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On Uniform Nonsquareness and Uniform Normal Structure in Banach Lattices

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We give a sufficient condition for uniform nonsquareness of a Banach lattice in terms of the characteristic of monotonicity and the Riesz angle. For a Köthe space X we prove that orthogonal uniform convexity implies uniform normal structure and uniform nonsquareness both in X and X^*.

Keywords: Uniform nonsquareness, uniform monotonicity, Riesz angle, orthogonal uniform convexity, uniform normal structure, fixed point property

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1. Introduction

In the theory of Banach spaces various coefficients and moduli scaling geometric properties are considered. Many of such properties and corresponding coefficients have been successfully applied to metric fixed point theory (see [14]). Uniform normal structure and uniform nonsquareness are among the basic ones. In particular, they guarantee existence of fixed points of nonexpansive mappings on bounded closed convex sets. This was the motivation for many authors to look for conditions which imply these properties. When dealing with a special class of Banach spaces we can find specific conditions of this kind.

In this paper we deal with Banach lattices, in particular with Köthe spaces. In this case we can use properties related to order, mainly uniform monotonicity and order uniform smoothness. Moduli corresponding to these properties can be found in the literature. Especially the modulus of monotonicity which corresponds to the first of them was intensively studied (see for instance [10], [9] and [5]). The coefficient related to this modulus is called the characteristic of monotonicity and denoted by $\varepsilon_{0,m}(X)$. The modulus corresponding to order uniform smoothness was defined by Kurc [18]. It is related to the Riesz angle $\alpha(X)$ which was introduced by Borwein and Sims [3] who used it to establish a fixed point

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theorem for Banach lattices. Recently applications of uniform monotonicity and the Riesz angle to metric fixed point theory were given in [2]. In this paper we prove that if for a Banach lattice X the inequality $2\varepsilon_{0,m}(X) + \alpha(X) < 2$ holds, then X is uniformly nonsquare. In case of Köthe spaces we study the property of orthogonal uniform convexity introduced by Kolwicz in [15]. We prove that it implies uniform normal structure and uniform nonsquareness both in the space and the dual space. As corollaries we obtain fixed point theorems for nonexpansive mappings.

2. Preliminaries

In this paper we will consider only real Banach spaces with dimension at least two. Let X be such space. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively.

The James constant, or the nonsquare constant of a Banach space X was defined by Gao and Lau in [6] as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

In this definition S_X can be replaced by B_X. In [7] Gao and Lau showed that, in general, $\sqrt{2} \leq J(X) \leq 2$. A space X is uniformly nonsquare if $J(X) < 2$. The dual space X^* is uniformly nonsquare if and only if X is uniformly nonsquare (see [12]). In [11] James proved that uniformly nonsquare spaces are superreflexive. Roughly speaking, the coefficient $J(X)$ shows how two dimensional subspaces of X are close to being isometric to the plane with the maximum norm.

In [21] the coefficient $G_1(X)$ was introduced. It shows how two dimensional subspaces of X are close to being isometric to the plane with a norm which coincides with the maximum norm in two symmetric quadrants. Let us recall that

$$G_1(X) = \inf\{\max\{\|x-y\|, 3 - \|2x-y\|, 3 - \|x-2y\|\} : x, y \in S_X\}.$$

It is easy to see that $1 \leq G_1(X) \leq 2$ and in [21] it was shown that if $G_1(X) > 1$, then X is uniformly nonsquare.

The above coefficients are related to normal structure, which is one of the basic concepts in metric fixed point theory. Let us recall the definition. Given a nonempty bounded subset A of a Banach space X, we put

$$r(A) = \inf\left\{\sup_{y\in A} \|x-y\| : x \in A\right\}.$$

This number is called the Chebyshev radius of A (with respect to A). A Banach space X has normal structure if

$$r(A) < \operatorname{diam} A$$

for every bounded convex subset A of X with $\operatorname{diam} A > 0$. If there exists $c \in (0,1)$ such that

$$r(A) \leq c \operatorname{diam} A$$

for every bounded convex subset A of X, the space X is said to have uniform normal structure. Uniformly nonsquare spaces need not have normal structure, but in [4] it was shown that X has uniform normal structure if $J(X) < \frac{1+\sqrt{5}}{2}$. Also, the condition $G_1(X) > 1$ implies uniform normal structure (see [21]).

One of the main applications of normal structure to metric fixed point theory is Kirk's theorem (see [13]) which assures that every reflexive space X with normal structure has the fixed point property for nonexpansive mappings, i.e., for every nonempty bounded closed convex subset C of X and every mapping $T : C \to C$ such that

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in C$ there exists $x \in C$ for which $T(x) = x$. In [8] it was proved that if X is uniformly nonsquare, then X has the fixed point property for nonexpansive mappings.

In this paper we deal with Banach lattices. For the definition and properties of such spaces the reader can consult [19] or [20]. A very useful property is that if an inequality involves lattice and algebraic operations, then it is enough to check its validity for real numbers to be sure that it holds for vectors in an arbitrary Banach lattice (see [19, p. 1]).

In the literature concerning Banach lattices specific coefficients and moduli related to order are considered. One of the basic moduli is the (lower) modulus of monotonicity $\delta_{m,X}$. Given a Banach lattice X and $\varepsilon \in [0,1]$, we have

$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : x, y \in X,\ 0 \leq y \leq x,\ \|x\| \leq 1,\ \|y\| \geq \varepsilon\}.$$

We say that X is uniformly monotone if $\delta_{m,X}(\varepsilon) > 0$ for all $\varepsilon \in (0,1]$. The coefficient

$$\varepsilon_{0,m}(X) = \sup\{\varepsilon \in [0,1] : \delta_{m,X}(\varepsilon) = 0\}$$

is called the characteristic of monotonicity of X. Clearly, X is uniformly monotone if and only if $\varepsilon_{0,m}(X) = 0$. Moreover, the condition $\varepsilon_{0,m}(X) < 1$ is equivalent to $\lim_{\varepsilon \to 1} \delta_{m,X}(\varepsilon) > 0$.

In [18] the modulus of order smoothness of a Banach lattice X was introduced as follows

$$\rho_{m,X}(t) = \sup\{\|x \vee ty\| - 1 : x, y \in B_X,\ x, y \geq 0\}$$

where $t \in [0,1]$. A lattice X is called order uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_{m,X}(t)}{t} = 0.$$

Both functions $\delta_{m,X}$ and $\rho_{m,X}$ are nondecreasing, convex and have value 0 at 0. Kurc [18] proved that $\delta_{m,X^{**}} = \delta_{m,X}$, $\rho_{m,X^{**}} = \rho_{m,X}$ and the following duality formulae hold:

$$\delta_{m,X}(\varepsilon) = \sup_{0\le t\le 1} (\varepsilon t - \rho_{m,X^*}(t)), \tag{1}$$

$$\rho_{m,X^*}(t) = \sup_{0\le \varepsilon\le 1} (\varepsilon t - \delta_{m,X}(\varepsilon))$$

for all $\varepsilon, t \in [0,1]$. This shows that X is uniformly monotone (resp. order uniformly smooth) if and only if the dual lattice X^* is order uniformly smooth (resp. uniformly monotone).

The number

$$\rho_{m,X}(1) + 1 = \sup\{\|x \vee y\| : x, y \in B_X,\ x, y \ge 0\}$$

is called the Riesz angle of X and denoted by $\alpha(X)$ (see [3]). The formula $\rho_{m,X^{**}} = \rho_{m,X}$ shows in particular that $\alpha(X^{**}) = \alpha(X)$. Clearly, $1 \le \alpha(X) \le 2$ and in metric fixed point theory lattices with $\alpha(X) < 2$ are considered (see [3] and [2]). This condition is equivalent to the following one

$$\lim_{t\to 0} \frac{\rho_{m,X}(t)}{t} < 1.$$

Indeed, $\rho_{m,X}$ is convex, so $\frac{\rho_{m,X}(t)}{t} \le \rho_{m,X}(1)$ if $t \in (0,1]$. Consequently, $\lim_{t\to 0} \frac{\rho_{m,X}(t)}{t} \le \alpha(X) - 1$. On the other hand, $x \vee y \le x \vee ty + (1-t)y$ for all $x, y \in B_X$ and every $t \in (0,1)$. Hence $\|x \vee y\| \le \|x \vee ty\| + 1 - t$ which shows that

$$\alpha(X) \le \rho_{m,X}(t) + 2 - t = 2 - t\left(1 - \frac{\rho_{m,X}(t)}{t}\right).$$

From this inequality it is easy to see that if $\lim_{t\to 0} \frac{\rho_{m,X}(t)}{t} < 1$, then $\alpha(X) < 2$. Observe also that from the duality formula (1) it follows that $\varepsilon_{0,m}(X) = \lim_{t\to 0} \frac{\rho_{m,X^*}(t)}{t}$. Hence $\alpha(X^*) < 2$ if and only if $\varepsilon_{0,m}(X) < 1$.

Let X be a σ-order complete Banach lattice, i.e., every sequence bounded from above has a least upper bound in X. Given $y \in X$ we can therefore consider the band projection given by the formula

$$P_y(x) = \bigvee_{n=1}^{\infty} \left[(n|y|) \wedge x^+\right] - \bigvee_{n=1}^{\infty} \left[(n|y|) \wedge x^-\right]$$

for every $x \in X$. We also put $Q_y = I - P_y$, where I is the identity operator. The maps P_y and Q_y are positive linear projections with the following properties (see [20], [23]):

(i) $P_y(y) = y$, $Q_y(y) = 0$,
(ii) $P_y(z) = 0$ for every $z \in X$ such that $|y| \wedge |z| = 0$,
(iii) $P_y(|x|) = |P_y(x)|$ and $Q_y(|x|) = |Q_y(x)|$ for every $x \in X$,
(iv) $|P_y(x)| \wedge |Q_y(z)| = 0$ for all $x, z \in X$,
(v) $P_y + P_z = P_{y+z}$ for all $y, z \in X$ such that $|y| \wedge |z| = 0$,
(vi) $\|P_y\| = 1 = \|Q_y\|$.

3. Results

The following result is essentially known, its partial case was established in [24].

Proposition 3.1. *If a Banach lattice X is σ-order complete, then*

$$\alpha(X) = \sup\{\|x \vee y\| : x, y \in B_X,\ x, y \geq 0,\ x \wedge y = 0\}.$$

Proof. Let

$$c = \sup\{\|x \vee y\| : x, y \in B_X,\ x, y \geq 0,\ x \wedge y = 0\}.$$

Clearly, $c \leq \alpha(X)$. To show the opposite inequality we take arbitrary $x, y \in B_X$ with $x, y \geq 0$ and put $u = x - x \wedge y$. Then $P_u(x), Q_u(y) \geq 0$, $P_u(x), Q_u(y) \in B_X$ and $P_u(x) \wedge Q_u(y) = 0$. Moreover, $u \wedge (y - x \wedge y) = 0$, so $P_u(y) - P_u(x \wedge y) = P_u(y - x \wedge y) = 0$. Hence $P_u(y) = P_u(x \wedge y)$ and

$$\begin{aligned} P_u(x) \vee Q_u(y) &= P_u(x) + Q_u(y) = P_u(x) + y - P_u(y) = P_u(x) + y - P_u(x \wedge y) \\ &= P_u(u) + y = u + y = x \vee y. \end{aligned}$$

Consequently, $\|x \vee y\| = \|P_u(x) \vee Q_u(y)\| \leq c$. This shows that $\alpha(X) \leq c$. □

Observe that if $|x| \wedge |y| = 0$, then $\||x| \vee |y|\| = \||x| + |y|\| = \||x| - |y|\|$. From Proposition 3.1 it therefore follows that $\alpha(X) \leq J(X)$ if X is σ-order complete. Actually, the assumption of σ-order completeness is superfluous. Indeed, the dual lattice is σ-order complete (see [19]), so given an arbitrary Banach lattice X we have $\alpha(X) = \alpha(X^{**}) \leq J(X^{**}) = J(X)$. We therefore see that if X is uniformly nonsquare, then $\alpha(X) < 2$. By duality we also conclude that if X is uniformly nonsquare, then $\varepsilon_{0,m}(X) < 1$.

Notice that $\varepsilon_{0,m}(l_1) = 0$, $\alpha(l_1) = 2$ and $\varepsilon_{0,m}(c_0) = 1$, $\alpha(c_0) = 1$ while l_1 and c_0 are not uniformly nonsquare. On the other hand, in [2] it was shown that if $\max\{2\varepsilon_{0,m}(X), \alpha(X)\} < 2$, then X is superreflexive. Our first result gives us an analogous sufficient condition for uniform nonsquareness.

Theorem 3.2. *Let X be a Banach lattice. If $2\varepsilon_{0,m}(X) + \alpha(X) < 2$, then X is uniformly nonsquare.*

Proof. Assume that $2\varepsilon_{0,m}(X) + \alpha(X) < 2$ and X is not uniformly nonsquare. Then for every $\gamma \in \left(0, 1 - \frac{\alpha(X)}{2}\right)$ there exist $x, y \in S_X$ such that

$$\|x - y\| > 2(1 - \gamma), \qquad \|x + y\| > 2(1 - \gamma).$$

We put

$$u = |x - y| - ||x| - |y||, \qquad v = |x + y| - ||x| - |y||.$$

The vectors u, v are nonnegative and

$$u \wedge v = |x - y| \wedge |x + y| - ||x| - |y|| = 0.$$

Therefore,

$$\begin{aligned} u+v = v \vee v &= |x-y| \vee |x+y| - ||x|-|y|| \\ &= |x|+|y| - ||x|-|y|| = 2(|x| \wedge |y|). \end{aligned}$$

By our assumption X is superreflexive and we can consider the projections P_u, P_v, Q_u, Q_v. We have $Q_u(u) = 0$, so

$$\begin{aligned} Q_u(|x-y|) = Q_u(||x|-|y||) &= Q_u(|x|+|y|-2(|x|\wedge|y|)) \\ &= |Q_u(x)| + |Q_u(y)| - 2Q_u(|x|\wedge|y|). \end{aligned}$$

Clearly,

$$P_u(|x-y|) = |P_u(x) - P_u(y)| \leq |P_u(x)| + |P_u(y)|$$

which shows that

$$\begin{aligned} |x-y| &= P_u(|x-y|) + Q_u(|x-y|) \\ &\leq |P_u(x)| + |P_u(y)| + |Q_u(x)| + |Q_u(y)| - 2Q_u(|x|\wedge|y|) \\ &= |x|+|y| - 2Q_u(|x|\wedge|y|). \end{aligned}$$

Similarly,

$$|x+y| \leq |x|+|y| - 2Q_v(|x|\wedge|y|).$$

Hence

$$\begin{aligned} 1-\gamma <\ & \min\left\{\left\|\frac{1}{2}(x-y)\right\|, \left\|\frac{1}{2}(x+y)\right\|\right\} \\ \leq\ & \min\left\{\left\|\frac{1}{2}(|x|+|y|) - Q_u(|x|\wedge|y|)\right\|, \left\|\frac{1}{2}(|x|+|y|) - Q_v(|x|\wedge|y|)\right\|\right\} \\ \leq\ & 1-\delta_{m,X}(\varepsilon) \end{aligned}$$

where $\varepsilon = \max\{\|Q_u(|x|\wedge|y|)\|, \|Q_v(|x|\wedge|y|)\|\}$. This gives us the inequality

$$\delta_{m,X}(\varepsilon) < \gamma. \tag{2}$$

But

$$P_u(|x|\wedge|y|) + P_v(|x|\wedge|y|) = P_{u+v}\left(\frac{1}{2}(u+v)\right) = \frac{1}{2}(u+v) = |x|\wedge|y|$$

and consequently,

$$Q_u(|x|\wedge|y|) + Q_v(|x|\wedge|y|) = |x|\wedge|y|.$$

Therefore,

$$\begin{aligned} 2(1-\gamma) < \|x+y\| &\leq \||x|\vee|y|\| + \||x|\wedge|y|\| \\ &\leq \alpha(X) + \|Q_u(|x|\wedge|y|)\| + \|Q_v(|x|\wedge|y|)\| \\ &\leq \alpha(X) + 2\varepsilon \end{aligned}$$

which shows that $\varepsilon > 1 - \gamma - \frac{\alpha(X)}{2}$. From this inequality and (2) we obtain

$$\delta_{m,X}\left(1 - \gamma - \frac{\alpha(X)}{2}\right) < \gamma$$

and passing to the limit with $\gamma \to 0$ we see that

$$\delta_{m,X}\left(1 - \frac{\alpha(X)}{2}\right) = 0.$$

This gives us the inequality $1 - \frac{\alpha(X)}{2} \leq \varepsilon_{0,m}(X)$ which contradicts our assumption. $\square$

If a Banach lattice X is uniformly monotone and l_1 is not finitely representable in X, then X has the fixed point property for nonexpansive mappings (see [1]). The same conclusion holds if $\alpha(X) < 2$ and X is weakly orthogonal (see [2]). Theorem 3.2 gives us another condition sufficient for the fixed point property for nonexpansive mappings.

Corollary 3.3. *Let X be a Banach lattice. If $2\varepsilon_{0,m}(X) + \alpha(X) < 2$, then X has the fixed point property for nonexpansive mappings.*

In our next result we deal with Köthe spaces, so let us recall the definition of these spaces. Let (T, Σ, μ) be a measure space with a σ-finite, complete measure μ. By L^0 we denote the space of all equivalence classes of real-valued measurable functions on T with respect to the relation of equality μ-almost everywhere. A Banach space X is said to be a Köthe space if X is a linear subspace of L^0 and the following conditions hold.

(1) If $x \in X$, $y \in L^0$ and $|y| \leq |x|$ μ-a.e., then $y \in X$ and $\|y\| \leq \|x\|$.
(2) There exists a function $x \in X$ which is strictly positive μ-a.e. in T.

For simplicity we usually consider elements of the Köthe space as functions on T.

Let X be a Köthe space. For $x, y \in X$ we put $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y = (\operatorname{supp} x \setminus \operatorname{supp} y) \cup (\operatorname{supp} y \setminus \operatorname{supp} x)$. The modulus of orthogonal convexity of X was defined by Kolwicz [15] in the following way

$$\delta_X^{\perp}(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : x, y \in S_X,\ \max\{\|x\chi_{A_{xy}}\|, \|y\chi_{A_{xy}}\|\} \geq \varepsilon\right\}$$

where $\varepsilon \in [0,1]$. It is easy to see that S_X can be replaced by B_X in this formula and $\delta_X^{\perp}(\varepsilon) \leq \frac{\varepsilon}{2}$ for every $\varepsilon \in [0,1]$. The space X is said to be orthogonally uniformly convex if $\delta_X^{\perp}(\varepsilon) > 0$ for every $\varepsilon \in (0,1]$. In [15] it was shown that if X is orthogonally uniformly convex, then X is uniformly monotone. Observe that if x, y are disjoint norm-one vectors in a Köthe space X, then $\max\{\|x\chi_{A_{xy}}\|, \|y\chi_{A_{xy}}\|\} = \max\{\|x\|, \|y\|\} = 1$ and consequently,

$$\|x \vee y\| = \|x + y\| \leq 2(1 - \delta_X^{\perp}(1)).$$

Since every Köthe space is σ-order complete (see [19]), from Proposition 3.1 it follows that

$$\alpha(X) \le 2(1 - \delta_X^{\perp}(1)).$$

In view of Theorem 3.2 this shows that if X is orthogonally uniformly convex, then X is uniformly nonsquare. However, our next theorem gives a stronger conclusion.

Theorem 3.4. *Let X be a Köthe space. If X is orthogonally uniformly convex, then $G_1(X) > 1$.*

Proof. Assume that X is orthogonally uniformly convex and $G_1(X) = 1$. We take an arbitrary $\eta \in (0, \frac{1}{8})$. Our assumption guarantees that $\delta_{m,X}$ and $\delta_X^{\perp}$ are increasing positive functions in $(0, 1]$ and we can choose $\gamma > 0$ such that $\gamma + \delta_{m,X}^{-1}(\gamma) < \delta_X^{\perp}(\eta)$.

Since $G_1(X) = 1$, there exist $x_1, x_2 \in S_X$ such that

$$\|x_1 - x_2\| < 1 + \gamma, \qquad \|2x_1 - x_2\| > 2(1 - \gamma^2), \qquad \|x_1 - 2x_2\| > 2(1 - \gamma^2).$$

We put $u_1 = \frac{1}{1+\gamma} x_1$, $u_2 = \frac{1}{1+\gamma} x_2$ and $u = u_2 - u_1$. Then $u_1, u_2, u \in B_X$,

$$\|u_1 - u\| > 2\frac{1 - \gamma^2}{1 + \gamma} = 2(1 - \gamma)$$

and similarly, $\|u + u_2\| > 2(1 - \gamma)$.

We consider elements of the space X as functions defined on a set T. We put

$$A_1 = \{t \in T : \operatorname{sign} u_1(t) = \operatorname{sign} u(t)\}, \qquad A_2 = \{t \in T : \operatorname{sign} u_2(t) \ne \operatorname{sign} u(t)\}$$

and $B_i = T \setminus A_i$ for $i = 1, 2$.

Then

$$|u_i(t) + (-1)^i u(t)| = ||u_i(t)| - |u(t)|| = |u_i(t)| + |u(t)| - 2(|u_i(t)| \wedge |u(t)|)$$

for every $t \in A_i$. Clearly, $|u_i(t) + (-1)^i u(t)| \le |u_i(t)| + |u(t)|$ for $t \in B_i$, so we see that

$$\frac{1}{2}|u_i(t) + (-1)^i u(t)| \le \frac{1}{2}(|u_i(t)| + |u(t)|) - (|u_i(t)| \wedge |u(t)|)\chi_{A_i}(t)$$

for every $t \in T$. Hence

$$\begin{aligned} 1 - \gamma < \left\| \frac{1}{2}(u_i + (-1)^i u) \right\| &\le \left\| \frac{1}{2}(|u_i| + |u|) - (|u_i| \wedge |u|)\chi_{A_i} \right\| \\ &\le 1 - \delta_{m,X}(\|(|u_i| \wedge |u|)\chi_{A_i}\|) \end{aligned}$$

which gives us the inequality $\delta_{m,X}(\max_{i=1,2} \|(|u_i| \wedge |u|)\chi_{A_i}\|) < \gamma$. Consequently,

$$\max_{i=1,2} \|(|u_i| \wedge |u|)\chi_{A_i}\| < \gamma_1$$

where $\gamma_1 = \delta_{m,X}^{-1}(\gamma)$. We put

$$w_i = |u_i| - (|u_i| \wedge |u|)\chi_{A_i},\ v_i = |u| - (|u_i| \wedge |u|)\chi_{A_i}.$$

Then $w_1, w_2, v_1, v_2 \in B_X$, $\|w_i - |u_i|\| < \gamma_1$, $\|v_i - |u|\| < \gamma_1$ and $w_i\chi_{A_i} \wedge v_i\chi_{A_i} = 0$ for $i = 1, 2$. The last equality shows that $C_i \cap D_i = \emptyset$, where $C_i = \operatorname{supp} w_i \cap A_i$ and $D_i = \operatorname{supp} v_i \cap A_i$. It follows that $S_i = \operatorname{supp} w_i \div \operatorname{supp} v_i \supset C_i \div D_i = C_i \cup D_i$ and therefore, $\|w_i\chi_{S_i}\| \geq \|w_i\chi_{A_i}\|$, $\|w\chi_{S_i}\| \geq \|v_i\chi_{A_i}\|$.

Hence

$$1 - \delta_X^{\perp}(\max\{\|w_i\chi_{A_i}\|, \|v_i\chi_{A_i}\|\}) \geq \left\|\frac{w_i + v_i}{2}\right\| \geq \left\|\frac{|u_i| + |u|}{2}\right\| - \gamma_1 > 1 - \gamma - \gamma_1$$

which shows that $\delta_X^{\perp}(\max\{\|w_i\chi_{A_i}\|, \|v_i\chi_{A_i}\|\}) < \gamma + \gamma_1 < \delta_X^{\perp}(\eta)$. Consequently,

$$\max\{\|w_i\chi_{A_i}\|, \|v_i\chi_{A_i}\|\} < \eta.$$

Let us consider the case where $i = 1$. We have

$$\|u_1\chi_{A_1}\| \leq \|w_1\chi_{A_1}\| + \||u_1|\chi_{A_1} - w_1\chi_{A_1}\| < \eta + \gamma_1 < \eta + \delta_X^{\perp}(\eta) < 2\eta$$

and

$$\|u_2\chi_{A_1}\| \leq \|u_1\chi_{A_1}\| + \|u\chi_{A_1}\| < 2\eta + \|v_1\chi_{A_1}\| + \gamma_1 < 3\eta + \gamma_1 < 4\eta.$$

Similarly, $\|u_i\chi_{A_2}\| < 4\eta$ for $i = 1, 2$. Let $A = A_1 \cup A_2$ and $B = T \setminus A = B_1 \cap B_2$. Then $\|u_i\chi_A\| < 8\eta$ for $i = 1, 2$.

We put $z_i = u_i\chi_B$ where $i = 1, 2$. Then $z_1, z_2, z_1 - z_2 \in B_X$ and $\|z_i - u_i\| = \|u_i\chi_A\| < 8\eta$. Hence

$$\|z_i\| \geq \|u_i\| - 8\eta = \frac{1}{1+\gamma} - 8\eta > \frac{1 - 9\eta}{1 + \eta}.$$

Moreover, if $t \in B$, then $\operatorname{sign} u_1(t) \neq \operatorname{sign} u(t) = \operatorname{sign} u_2(t)$. Consequently, $\operatorname{sign} u_1(t) \neq \operatorname{sign} u_2(t)$ which gives us the equality $|u_1(t) - u_2(t)| = |u_1(t)| + |u_2(t)|$. Therefore, $|z_1 - z_2| = |z_1| + |z_2| \geq |z_1|$ and $|z_1 - z_2| - |z_1| = |z_2|$. It follows that

$$\delta_{m,X}\left(\frac{1 - 9\eta}{1 + \eta}\right) \leq 1 - \|z_2\| \leq 1 - \frac{1 - 9\eta}{1 + \eta}$$

and passing to the limit with $\eta \to 0$, we see that $\delta_{m,X}(1) = 0$, which contradicts our assumption. □

Observe that if a Köthe space X is orthogonally uniformly convex, then the dual space X^* has uniform normal structure. Indeed, in [16] it was proved that orthogonally uniformly convex Köthe spaces are P-convex, so the conclusion follows from a result proved in [22]. This and Theorem 3.4 give us the following corollary.

Corollary 3.5. *Let X be a Köthe space. If X is orthogonally uniformly convex, then both X and X^* are uniformly nonsquare and have uniform normal structure.*

From Corollary 3.5 it follows that if a Köthe space X is orthogonally uniformly convex, then X and X^* have the fixed point property for nonexpansive mappings. This result for the space X has been already proved in [17].

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