for any $x\in S(E)$;*

(b) *$\lambda(x)=\lambda_+(x)$ for any $x\in S(E)_+$, where λ_+ is defined on $S(E)_+$ by the formula*

$$\lambda_+(x)=\sup\left\{\lambda\in[0,1]: x=\lambda e+(1-\lambda)y,\ e\in\operatorname{ext}B(X)_+,\ y\in B(X)\right\}.$$

Proof. (a) is clear since $|x| = (\operatorname{sign} x)\, x$, $(\operatorname{sign} x)\, e \in \operatorname{ext} B\,(X)$ and $(\operatorname{sign} x)\, y \in S(X)$, whenever $e \in \operatorname{ext} B\,(X)$ and $y \in S(X)$.

(b) Let $x \in S(X)_+$. By the definitions of $\lambda(x)$ and $\lambda_+(x)$ it is obvious that $\lambda(x) \geq \lambda_+(x)$. To prove the reverse inequality notice that if $\lambda\,(x) = 1$, then $e = x \in \operatorname{ext} B\,(X)_+$, whence $\lambda_+(x) = 1$, i.e. $\lambda_+(x) = \lambda(x)$ in this case. If $\lambda\,(x) = 0$, then $\lambda_+(x) = 0$ by $\lambda(x) \geq \lambda_+(x)$. Suppose now that $\lambda(x) \in (0, 1)$. Take $\varepsilon \in (0, \lambda(x))$. Then there is $\lambda_0 > \lambda(x) - \varepsilon$ such that $x = \lambda_0 e + (1 - \lambda_0) y$ with $e \in \operatorname{ext} B\,(X)$ and $y \in B\,(X)$. Since

$$1 = \|x\|_E \leq \lambda_0 \|e\|_E + (1 - \lambda_0) \|y\|_E \leq 1,$$

we obtain that $\|y\|_E = 1$. Moreover, $y = \frac{1}{1-\lambda_0} x - \frac{\lambda_0}{1-\lambda_0} e$. Define $y_1 = \frac{1}{1-\lambda_0} x - \frac{\lambda_0}{1-\lambda_0} |e|$. Obviously $|y| \geq |y_1|$, whence $\|y_1\|_E \leq \|y\|_E = 1$. On the other hand

$$1 = \|x\|_E = \|\lambda_0 |e| + (1 - \lambda_0) y_1\|_E \leq \lambda_0 \|e\|_E + (1 - \lambda_0) \|y_1\|_E \leq 1,$$

whence $\|y_1\|_E = 1$. Therefore, $\lambda_+(x) \geq \lambda_0 > \lambda(x) - \varepsilon$, so $\lambda_+(x) \geq \lambda(x)$, which finishes the proof. $\square$

Proposition 2.2. *Let $x \in L_\Phi \setminus \{0\}$. The set $K(x) = \varnothing$ if and only if $\lim_{u\to\infty} \Phi(u)/u = A < \infty$, the function $R(u) := A|u| - \Phi(u)$ is upper bounded and one of the following conditions is satisfied:*

(a) $\sup_{u>0} R(u) = 0$,
(b) $\mu\,(\operatorname{supp} x) < \frac{1}{c}$,
(c) $\mu\,(\operatorname{supp} x) = \frac{1}{c}$ *and* $u_\Phi = \infty$,
(d) $\mu\,(\operatorname{supp} x) = \frac{1}{c}$, $u_\Phi < \infty$ *and* $\operatorname{infess}_{t\in \operatorname{supp} x} |x(t)| = 0$,
where $c = \sup_{u>0} R(u)$.

Proof. *Necessity.* Suppose that $x \in L_\Phi \setminus \{0\}$ and $K(x) = \varnothing$. By Theorem 4 in [5] we conclude that $\lim_{u\to\infty} \Phi(u)/u = A < \infty$ and the function $R(u)$ is upper bounded. Then either $\sup_{u>0} R(u) = 0$ (i.e. (a) is satisfied) or $\sup_{u>0} R(u) > 0$.

Let $c = \sup_{u>0} R(u) > 0$. If $\mu\,(\operatorname{supp} x) > \frac{1}{c}$, then there is a positive number δ such that

$$\mu\,\{B_\delta\} := \mu\,\{t \in T : |x(t)| > \delta\} > \frac{1}{c}.$$

Since $R\,(u)$ is non-decreasing, we have

$$\sup_{u>0} R(u) = \lim_{u\to\infty} R(u) = c < \infty.$$

Consequently,

$$\Psi\,(A) = \sup_{v\geq 0} \{Av - \Phi\,(v)\} = \sup_{v\geq 0} \{Av - (Av - R\,(v))\} = \sup_{v\geq 0} R\,(v) = c \tag{1}$$

and

$$\begin{aligned}\Psi\,(u) = & \sup_{v\geq 0} \{uv - \Phi\,(v)\} = \sup_{v\geq 0} \{(u - A)\, v - R\,(v)\} \\ \geq & \lim_{v\to\infty} \{(u - A)\, v - R\,(v)\} = \infty\end{aligned}$$

for any $u > A$. Further, by the fact that $\lim_{u\to\infty} \Phi(u)/u = A < \infty$, we conclude that $\lim_{u\to\infty} p(u) = A$. Hence, by the continuity of Ψ on $[0, A]$,

$$I_\Psi \left(p \circ k\left|x\right|\right) \geq I_\Psi \left(p \circ k\left|x\right|\chi_{B_\delta}\right) > I_\Psi \left(p \circ k\delta\chi_{B_\delta}\right) = \Psi\left(p \circ k\delta\right)\mu\left(B_\delta\right) > 1$$

for $k > 0$ sufficiently large. Therefore

$$k^*\left(x\right) = \inf\left\{k > 0 : I_\Psi\left(p \circ k\left|x\right|\right) \geq 1\right\} < \infty.$$

It implies that $K(x) \neq \varnothing$, a contradiction. Hence the case $\mu\left(\operatorname{supp} x\right) > \frac{1}{c}$ is excluded.

Now suppose that in the case when $\mu\left(\operatorname{supp} x\right) = \frac{1}{c}$ and $u_\Phi < \infty$ there is $\gamma > 0$ such that $\mu\{B_\gamma\} = \frac{1}{c}$. The condition $u_\Phi < \infty$ means that $p(u) = A$ for $u > u_\Phi$. Take k_0 such that $k_0\gamma > u_\Phi$. Then

$$\begin{aligned} I_\Psi\left(p \circ k_0\left|x\right|\right) &= I_\Psi\left(p \circ k_0\left|x\right|\chi_{B_\gamma}\right) = I_\Psi\left(p \circ k_0\gamma\chi_{B_\gamma}\right) \\ &= \Psi\left(p \circ k_0\gamma\right)\mu\left(B_\gamma\right) = \Psi\left(A\right)\frac{1}{c} = 1, \end{aligned}$$

whence $k_0 \in K(x)$ i.e. $K(x)$ cannot be empty, which finishes the proof of necessity.

Sufficiency. Assume that $\sup_{u>0} R(u) = c$. Let $x \in L_\Phi \diagdown \{0\}$. If $c = 0$, then $\Phi(u) = A\left|u\right|$ and

$$\begin{aligned} \left\|x\right\|_\Phi &= \inf_{k>0}\frac{1}{k}\left(1 + I_\Phi(kx)\right) = \inf_{k>0}\frac{1}{k}\left(1 + A\int_T k\left|x(t)\right|d\mu\right) \\ &= \inf_{k>0}\left(\frac{1}{k} + A\int_T \left|x(t)\right|d\mu\right) = A\int_T \left|x(t)\right|d\mu, \end{aligned}$$

so $K(x) = \varnothing$. Let $c > 0$. If $\mu\left(\operatorname{supp} x\right) < \frac{1}{c}$, then

$$I_\Psi\left(p \circ k\left|x\right|\right) \leq I_\Psi\left(A\chi_{\operatorname{supp} x}\right) = \Psi(A)\mu(\operatorname{supp} x) < 1$$

for any $k > 0$. Now, suppose that $\mu\left(\operatorname{supp} x\right) = \frac{1}{c}$. If $u_\Phi = \infty$, then $p(u) < A$ for any $u > 0$. Hence, by (1),

$$I_\Psi\left(p \circ k\left|x\right|\right) < \Psi(A)\mu(\operatorname{supp} x) = 1$$

for any $k > 0$. If $u_\Phi < \infty$ and $\operatorname{infess}_{t\in\operatorname{supp} x}\left|x(t)\right| = 0$, then $\mu\left\{B_a\right\} < \frac{1}{c}$ for any $a > 0$. Moreover $p(u) = A$ for any $u \geq u_\Phi$ and $p(u) < A$ for $u \in (0, u_\Phi)$. Consequently,

$$\begin{aligned} I_\Psi\left(p \circ k\left|x\right|\right) &= I_\Psi\left(p \circ \left(k\left|x\right|\chi_{B_{u_\Phi/k}} + k\left|x\right|\chi_{\operatorname{supp} x\setminus B_{u_\Phi/k}}\right)\right) \\ &= I_\Psi\left(p \circ \left(k\left|x\right|\chi_{B_{u_\Phi/k}}\right)\right) + I_\Psi\left(p \circ \left(k\left|x\right|\chi_{\operatorname{supp} x\setminus B_{u_\Phi/k}}\right)\right) \\ &< \Psi(A)\mu(B_{u_\Phi/k}) + \Psi(A)\mu(\operatorname{supp} x \setminus B_{u_\Phi/k}) \\ &= \Psi(A)\mu(B_{u_\Phi/k}) + \Psi(A)\mu(\operatorname{supp} x) - \Psi(A)\mu(B_{u_\Phi/k}) = 1 \end{aligned}$$

for every $k > 0$. Since under each of assumptions (b), (c) and (d), we obtain that $I_\Psi\left(p \circ k\left|x\right|\right) < 1$ for any $k > 0$, we have $k^*\left(x\right) = \infty$, whence $K(x) = \varnothing$, which completes the proof of the preposition. $\square$

Proposition 2.3. *Let Φ be an Orlicz function such that $\lim_{u\to\infty}\Phi(u)/u = A < \infty$ and let $x \in L_\Phi \diagdown \{0\}$. Then $\|x\|_\Phi = A\int_T |x(t)|\,d\mu$ if and only if either $\sup_{u>0} R(u) = 0$ or $0 < \sup_{u>0} R(u) = c < \infty$ and $\mu\,(\operatorname{supp} x) \le \frac{1}{c}$.*

Proof. Suppose that $\lim_{u\to\infty}\Phi(u)/u = A < \infty$. Then

$$\begin{aligned}\|x\|_\Phi &= \inf_{k>0}\frac{1}{k}\left(1+\int_T \Phi(kx(t))d\mu\right)\\ &= \inf_{k>0}\frac{1}{k}\left(1+\int_T Ak\,|x(t)|\,d\mu - \int_T R\,(kx(t))\,d\mu\right)\\ &= A\int_T |x(t)|\,d\mu + \inf_{k>0}\frac{1}{k}\left(1-\int_T R\,(kx(t))\,d\mu\right).\end{aligned}$$

Sufficiency. If $\sup_{u>0} R(u) = 0$, then $\Phi(u) = A\,|u|$ and

$$\|x\|_\Phi = A\int_T |x(t)|\,d\mu + \inf_{k>0}\frac{1}{k} = A\int_T |x(t)|\,d\mu.$$

Suppose that $0 < \sup_{u>0} R(u) = c < \infty$ and $\mu\,(\operatorname{supp} x) \le \frac{1}{c}$. Then

$$\int_T R\,(kx(t))\,d\mu = \int_{\operatorname{supp} x} R\,(kx(t))\,d\mu \le \sup_{u>0} R(u)\mu\,(\operatorname{supp} x) = 1,$$

whence

$$\inf_{k>0}\frac{1}{k}\left(1-\int_T R\,(kx(t))\,d\mu\right) = 0,$$

i.e. $\|x\|_\Phi = A\int_T |x(t)|\,d\mu$.

Necessity. Let $x \in L_\Phi \diagdown \{0\}$ be such that $\|x\|_\Phi = A\int |x(t)|\,d\mu$, where $\lim_{u\to\infty}\Phi(u)/u = A < \infty$. Hence

$$\inf_{k>0}\frac{1}{k}\left(1-\int_T R\,(kx(t))\,d\mu\right) = 0.$$

It implies that $\int_T R\,(kx(t))\,d\mu \le 1$ for any $k > 0$. Consequently, by the fact that $\mu(\operatorname{supp} x) > 0$, $R(u)$ is upper bounded. Notice that, by the convexity of Φ, $R(u)$ is nondecreasing. Hence

$$\sup_{u>0} R(u) = \lim_{u\to\infty} R(u) = c < \infty$$

Take $\varepsilon > 0$. Then there is $a_\varepsilon > 0$ such that

$$\mu\,(B_{a_\varepsilon}) = \mu\,\{t \in T : |x(t)| > a_\varepsilon\} > \mu(\operatorname{supp} x) - \frac{\varepsilon}{c}.$$

Hence

$$
\begin{aligned}
1 \geq \int_T R(kx(t))\, d\mu &\geq \int_{B_{a_\varepsilon}} R(ka_\varepsilon)\, d\mu = R(ka_\varepsilon)\, \mu(B_{a_\varepsilon}) \\
&> R(ka_\varepsilon)\left(\mu(\operatorname{supp} x) - \frac{\varepsilon}{c}\right)
\end{aligned}
$$

for any $k > 0$. Now, passing with $k \to \infty$, we get $c\mu(\operatorname{supp} x) - \varepsilon \leq 1$. Thus $\mu(\operatorname{supp} x) \leq \frac{1}{c}$ for $c \neq 0$ or $\mu(\operatorname{supp} x)$is arbitrary for $c = 0$. If $c = 0$, then $\Phi(u) = A\,|u|$, which completes the proof of necessity. □

Lemma 2.4. *Let $x \in L_\Phi$.*

(a) *If $u_\Phi < \infty$, $(u_\Phi, \infty) = AF(\Phi)_+$ and $\mu(\operatorname{supp} x) > \frac{1}{\Psi(p(u_\Phi))}$, then $K(x)$ is a singleton.*

(b) *If $a_\Phi > 0$ and $[0, a_\Phi) = AF(\Phi)_+$, then $K(x)$ is a singleton.*

Proof. (a) If $u_\Phi < \infty$, then $\Phi(u) = Au - c$ for $u \geq u_\Phi$, where A, c are positive constant. Hence $p(u) = A$ for $u \geq u_\Phi$ and $\Psi(A) = c$. For $x \in L_\Phi$ with $\mu(\operatorname{supp} x) > \frac{1}{c}$, by Proposition 2.2, $K(x) \neq \varnothing$. Suppose that $K(x)$ has more than one point and let $k_1, k_2 \in (k^*(x), k^{**}(x))$ be such that $k_1 < k_2$. First notice that $\mu(\{t \in T : k_1 |x(t)| > u_\Phi\}) \leq \frac{1}{c}$. If not, then denoting the considered set by B_{u_Φ/k_1} we have

$$
I_\Psi(p \circ k_1 |x|) \geq I_\Psi\left(p \circ k_1 |x|\, \chi_{B_{u_\Phi/k_1}}\right) = \mu(B_{u_\Phi/k_1})\Psi(p(u_\Phi)) > 1,
$$

i.e. $k_1 \notin K(x)$. Consequently, $\mu(\{t \in T : k_1 |x(t)| \leq u_\Phi\}) > 0$. Since p is increasing on $[0, u_\Phi]$, we have

$$
1 \leq I_\Psi(p \circ k_1 |x|) < I_\Psi(p \circ k_2 |x|) \leq 1,
$$

which is a contradiction.

(b) Suppose that $a_\Phi > 0$ and $[0, a_\Phi) = AF(\Phi)_+$. Then $p(u) = 0$ for $u \in (0, a_\Phi)$, p is increasing on (a_Φ, b_Φ) ($b_\Phi \leq \infty$) and $a_\Psi = 0$. By Proposition 2.2, $K(x) \neq \varnothing$ for any $x \in L_\Phi$. Similarly as in (a), suppose that $k_1, k_2 \in (k^*(x), k^{**}(x))$ and $k_1 < k_2$. It follows, by the definition of numbers $k^*(x)$ and $k^{**}(x)$ that $I_\Psi(p \circ k_1 |x|) = I_\Psi(p \circ k_2 |x|) = 1$. Hence $k_1 |x(t)| \leq k_2 |x(t)| \leq b_\Phi$ for a.e. $t \in T$ and

$$
\mu(B_{a_\Phi/k_1}) = \mu(\{t \in T : k_1 |x(t)| > a_\Phi\}) > 0.
$$

Consequently, $a_\Phi < k_1 |x(t)| < k_2 |x(t)|$ for a.e. $t \in B_{a_\Phi/k_1}$, whence $p(k_1 |x(t)|) < p(k_2 |x(t)|)$ for a.e. $t \in B_{a_\Phi/k_1}$. Therefore $I_\Psi(p \circ k_1 |x|) < I_\Psi(p \circ k_2 |x|)$. The obtained contradiction completes the proof. □

Theorem 2.5. *A point $x \in S(L_\Phi)$ is a λ-point of $B(L_\Phi)$ if and only if $I_\Psi(p \circ u_\Phi \chi_T) > 1$, $K(x) \neq \varnothing$ and either $a_\Phi > 0$ or $a_\Phi = 0$ and $k^{**}(x) < \infty$.*

Proof. *Necessity.* By Remark 1.2 and Proposition 2.1, without loss of generality, we can assume that $x \in S(L_\Phi)_+$. Suppose that $K(x) = \varnothing$ or $I_\Psi\left(p \circ u_\Phi \chi_T\right) \le 1$. If $I_\Psi\left(p \circ u_\Phi \chi_T\right) \le 1$, then, by Proposition 1.4, the set $\operatorname{ext} B(L_\Phi) = \varnothing$ and consequently $\lambda\left[B(L_\Phi)\right] = \varnothing$. In particular, x cannot be a λ-point of $S(L_\Phi)$.

Assume now that $K(x) = \varnothing$. By Proposition 2.2, the Orlicz space L_Φ is generated by an Orlicz function Φ such that the function $R(u) = Au - \Phi(u)$ is upper bounded, where $A = \lim_{u\to\infty} \Phi(u)/u$. It is shown in the proof of Proposition 2.2 that $\Psi(A) = \lim_{u\to\infty} R(u) = c$, $\Psi(u) = \infty$ for any $u > A$ and $p(u_\Phi) = A$. Moreover, one of the conditions (a), (b), (c) or (d) of Proposition 2.2 is satisfied.

If $\sup_{u>0} R(u) = 0$ (i.e. (a) is satisfied), then $\Phi(u) = A\,|u|$ and consequently $L_\Phi = L^1$ equipped with the norm $A\left\|\cdot\right\|_{L^1}$. It is well known that $\operatorname{ext} B(L^1) = \varnothing$. Thus x cannot be a λ-point.

Assume that $0 < \sup_{u>0} R(u) = c$. Notice that under this condition $\operatorname{ext} B(L_\Phi) \neq \varnothing$ if and only if $\mu(T) > \frac{1}{c}$. Really, if $\mu(T) \le \frac{1}{c}$, then

$$I_\Psi\left(p \circ u_\Phi \chi_T\right) = \int_T \Psi\left(p(u_\Phi)\right) d\mu = \Psi(A)\mu(T) \le 1,$$

whence, by Proposition 1.4, $\operatorname{ext} B(L_\Phi) = \varnothing$. Conversely, for $\mu(T) > \frac{1}{c}$ we have

$$I_\Psi\left(p \circ u_\Phi \chi_T\right) = \Psi(A)\mu(T) > 1.$$

By Proposition 1.4 we conclude that $\operatorname{ext} B(L_\Phi) \neq \varnothing$. Therefore it makes sense to consider only the case when one of the conditions (b), (c) or (d) in Proposition 2.2 is satisfied and $\mu(T) > \frac{1}{c}$.

Let $e \in \operatorname{ext} B\left(L_\Phi\right) \cap B\left(L_\Phi\right)_+$. By Proposition 1.3 the set $K(e) = \{k_0\}$ and $k_0 e(t) \in \mathrm{SC}(\Phi)$ for μ-a.e. $t \in T$. By the definition of $K(e)$, we conclude that $I_\Psi\left(p \circ ke\right) > 1$ for any $k > k_0$. Hence

$$\mu(\operatorname{supp} e) = \mu(\operatorname{supp} ke) = \frac{1}{c}\Psi(A)\mu(\operatorname{supp} ke) \ge \frac{1}{c} I_\Psi\left(p \circ k\,|e|\right) > \frac{1}{c}.$$

Now, we will show that if one of conditions (b) or (c) in Proposition 2.2 is satisfied, then $\left\|e\chi_{\operatorname{supp} x}\right\|_\Phi < 1$. Suppose that the condition (b) is satisfied, i.e. $\mu\left(\operatorname{supp} x\right) < \frac{1}{c}$. Then

$$\mu(\operatorname{supp}\left(e\chi_{\operatorname{supp} x}\right)) \le \mu\left(\operatorname{supp} x\right) < \frac{1}{c},$$

whence, by Proposition 2.2, $K\left(e\chi_{\operatorname{supp} x}\right) = \varnothing$.

Let us assume (c), i.e. $\mu\left(\operatorname{supp} x\right) = \frac{1}{c}$ and $u_\Phi = \infty$. Then $\mu(\operatorname{supp}\left(e\chi_{\operatorname{supp} x}\right)) \le \frac{1}{c}$. By Proposition 2.2, $K\left(e\chi_{\operatorname{supp} x}\right) = \varnothing$. Consequently, for both cases, we have

$$\begin{aligned}\left\|e\chi_{\operatorname{supp} x}\right\|_\Phi &= \inf_{k>0} \frac{1}{k}\left(1 + I_\Phi(ke\chi_{\operatorname{supp} x})\right) < \frac{1}{k_0}\left(1 + I_\Phi(k_0 e\chi_{\operatorname{supp} x})\right) \\ &\le \frac{1}{k_0}\left(1 + I_\Phi(k_0 e\chi_{\operatorname{supp} x}) + I_\Phi(k_0 e\chi_{\operatorname{supp} e \setminus \operatorname{supp} x})\right) = \|e\|_\Phi = 1.\end{aligned}$$

We will prove that if (d) from Proposition 2.2 is satisfied, then either $\|e\chi_{\operatorname{supp} x}\|_\Phi < 1$ or $e\chi_{T\setminus \operatorname{supp} x} = \frac{a_\Phi}{k_0}\chi_{T\setminus \operatorname{supp} x} > 0$.

Suppose that $\mu(\operatorname{supp} x) = \frac{1}{c}$, $u_\Phi < \infty$ and $\operatorname{infess}_{t\in \operatorname{supp} x} x(t) = 0$. Then

$$\mu(B_a) = \mu(\{t \in T : x(t) > a\}) < \frac{1}{c}$$

for any $a > 0$. Since

$$1 = \|e\|_\Phi = \frac{1}{k_0}(1 + I_\Phi(k_0 e)) = \frac{1}{k_0}\left[1 + I_\Phi(k_0 e\chi_{\operatorname{supp} x}) + I_\Phi(k_0 e\chi_{\operatorname{supp} e\setminus \operatorname{supp} x})\right],$$

by the definition of Orlicz norm, we have

$$\begin{aligned} \|e\chi_{\operatorname{supp} x}\|_\Phi &= \inf_{k>0}\frac{1}{k}(1 + I_\Phi(ke\chi_{\operatorname{supp} x})) \\ &\le \frac{1}{k_0}(1 + I_\Phi(k_0 e\chi_{\operatorname{supp} x})) = 1 - \frac{1}{k_0} I_\Phi(k_0 e\chi_{\operatorname{supp} e\setminus \operatorname{supp} x}). \end{aligned} \tag{2}$$

Since $\mu(\operatorname{supp} e) > \frac{1}{c}$, we have $\mu(\operatorname{supp} e \setminus \operatorname{supp} x) > 0$. Obviously, if $a_\Phi = 0$, then

$$I_\Phi(k_0 e\chi_{\operatorname{supp} e\setminus \operatorname{supp} x}) > 0.$$

Hence, by the inequality (2), $\|e\chi_{\operatorname{supp} x}\|_\Phi < 1$. Suppose for the moment that $a_\Phi > 0$. Then $k_0 e(t) \ge a_\Phi = \inf \mathrm{SC}(\Phi)$ for μ-a.e. $t \in T$. Consequently, $T = \operatorname{supp} e$. If there exists a set $D \subset T\setminus \operatorname{supp} x$ of positive measure such that $k_0 e(t) > a_\Phi$ for μ-a.e. $t \in D$, then $I_\Phi(k_0 e\chi_{\operatorname{supp} e\setminus \operatorname{supp} x}) > 0$. Therefore again $\|e\chi_{\operatorname{supp} x}\|_\Phi < 1$. The case when $k_0 e(t) = a_\Phi$ for μ-a.e. $t \in T\setminus \operatorname{supp} x$ we will consider separately in the last part of the proof of necessity.

Now, suppose that $K(x) \ne \varnothing$, $a_\Phi = 0$ and $k^{**}(x) = \infty$. Then $k^*(x) < \infty$ and it follows that $I_\Psi(p \circ kx) = 1$ for any $k > k^*(x)$. Hence $\operatorname{infess}_{t\in \operatorname{supp} x} x(t) > 0$ and $p(u)$ is constant for every $u > k^*(x) \operatorname{infess}_{t\in \operatorname{supp} x} x(t)$. Consequently,

$$u_\Phi = \sup\{u \ge a_\Phi : u \in \mathrm{SC}(\Phi)\} = k^*(x) \operatorname*{infess}_{t\in \operatorname{supp} x} x(t)$$

and $\Phi(u) = Au - c$ for $u \ge u_\Phi$, where A and c are positive. Obviously, $R(u) = c$ for $u \ge u_\Phi$ and for any $k > k^*(x)$ we have

$$1 = I_\Psi(p \circ kx) = \Psi(A)\mu(\operatorname{supp} x) = c\mu(\operatorname{supp} x),$$

whence $\mu(\operatorname{supp} x) = \frac{1}{c}$. Since $\mu(\operatorname{supp} e) > \frac{1}{c}$ for any $e \in \operatorname{ext} B(X)$ and the space L_Φ is strictly monotone whenever $a_\Phi = 0$, we have again that

$$\|e\chi_{\operatorname{supp} x}\|_\Phi < \|e\|_\Phi = 1.$$

Let $y \in B(L_\Phi)$ be such that $x = \lambda e + (1-\lambda)y$ for $\lambda \in (0,1)$. Then

$$y = \frac{x - \lambda e}{1-\lambda} = \frac{\lambda}{1-\lambda}(x - e\chi_{\operatorname{supp} x}) + x - \frac{\lambda}{1-\lambda} e\chi_{\operatorname{supp} e\setminus \operatorname{supp} x}. \tag{3}$$

Hence, in all considered cases, except the one when $e\chi_{T\setminus \operatorname{supp} x} = \frac{a_\Phi}{k_0}\chi_{T\setminus \operatorname{supp} x} > 0$, we get

$$\begin{aligned}
\|y\|_\Phi &= \left\| \frac{\lambda}{1-\lambda}\left(x - e\chi_{\operatorname{supp} x}\right) + x - \frac{\lambda}{1-\lambda} e\chi_{\operatorname{supp} e\setminus \operatorname{supp} x} \right\|_\Phi \\
&\geq \left\| \frac{\lambda}{1-\lambda}\left(x - e\chi_{\operatorname{supp} x}\right) + x \right\|_\Phi \\
&= \left\| \frac{1}{1-\lambda} x - \frac{\lambda}{1-\lambda} e\chi_{\operatorname{supp} x} \right\|_\Phi \\
&\geq \frac{1}{1-\lambda}\|x\|_\Phi - \frac{\lambda}{1-\lambda}\|e\chi_{\operatorname{supp} x}\|_\Phi \\
&> \frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda} = 1,
\end{aligned}$$

which contradicts to the fact that $y \in B(L_\Phi)$. Thus x cannot be a λ-point.

Finally, suppose that $\mu(\operatorname{supp} x) = \frac{1}{c}$, $u_\Phi < \infty$, $\operatorname{infess}_{t\in \operatorname{supp} x} x(t) = 0$ and e is an extreme point of $B(L_\Phi)$ such that $k_0 e(t) = a_\Phi > 0$ for μ-a.e. $t \in T \setminus \operatorname{supp} x$. Then $\mu\{B_a\} < \frac{1}{c}$ for any $a > 0$. Choose a measurable set

$$D_\lambda \subset \operatorname{supp} x \setminus B_{\lambda a_\Phi / k_0} = \left\{ t \in \operatorname{supp} x : |x(t)| < \frac{\lambda a_\Phi}{k_0} \right\}$$

such that $0 < \mu(D_\lambda) < \mu(T) - \frac{1}{c}$. Let $E_\lambda \subset T \setminus \operatorname{supp} x$ be such that $\mu(E_\lambda) = \mu(D_\lambda)$. By Proposition 2.3, we have

$$\begin{aligned}
1 = \|x\|_\Phi = \int_T x(t)dt &= \int_{\operatorname{supp} x\setminus D_\lambda} x(t)dt + \int_{D_\lambda} x(t)dt \\
&< \int_{\operatorname{supp} x\setminus D_\lambda} x(t)dt + \frac{\lambda a_\Phi}{k_0}\mu(D_\lambda) := \eta_\lambda.
\end{aligned}$$

Take y as above. Then

$$\begin{aligned}
\|y\|_\Phi &= \left\| \frac{1}{1-\lambda} x - \frac{\lambda}{1-\lambda} e\chi_{\operatorname{supp} x} - \frac{\lambda a_\Phi}{(1-\lambda)k_0}\chi_{T\setminus \operatorname{supp} x} \right\|_\Phi \\
&\geq \left\| \frac{1}{1-\lambda} x\chi_{\operatorname{supp} x\setminus D_\lambda} - \frac{\lambda}{1-\lambda} e\chi_{\operatorname{supp} x\setminus D_\lambda} - \frac{\lambda a_\Phi}{(1-\lambda)k_0}\chi_{E_\lambda} \right\|_\Phi \\
&\geq \left\| \frac{1}{1-\lambda} x\chi_{\operatorname{supp} x\setminus D_\lambda} - \frac{\lambda a_\Phi}{(1-\lambda)k_0}\chi_{E_\lambda} \right\|_\Phi - \left\| \frac{\lambda}{1-\lambda} e\chi_{\operatorname{supp} x\setminus D_\lambda} \right\|_\Phi \\
&= \frac{1}{1-\lambda}\left(\int_{\operatorname{supp} x\setminus D_\lambda} x(t)dt + \frac{\lambda a_\Phi}{k_0}\mu(E_\lambda) \right) - \frac{\lambda}{1-\lambda}\|e\chi_{\operatorname{supp} x\setminus D_\lambda}\|_\Phi \\
&\geq \frac{\eta_\lambda}{1-\lambda} - \frac{\lambda}{1-\lambda} = \frac{\eta_\lambda - \lambda}{1-\lambda} > 1,
\end{aligned}$$

i.e. $y \notin B(L_\Phi)$, so x is not a λ-point of $S(L_\Phi)$.

Sufficiency. Suppose that $x \in B(L_\Phi)$, $K(x) \neq \varnothing$ and $I_\Psi(p \circ u_\Phi \chi_T) > 1$. By Proposition 1.4, $\operatorname{ext} B(L_\Phi) \neq \varnothing$. If $x \in \operatorname{ext} B(L_\Phi)$, then $\lambda(x) = 1$, so x is a λ-point of $B(L_\Phi)$. Hence, without loss of generality, we may assume that $x \in S(L_\Phi) \setminus \operatorname{ext} B(L_\Phi)$ and $x(t) \geq 0$ for all $t \in T$. We have

$$AF(\Phi)_+ = [0, a_\Phi) \cup \left(\bigcup_{i=1}^{r} (a_i, b_i) \right) \cup (u_\Phi, \infty),$$

where $r \in \mathbb{N}$ or $r = \infty$ and $a_i, b_i \in \mathrm{SC}(\Phi)$. If $a_\Phi = 0$ ($u_\Phi = \infty$), then we assume that $[0, a_\Phi) = \varnothing$ ($(u_\Phi, \infty) = \varnothing$), respectively. Consider two possible cases.

Case 1. Suppose that $K(x)$ is a singleton, i.e. $K(x) = \{k_0\}$. Since x is not an extreme point of $B(L_\Phi)$, by Proposition 1.3, $\mu\{t \in T : k_0 x(t) \in AF(\Phi)_+\} > 0$. Obviously, $k_0 x(t) \leq b_\Phi$ for μ-a.e. $t \in T$. For any $1 \leq i \leq r$ denote

$$T_1^i = \left\{ t \in T : a_i < k_0 x(t) \leq \frac{a_i + b_i}{2} \right\},$$

$$T_2^i = \left\{ t \in T : \frac{a_i + b_i}{2} < k_0 x(t) < b_i \right\}.$$

Divide the set T into five disjoint subsets T_1, T_2, T_3, T_4, T_5 such that $T_1 = \bigcup_{i=1}^r T_1^i$, $T_2 = \bigcup_{i=1}^r T_2^i$,

$$T_3 = \{t \in T : k_0 x(t) < a_\Phi\},$$
$$T_4 = \{t \in T : k_0 x(t) > u_\Phi\}$$

and $T_5 = T \setminus \bigcup_{j=1}^4 T_j$. If $a_\Phi = 0$ ($u_\Phi = \infty$), then $T_3 = \varnothing$ ($T_4 = \varnothing$), respectively. Basing on the function x, we will construct $\widetilde{e} \in \operatorname{ext} B(X)_+$. To this end define

$$e(t) = \begin{cases} \frac{1}{k_0} a_i & \text{if } t \in T_1^i \text{ and } 1 \leq i \leq r, \\ \frac{1}{k_0} b_i & \text{if } t \in T_2^i \text{ and } 1 \leq i \leq r, \\ \frac{1}{k_0} a_\Phi & \text{if } t \in T_3, \\ \frac{1}{k_0} u_\Phi & \text{if } t \in T_4, \\ x(t) & \text{if } t \in T_5. \end{cases}$$

The first claim is that $\mu(\operatorname{supp} e) > 0$. Notice that if $\mu(\operatorname{supp} e) = 0$, then $a_\Phi > 0$ and $0 \leq k_0 x(t) \leq \frac{1}{2} a_\Phi$ for μ-a.e. $t \in T$. Take $k_1 = \frac{3}{2} k_0$. Since $0 \leq k_1 x(t) \leq \frac{3}{4} a_\Phi$, by the linearity of Φ on $(0, a_\Phi)$, we have

$$\frac{1}{k_1}(1 + I_\Phi(k_1 x)) = \frac{1}{k_1} + I_\Phi(x) < \frac{1}{k_0} + I_\Phi(x) = \frac{1}{k_0}(1 + I_\Phi(k_0 x),$$

whence $k_0 \notin K(x)$. A contradiction.

The next claim is that $K(e) \neq \varnothing$. To show this, suppose that $K(e) = \varnothing$. By Proposition 2.2, it is enough to consider an Orlicz function such that

$\lim_{u\to\infty}\Phi(u)/u = A < \infty$, and $\sup_{u>0} R(u) = c < \infty$. Since $K(x)$ is a singleton, we conclude that $\mu(\operatorname{supp} x) > \frac{1}{c}$. If $a_\Phi > 0$ or $a_\Phi = 0$ and $(0,\delta)\cap \mathrm{SC}(\Phi) \neq \varnothing$ for any $\delta > 0$, then $\operatorname{supp} x \subset \operatorname{supp} e$. Hence $\mu(\operatorname{supp} e) > \frac{1}{c}$ and, by Proposition 2.2, $K(e) \neq \varnothing$. Therefore, the set $K(e)$ can be empty only in the case when $a_\Phi = 0$ and there exists $b_1 > 0$ such that $(0, b_1] \cap \mathrm{SC}(\Phi) = \{b_1\}$. Then there is $B < A$ such that

$$\Phi(u) = Bu \text{ for } u \in [0, b_1],$$

$$p(u) = B \text{ for } u \in [0, b_1), \quad p(u) > B \text{ for } u > b_1, \quad \lim_{u\to\infty} p(u) = A,$$

$$\Psi(u) = 0 \text{ for } u \in [0, B], \quad \Psi(A) = c \quad \text{and} \quad \Psi(u) = \infty \text{ for } u > A.$$

To get a contradiction, we will show that $\mu(\operatorname{supp} e) > \frac{1}{c}$. By the definition of e, it is easy to see that

$$\operatorname{supp} e = \operatorname{supp} x \setminus T_1^1,$$

where $T_1^1 = \left\{t \in T : 0 < k_0 x(t) \le \frac{1}{2} b_1\right\}$. Let $k_1 = \frac{3}{2} k_0$. Then

$$p\left(k_0 x(t)\right) = p\left(k_1 x(t)\right) = B$$

for any $t \in T_1^1$. Since $K(x) = \{k_0\}$, by the definitions of $k^*(x)$ and $k^{**}(x)$, it follows that $I_\Psi\left(p \circ k_1 x\right) > 1$. Therefore

$$\begin{aligned} c\mu(\operatorname{supp} e) &= \Psi(A)\mu(\operatorname{supp} x \setminus T_1^1) + \Psi(B)\mu(T_1^1) \\ &\ge \int_T \Psi\left(p(k_1 x(t))\right) d\mu > 1, \end{aligned}$$

so $\mu(\operatorname{supp} e) > \frac{1}{c}$. Hence, again by Proposition 2.2, $K(x) \neq \varnothing$, as required for the claim.

Choose $\varepsilon \in (0,1)$. Clearly, if $t \in T \setminus (T_1 \cup T_4)$, then $e(t) \ge x(t)$ and consequently

$$I_\Psi\left[p\left((1+\varepsilon)k_0 e\right)\chi_{T\setminus(T_1\cup T_4)}\right] \ge I_\Psi\left[p\left(\left(1+\frac{\varepsilon}{4}\right)k_0 x\right)\chi_{T\setminus(T_1\cup T_4)}\right]. \tag{4}$$

If $t \in T_4$, then

$$I_\Psi\left[p\left((1+\varepsilon)k_0 e\right)\chi_{T_4}\right] = I_\Psi\left[p\left(\left(1+\frac{\varepsilon}{4}\right)k_0 x\right)\chi_{T_4}\right]. \tag{5}$$

Now, consider $t \in T_1$. For any fixed i, either

$$(1+\varepsilon)\,a_i < b_i \text{ or } (1+\varepsilon)\,a_i \ge b_i.$$

If the first inequality is satisfied, then

$$\begin{aligned} \left(1+\frac{\varepsilon}{4}\right)k_0 x(t) &\le \left(1+\frac{\varepsilon}{4}\right)\frac{a_i+b_i}{2} \\ &< \frac{1}{2}\left(1+\frac{\varepsilon}{4}\right)\left(\frac{b_i}{1+\varepsilon}+b_i\right) = \frac{(4+\varepsilon)(2+\varepsilon)}{8(1+\varepsilon)} b_i < b_i. \end{aligned}$$

Suppose that $(1+\varepsilon)\,a_i \geq b_i$. We have

$$\begin{aligned}\left(1+\frac{\varepsilon}{4}\right) k_0 x(t) \leq\ & \left(1+\frac{\varepsilon}{4}\right) \frac{a_i+b_i}{2} \\ \leq\ & \frac{(4+\varepsilon)\,(2+\varepsilon)}{8} a_i < (1+\varepsilon)\,a_i,\end{aligned}$$

Since p is constant on each (a_i, b_i), we conclude that

$$p\left((1+\varepsilon)\,k_0 e\right) \chi_{T_1} \geq p\left(\left(1+\frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T_1}. \tag{6}$$

in both considered cases. Combining (4) with (6) and (5), we get

$$I_\Psi\left[p\left((1+\varepsilon)\,k_0 e\right)\right] \geq I_\Psi\left[p\left(\left(1+\frac{\varepsilon}{4}\right) k_0 x\right)\right] > 1. \tag{7}$$

Hence, $k_0 \geq k^{**}(e)$.

Analogously, by $e(t) \leq x(t)$ for $t \in T \setminus (T_2 \cup T_3)$, we have

$$p\left((1-\varepsilon)\,k_0 e\right) \chi_{T\setminus(T_2\cup T_3)} \leq p\left(\left(1-\frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T\setminus(T_2\cup T_3)}. \tag{8}$$

Moreover, it is obvious that

$$p\left((1-\varepsilon)\,k_0 e\right) \chi_{T_3} = p\left(\left(1-\frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T_3} = 0. \tag{9}$$

Now, assume that $t \in T_2$. Then either

$$(1-\varepsilon)\,b_i > a_i \quad \text{or} \quad (1-\varepsilon)\,b_i \leq a_i.$$

If $(1-\varepsilon)\,b_i > a_i$, then

$$\begin{aligned}\left(1-\frac{\varepsilon}{4}\right) k_0 x(t) \geq\ & \left(1-\frac{\varepsilon}{4}\right) \frac{a_i+b_i}{2} \\ >\ & \frac{1}{2}\left(1-\frac{\varepsilon}{4}\right)\left(\frac{a_i}{1-\varepsilon} + a_i\right) = \frac{(4-\varepsilon)\,(2-\varepsilon)}{8\,(1-\varepsilon)} a_i > a_i.\end{aligned}$$

When the second inequality holds, we get

$$\begin{aligned}\left(1-\frac{\varepsilon}{4}\right) k_0 x(t) \geq\ & \left(1-\frac{\varepsilon}{4}\right) \frac{a_i+b_i}{2} \\ \geq\ & \frac{(4-\varepsilon)\,(2-\varepsilon)}{8} b_i > (1-\varepsilon)\,b_i.\end{aligned}$$

Therefore, taking into account both cases, we obtain

$$p\left((1-\varepsilon)\,k_0 e\right) \chi_{T_2} \leq p\left(\left(1-\frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T_2}. \tag{10}$$

The inequalities (8), (9) and (10) imply that

$$I_\Psi\left[p\left(\left(1-\varepsilon\right)k_0 e\right)\right] \le I_\Psi\left[p\left(\left(1-\frac{\varepsilon}{4}\right)k_0 x\right)\right] < 1, \tag{11}$$

whence $k_0 \le k^*(e)$. Thus, by the definition of the set $K(e)$, we conclude that $K(e) = \{k_0\}$. Define $\widetilde{e} = \frac{e}{\|e\|_\Phi}$. Then

$$K(\widetilde{e}) = K\left(\frac{e}{\|e\|_\Phi}\right) = \{\|e\|_\Phi\, k_0\} = \{\widetilde{k}_0\}.$$

Moreover, $\widetilde{k}_0\widetilde{e}(t) = k_0 e(t) \in \mathrm{SC}(\Phi)$ for all $t \in T$. By Proposition 1.3, $\widetilde{e}$ is an extreme point of $B\left(L_\Phi\right)$.

Let $y = 2x - e$ and $\widetilde{y} = \frac{y}{\|y\|_\Phi}$. Then

$$\begin{gathered} e\chi_{T_5} = y\chi_{T_5} = x\chi_{T_5}, \\ a_i\chi_{T_1} = k_0 e\chi_{T_1^i} < k_0 x\chi_{T_1^i} < k_0 y\chi_{T_1^i} \le b_i\chi_{T_1^i},\ i = 1, 2, ..., r, \\ a_i\chi_{T_2} < k_0 y\chi_{T_2} < k_0 x\chi_{T_2} < k_0 e\chi_{T_2} = b_i\chi_{T_2}, \\ k_0\left|y\right|\chi_{T_3} < a_\Phi\chi_{T_3} = k_0 e\chi_{T_3}, \end{gathered}$$

and

$$u_\Phi\chi_{T_4} = k_0 e\chi_{T_4} < k_0 y\chi_{T_4}.$$

Since $k_0 e$ differs from $k_0 y$ only on $AF(\Phi)$, we have

$$\Phi\left(\frac{k_0 e(t) + k_0 y(t)}{2}\right) = \frac{1}{2}\Phi\left(k_0 e(t)\right) + \frac{1}{2}\Phi\left(k_0 y(t)\right)$$

for all $t \in T$. Hence

$$\begin{aligned} \frac{1}{2}\left\|e\right\|_\Phi + \frac{1}{2}\left\|y\right\|_\Phi \ge \left\|\frac{e+y}{2}\right\|_\Phi &= \left\|x\right\|_\Phi = \frac{1}{k_0}\left(1 + I_\Phi(k_0 x)\right) \\ &= \frac{1}{k_0}\left(1 + I_\Phi\left(k_0\frac{e+y}{2}\right)\right) \\ &= \frac{1}{2k_0}\left(1 + I_\Phi\left(k_0 e\right)\right) + \frac{1}{2k_0}\left(1 + I_\Phi\left(k_0 y\right)\right) \\ &= \frac{1}{2}\left\|e\right\|_\Phi + \frac{1}{2k_0}\left(1 + I_\Phi\left(k_0 y\right)\right) \end{aligned}$$

and consequently

$$\frac{1}{k_0}\left(1 + I_\Phi\left(k_0 y\right)\right) \le \left\|y\right\|_\Phi = \inf_{k>0}\frac{1}{k}\left(1 + I_\Phi\left(k y\right)\right).$$

Therefore

$$\left\|y\right\|_\Phi = \frac{1}{k_0}\left(1 + I_\Phi\left(k_0 y\right)\right),$$

and $k_0 \in K(y)$. Moreover,

$$1 = \|x\|_\Phi = \frac{1}{2}\|e\|_\Phi + \frac{1}{2}\|y\|_\Phi .$$

Taking $\lambda = \frac{1}{2}\|e\|_\Phi$, we have

$$\lambda\widetilde{e} + (1-\lambda)\widetilde{y} = \frac{1}{2}\|e\|_\Phi \frac{e}{\|e\|_\Phi} + \left(1 - \frac{1}{2}\|e\|_\Phi\right)\frac{y}{\|y\|_\Phi} = \frac{1}{2}e + \frac{1}{2}y = x,$$

whence $\lambda(x) \geq \frac{1}{2}\|e\|_\Phi > 0$.

Case 2. Suppose that $K(x)$ contains more than one point.

Subcase 2.1. Assume that $a_\Phi > 0$, $k^*(x) = k_0$ and $k^{**}(x) = \infty$, i.e. $K(x) = [k_0, \infty)$. Then, by the definitions of $k^*(x)$ and $k^{**}(x)$, we have $I_\Psi(p \circ kx) = 1$ for any $k > k_0$. It implies that

$$p(u) = 0 \text{ for } u \in [0, a_\Phi), \qquad 0 < p(u) < A \text{ for } u \in (a_\Phi, u_\Phi)$$
$$\text{and} \quad p(u) = A \text{ for } u \geq u_\Phi,$$

where $A > 0$ and $u_\Phi := k_0 \operatorname{infess}_{t \in \operatorname{supp} x} x(t) > 0$. Moreover, it is clear that there is $c > 0$ such that $\Phi(u) = Au - c$ for $u \geq u_\Phi$, $\mu(\operatorname{supp} x) = \frac{1}{c}$ and $\mu(T) > \frac{1}{c}$. Define

$$e = \frac{1}{k_0}\left(a_\Phi \chi_{T \setminus \operatorname{supp} x} + u_\Phi \chi_{\operatorname{supp} x}\right).$$

Take $\varepsilon > 0$. Since $a_\Phi > 0$ implies $a_\Psi = 0$, we have

$$\begin{aligned} I_\Psi(p \circ (1+\varepsilon) k_0 e) &= I_\Psi(p \circ (1+\varepsilon) u_\Phi \chi_{\operatorname{supp} x}) + I_\Psi(p \circ (1+\varepsilon) a_\Phi \chi_{T \operatorname{supp} x}) \\ &> \Psi(A)\mu(\operatorname{supp} x) = 1 \end{aligned}$$

and

$$\begin{aligned} I_\Psi(p \circ (1-\varepsilon) k_0 e) &= I_\Psi(p \circ (1-\varepsilon) u_\Phi \chi_{\operatorname{supp} x}) + I_\Psi(p \circ (1-\varepsilon) a_\Phi \chi_{T \operatorname{supp} x}) \\ &= \Psi(p((1-\varepsilon) u_\Phi))\mu(\operatorname{supp} x) < \Psi(A)\mu(\operatorname{supp} x) = 1. \end{aligned}$$

Therefore $K(e) = \{k_0\}$. It also follows that

$$\begin{aligned} \|e\|_\Phi &= \frac{1}{k_0}(1 + I_\Phi(k_0 e)) = \frac{1}{k_0}(1 + I_\Phi(u_\Phi \chi_{\operatorname{supp} x})) \\ &\leq \frac{1}{k_0}(1 + I_\Phi(k_0 x)) = \|x\|_\Phi = 1 \end{aligned}$$

Hence, similarly as in the Case 1, $\widetilde{e} = \frac{e}{\|e\|_\Phi} \in \operatorname{ext} B(L_\Phi)$, $K(\widetilde{e}) = \left\{\widetilde{k_0}\right\}$. Taking $y = 2x - e$ and $\widetilde{y} = \frac{y}{\|y\|_\Phi}$, notice that

$$y\chi_{T \setminus \operatorname{supp} x} = -\frac{1}{k_0} a_\Phi \chi_{T \setminus \operatorname{supp} x} \quad \text{and} \quad k_0 y \chi_{\operatorname{supp} x} \geq k_0 x \geq u_\Phi \chi_{\operatorname{supp} x}.$$

Hence

$$\Phi\left(\frac{k_0 e(t)+k_0 y(t)}{2}\right)=\frac{1}{2}\Phi\left(k_0 e(t)\right)+\frac{1}{2}\Phi\left(k_0 y(t)\right)$$

for all $t\in T$. Repeating the same arguments as in the Case 1, it implies that $k_0\in K(y)$ and

$$1=\|x\|_\Phi=\frac{1}{2}\|e\|_\Phi+\frac{1}{2}\|y\|_\Phi\,.$$

Taking $\lambda=\frac{1}{2}\|e\|_\Phi$, we get $x=\lambda\widetilde{e}+(1-\lambda)\widetilde{y}$, i.e. x is a λ-point.

Subcase 2.2. Suppose that $K(x)=[k^*(x),k^{**}(x)]$ with $k^*(x)<k^{**}(x)<\infty$. By Lemma 2.4, without loss of generality, we can assume that if $u_\Phi<\infty$ $(a_\Phi>0)$, then $AF(\Phi)_+\setminus(u_\Phi,\infty)\neq\varnothing$ $(AF(\Phi)_+\setminus(0,a_\Phi)\neq\varnothing)$, respectively. This means that if $K(x)$ is an interval of positive measure, then the set $AF(\Phi)_+\setminus(u_\Phi,\infty)$ (resp. $AF(\Phi)_+\setminus(0,a_\Phi)$) contains at least one interval. Now, for any $k\in K(x)$ the set T can be divided into five disjoint subsets $T_j(k)$ $(j=1,2,3,4,5)$ such that $T_j(k)=\bigcup_{i=1}^r T_j^i(k)$ $(j=1,2)$, where $T_j^i(k)$ $(j=1,2;\ i=1,2,...,r)$, $T_j(k)$ $(j=3,4,5)$ are defined in the same way as sets T_j^i $(j=1,2;\ i=1,2,...,r)$, T_j $(j=3,4,5)$ in Case 1 by replacing k_0 with k. Now, choose $k_0\in K(x)$ such that neither $T_1(k_0)\cup T_4(k_0)$ nor $T_2(k_0)\cup T_3(k_0)$ is a null set. Define $e(t)$ in the same way as in Case 1. Fix $\varepsilon>0$. By the definitions of $k^*(x)$ and $k^{**}(x)$, we have that

$$I_\Psi\left[p\left(\left(1+\frac{\varepsilon}{4}\right)k_0x\right)\right]\geq 1\quad\text{and}\quad I_\Psi\left[p\left(\left(1-\frac{\varepsilon}{4}\right)k_0x\right)\right]\leq 1.$$

In contrast to the situation in Case 1, the both inequalities need not be strict. Repeating the same arguments as in the proof of Case 1, we can show that

$$p\left((1+\varepsilon)k_0e\right)\geq p\left(\left(1+\frac{\varepsilon}{4}\right)k_0x\right)$$

and

$$p\left((1-\varepsilon)k_0e\right)\leq p\left(\left(1-\frac{\varepsilon}{4}\right)k_0x\right),$$

where k_0 is chosen in such a way that at least one of the sets $T_2(k_0)$, $T_3(k_0)$ is of positive measure. If $\mu\left(T_2(k_0)\right)>0$, then there exists i_0 such that $e(t)=\frac{1}{k_0}b_{i_0}$ for $t\in T_2^{i_0}(k_0)$. It follows that for ε small enough there is a set $S_2^{i_0}\subset T_2^{i_0}(k_0)$ with $\mu(S_2^{i_0})>0$ such that

$$p\left((1+\varepsilon)b_{i_0}\right)\chi_{S_2^{i_0}}>p\left(\left(1+\frac{\varepsilon}{4}\right)k_0x\right)\chi_{S_2^{i_0}}.$$

Similarly, if $\mu\left(T_3(k_0)\right)>0$, then

$$p\left((1+\varepsilon)b_{i_0}\right)\chi_{S_3}>p\left(\left(1+\frac{\varepsilon}{4}\right)k_0x\right)\chi_{S_3},$$

where $S_3\subset T_3(k_0)$ and $\mu(S_3)>0$. Consequently,

$$I_\Psi\left[p\left((1+\varepsilon)k_0e\right)\right]>I_\Psi\left[p\left(\left(1+\frac{\varepsilon}{4}\right)k_0x\right)\right]\geq 1.$$

Since $\mu\left(T_1(k_0) \cup T_4(k_0)\right) > 0$, in a completely analogous way, we can show that

$$I_\Psi\left[p\left((1-\varepsilon)\,k_0 e\right)\right] < I_\Psi\left[p\left(\left(1-\frac{\varepsilon}{4}\right)k_0 x\right)\right] \leq 1.$$

Therefore $K(e) = \{k_0\}$. Repeating the same arguments as in Case 1, we conclude that x is a λ-point. □

3. Corollaries

Theorem 2.5 has several important consequences for the λ-property in Orlicz spaces

Corollary 3.1. *The following are equivalent:*

(a) *The space L_Φ has the λ-property;*

(b) *Either* $\lim_{u\to\infty}\frac{\Phi(u)}{u} = \infty$ *or* $\lim_{u\to\infty}\frac{\Phi(u)}{u} < \infty$ *and* $\sup_u R(u) = \infty$*;*

(c) *The space L_Φ does not contain an order isometric copy of l^1.*

Proof. The equivalence of (a) and (b) follows immediately from Theorem 2.5 and the definition of the λ-property. The equivalence of (b) and (c) is a consequence of Theorem 2.5 and Theorem 5 in [5]. □

Now, we can apply Theorem 2.5 to the classical interpolation space $L^1 + L^\infty$ equipped with the norm

$$\|x\|_{L^1+L^\infty} = \inf\left\{\|y\|_1 + \|z\|_\infty : y + z = x,\ y \in L^1,\ z \in L^\infty\right\}$$

(see [3], [15]).

Corollary 3.2. *A point $x \in S(L^1 + L^\infty)$ is a λ-point if and only if $\mu(T) > 1$ and either $\mu(\operatorname{supp} x) > 1$ or $\mu(\operatorname{supp} x) = 1$ and* $\operatorname{infess}_{t\in\operatorname{supp} x}|x(t)| > 0$.

Proof. It is known (see [12]) that the space $L^1 + L^\infty$ is an Orlicz space generated by the Orlicz function Φ defined by the formula $\Phi(u) = \max\{0, |u| - 1\}$. Clearly $a_\Phi = u_\Phi = 1$. Moreover, $\|\cdot\|_{L^1+L^\infty} = \|\cdot\|_\Phi$. It follows that $R(u) = \min\{u, 1\}$ for any $u > 0$.

Necessity. Suppose that x is a λ-point. By Theorem 2.5, $I_\Psi\left(p \circ u_\Phi \chi_T\right) > 1$ and $K(x) \neq \varnothing$. Since

$$p(u) = \begin{cases} 0 & \text{for } u \in [0,1) \\ 1 & \text{for } u \geq 1 \end{cases} \quad \text{and} \quad \Psi(u) = \begin{cases} |u| & \text{for } u \in [-1,1] \\ \infty & \text{for } |u| > 1, \end{cases}$$

we have

$$\mu(T) = \int_T \Psi\left(p(1)\right) d\mu = I_\Psi\left(p \circ u_\Phi \chi_T\right) > 1.$$

Moreover, by Proposition 2.2, it follows that either $\mu(\operatorname{supp} x) > 1$ or $\mu(\operatorname{supp} x) = 1$ and $\operatorname{infess}_{t\in\operatorname{supp} x}|x(t)| > 0$.

Sufficiency. Obviously, $I_\Psi\left(p \circ u_\Phi \chi_T\right) > 1$, whenever $\mu(T) > 1$. By Proposition 2.2, $K(x) \neq \varnothing$, so by Theorem 2.5, s is a λ-point. □

Consider the space $L^1 \cap L^\infty$ equipped with the norm $\|\cdot\|_{L^1 \cap L^\infty} = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty}$.

Corollary 3.3. *The space $L^1 \cap L^\infty$ has the λ-property.*

Proof. $L^1 \cap L^\infty$ is the Orlicz space L_Φ with the equality of the norms, where

$$\Phi(u) = \begin{cases} |u| & \text{for } u \in [-1,1] \\ \infty & \text{for } |u| > 1 \end{cases}$$

(see [13]). Since $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$, by Corollary 3.1, we get the thesis. □

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