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Some More Applications of the Hahn-Banach Theorem

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We give a new proof of the fact that equivalent norms on subspaces can be extended. This new proof is based on the Hahn-Banach Extension Theorem. We also give new characterizations for an equivalent norm on a dual space to be a dual norm. Finally, a new proof of a particular case of the Hahn-Banach Separation Theorem is provided without involving the Axiom of Choice.

Keywords: Equivalent norm, dual norm, Hahn-Banach, axiom of choice

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1. Extensions of equivalent norms

Extensions of equivalent norms have a lot of applications in the Theory of Banach Spaces. For this we refer the reader to [3, Lemma 8.1], where the following was proved.

Lemma 1.1 (Deville, Godefroy, and Zizler, 1993). *Let X be a real Banach space. Let Y be a subspace of X. If $|\cdot|_Y$ is an equivalent norm on Y, then there exists an equivalent norm $|\cdot|_X$ on X such that $|\cdot|_X|_Y = |\cdot|_Y$.*

Even though the proof of this lemma appears in [3, Lemma 8.1], here we will show it with little modifications. We recall the reader that a subset of a real vector space is said to be absolutely convex provided that it is balanced and convex.

Proof of Lemma 1.1. Observe that we may assume that $\mathsf{B}_{|\cdot|_Y} \supseteq \mathsf{B}_Y$. Take $|\cdot|_X$

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to be the norm on X whose unit ball is $\mathsf{B}_{|\cdot|_X} := \overline{\text{co}}\left(\mathsf{B}_X \cup \mathsf{B}_{|\cdot|_Y}\right)$. It is clear that $\mathsf{B}_{|\cdot|_X}$ is bounded, closed, absolutely convex, and contains B_X, therefore it is the unit ball of an equivalent norm on X. All is left to show is that $\mathsf{B}_{|\cdot|_X} \cap Y = \mathsf{B}_{|\cdot|_Y}$. By construction we have that $\mathsf{B}_{|\cdot|_X} \cap Y \supseteq \mathsf{B}_{|\cdot|_Y}$. An element y of $\mathsf{B}_{|\cdot|_X} \cap Y$ is the limit of a sequence $(tx_n + (1-t)\,y_n)_{n\in\mathbb{N}}$ where $t \in [0,1]$ and for all $n \in \mathbb{N}$ we have $x_n \in \mathsf{B}_X$ and $y_n \in \mathsf{B}_{|\cdot|_Y}$. If $t = 0$, then $y \in \mathsf{B}_{|\cdot|_Y}$ because $\mathsf{B}_{|\cdot|_Y}$ is closed in Y. Assume then that $t \in (0,1]$. Note that the sequence

$$\left(x_n - \left(\frac{1}{t}y - \frac{1-t}{t}y_n\right)\right)_{n\in\mathbb{N}}$$

converges to 0. This leaves us two possibilities:

(1) There exists a subsequence $\left(\frac{1}{t}y - \frac{1-t}{t}y_{n_k}\right)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$ we have that $\left\|\frac{1}{t}y - \frac{1-t}{t}y_{n_k}\right\| \leq 1$ and hence $\frac{1}{t}y - \frac{1-t}{t}y_{n_k} \in \mathsf{B}_{|\cdot|_Y}$. In this case, the sequence

$$\left(t\left(\frac{1}{t}y - \frac{1-t}{t}y_{n_k}\right) + (1-t)\,y_{n_k}\right)_{k\in\mathbb{N}}$$

converges to y and is contained in $\mathsf{B}_{|\cdot|_Y}$, which means that $y \in \mathsf{B}_{|\cdot|_Y}$ because $\mathsf{B}_{|\cdot|_Y}$ is closed in Y.

(2) There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $\left\|\frac{1}{t}y - \frac{1-t}{t}y_n\right\| \searrow 1$. In this case, the sequence

$$\left(t\left(\frac{\frac{1}{t}y - \frac{1-t}{t}y_n}{\left\|\frac{1}{t}y - \frac{1-t}{t}y_n\right\|}\right) + (1-t)\,y_n\right)_{n\in\mathbb{N}}$$

converges to y and is contained in $\mathsf{B}_{|\cdot|_Y}$, which means that $y \in \mathsf{B}_{|\cdot|_Y}$ because $\mathsf{B}_{|\cdot|_Y}$ is closed in Y. □

We present here a new proof of Lemma 1.1 that relies on the Hahn-Banach Extension Theorem. Notice that this new proof is legitimate because the original proof of the Hahn-Banach Extension Theorem is not based upon extending any norm.

New Proof of Lemma 1.1. We can assume without any loss of generality that $\mathsf{B}_{|\cdot|_Y} \subseteq \mathsf{B}_Y$. Take $|\cdot|_X$ to be the norm on X whose unit ball is

$$\mathsf{B}_{|\cdot|_X} := \mathsf{B}_X \cap \bigcap_{f\in T} f^{-1}([-1,1])$$

where

$$T := \left\{f \in X^* : |f|_Y|_Y^* = 1, \|f|_Y\|^* = \|f\|^*\right\}.$$

Observe the following:

(1) $|\cdot|_X$ is an equivalent norm on X. Indeed, $\mathsf{B}_{|\cdot|_X}$ is closed, bounded, and absolutely convex. It suffices to prove that $\mathsf{B}_{|\cdot|_X}$ has non-empty interior. Since $|\cdot|_Y^*$ is an equivalent norm on Y^*, there must exist a constant $K > 0$ such that $|\cdot|_Y^* \geq K \|\cdot\|^*$ on Y^*. We will show that $\mathsf{B}_X(0, \min\{1, K\}) \subseteq \mathsf{B}_{|\cdot|_X}$. It is obvious that $\mathsf{B}_X(0, \min\{1, K\}) \subseteq \mathsf{B}_X$. Finally, if $x \in \mathsf{B}_X(0, \min\{1, K\})$ and $f \in T$, then

$$|f(x)| \leq \|f\|^* \|x\| = \|f|_Y\|^* \|x\| \leq \frac{1}{K} |f|_Y|_Y^* \|x\| \leq 1.$$

(2) $\mathsf{B}_{|\cdot|_X} \cap Y = \mathsf{B}_{|\cdot|_Y}$. Indeed, let $y \in \mathsf{B}_{|\cdot|_X} \cap Y$ and assume that $|y|_Y > 1$. We can find $f \in Y^*$ such that $|f|_Y^* = 1$ and $f(y) = |y|_Y$. In accordance to the Hahn-Banach Extension Theorem, we can extend f linearly and continuously to the whole of X and preserving its $\|\cdot\|^*$-norm. We will keep denoting this extension by f. By construction, $f \in T$. Now we get the contradiction that $|y|_Y = |f(y)| \leq 1$. Conversely, let $y \in \mathsf{B}_{|\cdot|_Y}$. If $f \in T$, then $|f|_Y|_Y^* = 1$ so we obviously have that $|f(y)| \leq 1$. $\square$

In order to extend dual norms, we will follow a similar process as in the original proof of [3, Lemma 8.1].

Lemma 1.2. *Let X be a real Banach space. Let Y be a w^*-closed subspace of X^*. If $|\cdot|_Y$ is an equivalent norm on Y, then there exists an equivalent dual norm $|\cdot|_{X^*}$ on X^* such that $|\cdot|_{X^*}|_Y = |\cdot|_Y$.*

Proof. Assume that $\mathsf{B}_{|\cdot|_Y} \supseteq \mathsf{B}_Y$ and take $|\cdot|_{X^*}$ to be the norm on X^* whose unit ball is $\mathsf{B}_{|\cdot|_{X^*}} := \operatorname{co}\left(\mathsf{B}_{X^*} \cup \mathsf{B}_{|\cdot|_Y}\right)$. We can easily see that $\mathsf{B}_{|\cdot|_{X^*}}$ is w^*-compact, bounded, and absolutely convex. Therefore, it defines a new equivalent dual norm on X^*. It remains to see that $\mathsf{B}_{|\cdot|_{X^*}} \cap Y = \mathsf{B}_{|\cdot|_Y}$. Obviously, $\mathsf{B}_{|\cdot|_{X^*}} \cap Y \supseteq \mathsf{B}_{|\cdot|_Y}$. An element z of $\mathsf{B}_{|\cdot|_{X^*}} \cap Y$ is of the form $tx^* + (1-t)y$ where $t \in [0,1]$, $x^* \in \mathsf{B}_{X^*}$ and $y \in \mathsf{B}_{|\cdot|_Y}$. If $t = 0$, then z is trivially in $\mathsf{B}_{|\cdot|_Y}$. Assume then that $t \neq 0$. Since $z \in Y$, we must have that $x^* \in Y$, so $x^* \in \mathsf{B}_Y \subseteq \mathsf{B}_{|\cdot|_Y}$. Therefore, $z \in \mathsf{B}_{|\cdot|_Y}$. $\square$

The next point here is to adapt the new proof of Lemma 1.1 to extend equivalent dual norms. However, we will show now that it cannot be adapted quite well.

Remark 1.3. Let X be a real Banach space. Let Y be a w^*-closed subspace of X^*. Let $|\cdot|_Y$ be an equivalent norm on Y. As in the new proof of Lemma 1.1, we may assume without any loss of generality that $\mathsf{B}_{|\cdot|_Y} \subseteq \mathsf{B}_Y$. Take $|\cdot|_{X^*}$ to be the norm on X^* whose unit ball is

$$\mathsf{B}_{|\cdot|_{X^*}} := \mathsf{B}_{X^*} \cap \bigcap_{f \in T} f^{-1}([-1,1])$$

where

$$T := \left\{f \in X^{**} : f \text{ is } w^*\text{-continuous}, \ |f|_Y|_Y^* = 1, \|f|_Y\|^{**} = \|f\|^{**}\right\}.$$

Following a similar way as in the new proof of Lemma 1.1, it can be shown that $|\cdot|_{X^*}$ is an equivalent norm on X^* and $\mathsf{B}_{|\cdot|_{X^*}} \cap Y \supseteq \mathsf{B}_{|\cdot|_Y}$. In order to prove that $\mathsf{B}_{|\cdot|_{X^*}} \cap Y \subseteq \mathsf{B}_{|\cdot|_Y}$ we need to verify two things:

(1) Given any $y \in Y$ with $|y|_Y > 1$, there must exist a w^* continuous $f \in Y^*$ with $|f|_Y^* = 1$ and $f(y) = |y|_Y$.

(2) Given any w^* continuous $f \in Y^*$, there must exist a w^* continuous linear extension to the whole of X^{**} preserving its $\|\cdot\|^{**}$-norm.

The previous remark motivates the following question:

Question 1.4. In the settings of Remark 1.3, what conditions does Y require in order for (1) and (2) to be verified?

In [3, Proposition 8.2] it is also proved the following result:

Proposition 1.5 (Deville, Godefroy, and Zizler, 1993). *Let X be a separable real Banach space. Let Y be a subspace of X. If $|\cdot|_Y$ is an equivalent locally uniformly rotund norm on Y, then there exists an equivalent locally uniformly rotund norm $|\cdot|_X$ on X such that $|\cdot|_X|_Y = |\cdot|_Y$.*

The next results show that we can extend equivalent norms preserving other geometric characteristics.

Theorem 1.6. *Let X be a real Banach space. Let Y be a closed subspace of X. Assume that there exists an equivalent norm $|\cdot|_Y$ on Y whose unit sphere $\mathsf{S}_{|\cdot|_Y}$ contains a face C with non-empty interior relative to $\mathsf{S}_{|\cdot|_Y}$. There exists an equivalent norm $|\cdot|_X$ on X such that:*

(1) The unit sphere $\mathsf{S}_{|\cdot|_X}$ contains a face D with non-empty interior relative to $\mathsf{S}_{|\cdot|_X}$ such that $D \cap Y = C$.

(2) The norm $|\cdot|_X|_Y = |\cdot|_Y$.

Proof. The Hahn-Banach Separation Theorem assures the existence of an element $f \in Y^*$ such that $|f|_Y^* = 1$ and $C = f^{-1}(1) \cap \mathsf{B}_{|\cdot|_Y}$. The Hahn-Banach Extension Theorem assures the existence of a linear continuous extension of f to the whole of X preserving its $\|\cdot\|^*$-norm. We will keep denoting this extension by f. By hypothesis, there exists $y \in \mathsf{S}_{|\cdot|_Y}$ and $\varepsilon' > 0$ such that $\mathsf{B}_Y(y, \varepsilon') \cap f^{-1}(1) \subseteq C$. Denote $F := \mathsf{B}_X(y, \varepsilon) \cap f^{-1}(1)$ where $0 < \varepsilon < \varepsilon'$. We will consider the closed absolutely convex set $\overline{\text{aco}}\left(\mathsf{B}_{|\cdot|_Y} \cup F\right)$. This set verifies the following properties:

(1) $\overline{\text{aco}}\left(\mathsf{B}_{|\cdot|_Y} \cup F\right)$ has non-empty interior. Indeed, choose

$$0 < \tau < \min\left\{\frac{1}{3\|f\|^*}, \frac{\varepsilon}{3(1 + \|f\|^*\|y\|)}\right\}.$$

We will show that $\mathsf{B}_X(0, \tau) \subseteq \text{aco}\left(\mathsf{B}_{|\cdot|_Y} \cup F\right)$. Let $x \in \mathsf{B}_X(0, \tau)$. Note that

$$x = \left(f(x) + \frac{1}{3}\right) y - \frac{1}{3}\left(y + 3\left(x - f(x)\, y\right)\right).$$

Observe that $y+3\,(x-f\,(x)\,y)\in F$ and that $\left|f\,(x)+\frac{1}{3}\right|+\left|\frac{1}{3}\right|\leq 1$, therefore $x\in\operatorname{aco}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)$. Notice that we have proved that $\overline{\operatorname{aco}}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)$ is the unit ball of an equivalent norm on X that we will denote by $|\cdot|_X$.

(2) $\overline{\operatorname{aco}}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)\cap Y=\mathsf{B}_{|\cdot|_Y}$. Indeed, obviously $\overline{\operatorname{aco}}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)\cap Y\supseteq\mathsf{B}_{|\cdot|_Y}$. An element y of $\overline{\operatorname{aco}}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)\cap Y$ is the limit of a sequence $(ty_n+sx_n)_{n\in\mathbb{N}}$ in which $|t|+|s|\leq 1$ and for all $n\in\mathbb{N}$ it is $y_n\in\mathsf{B}_{|\cdot|_Y}$ and $x_n\in F$. If $s=0$, then y trivially belongs to $\mathsf{B}_{|\cdot|_Y}$, so we may assume that $s\neq 0$. Observe that the sequence

$$\left(x_n-\frac{\frac{1}{s}y-\frac{t}{s}y_n}{f\left(\frac{1}{s}y-\frac{t}{s}y_n\right)}\right)_{n\in\mathbb{N}}$$

converges to 0. Since $\varepsilon<\varepsilon'$, we eventually have that

$$\frac{\frac{1}{s}y-\frac{t}{s}y_n}{f\left(\frac{1}{s}y-\frac{t}{s}y_n\right)}\in\mathsf{B}_Y\left(y,\varepsilon'\right)\cap f^{-1}\left(1\right)\subseteq C\subseteq\mathsf{B}_{|\cdot|_Y}$$

for large n's, which means that

$$y=\lim_{n\to\infty}\left(ty_n+s\frac{\frac{1}{s}y-\frac{t}{s}y_n}{f\left(\frac{1}{s}y-\frac{t}{s}y_n\right)}\right)\in\mathsf{B}_{|\cdot|_Y}.$$

(3) The boundary of $\overline{\operatorname{aco}}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)$ has a face D with non-empty interior relative to that boundary such that $D\cap Y=C$. Indeed, by construction F is a convex set with non-empty interior relative to the boundary of $\overline{\operatorname{aco}}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)$ (to see this it is sufficient to notice that $|f|_X^*=\sup f\left(\operatorname{aco}\left(\mathsf{B}_{|\cdot|_Y}\cup F\right)\right)=1$). Now take $D:=f^{-1}\left(1\right)\cap\mathsf{B}_{|\cdot|_X}$. Finally,

$$D\cap Y=f^{-1}\left(1\right)\cap\mathsf{B}_{|\cdot|_X}\cap Y=f^{-1}\left(1\right)\cap\mathsf{B}_{|\cdot|_Y}=C. \qquad \square$$

To finish this section we make the reader notice without a proof that the dual version of the previous theorem also holds.

Theorem 1.7. *Let X be a real Banach space. Let Y be a w^*-closed subspace of X^*. Assume that there exists an equivalent norm $|\cdot|_Y$ on Y whose unit sphere $\mathsf{S}_{|\cdot|_Y}$ contains a face C with non-empty interior relative to $\mathsf{S}_{|\cdot|_Y}$. There exists an equivalent dual norm $|\cdot|_{X^*}$ on X^* such that:*

(*1*) *The unit sphere $\mathsf{S}_{|\cdot|_{X^*}}$ contains a face D with non-empty interior relative to $\mathsf{S}_{|\cdot|_{X^*}}$ such that $D\cap Y=C$.*

(*2*) *The norm $|\cdot|_{X^*}|_Y=|\cdot|_Y$.*

2. Equivalent dual norms

The following characterization of equivalent dual norms can be found in [3, Fact 5.4]:

Fact 2.1 (Deville, Godefroy, and Zizler, 1993). *Let X be a real Banach space. Let $|\cdot|$ be an equivalent norm on X^*. The following conditions are equivalent:*

(*1*) *The norm $|\cdot|$ is a dual norm.*
(*2*) *The Banach-Aloaglu Theorem is verified, in other words, $\mathsf{B}_{|\cdot|}$ is w^*-closed.*
(*3*) *The norm $|\cdot| : (X^*, w^*) \to \mathbb{R}$ is lower semi-continuous.*

Our contribution to this topic consists of showing that an equivalent norm on a dual space is a dual norm if and only if the Goldstine Theorem is verified. This will be bring interesting consequences on the extension of dual norms.

Remark 2.2. Let X be a real Banach space. Let $|\cdot|$ be an equivalent norm on X^*. Consider the equivalent norm on X given by $\lfloor\cdot\rfloor := |\cdot|^*|_X$. We have the following basic facts:

- $\mathsf{B}_{\lfloor\cdot\rfloor} = \mathsf{B}_{|\cdot|^*} \cap X$.
- $|\cdot|$ is a dual norm on X if and only if $|\cdot| = \lfloor\cdot\rfloor^*$.
- $\lfloor\cdot\rfloor^{**}|_X = |\cdot|^*|_X$.

By taking into consideration Remark 2.2 we are capable of carrying out the desired characterizations.

Theorem 2.3. *Let X be a real Banach space. Let $|\cdot|$ be an equivalent norm on X^*. The following conditions are equivalent:*

(*1*) *The norm $|\cdot|$ is a dual norm.*
(*2*) *The Goldstine Theorem is verified, in other words,*

$$\mathrm{cl}_{w^*}\left(\mathsf{B}_{|\cdot|^*} \cap X\right) = \mathsf{B}_{|\cdot|^*}.$$

Proof. Let $\lfloor\cdot\rfloor := |\cdot|^*|_X$. For every $x^* \in X^*$ we have that

$$\begin{aligned}\lfloor x^*\rfloor^* &= \sup\left\{x^*(x) : x \in \mathsf{B}_{\lfloor\cdot\rfloor}\right\}\\ &= \sup\left\{x(x^*) : x \in \mathsf{B}_{\lfloor\cdot\rfloor}\right\}\\ &= \sup\left\{x(x^*) : x \in \mathsf{B}_{|\cdot|^*} \cap X\right\}\\ &= \sup\left\{x^{**}(x^*) : x^{**} \in \mathsf{B}_{|\cdot|^*}\right\} \quad \text{(Goldstine Theorem)}\\ &= |x^*|. \end{aligned}$$ □

The following characterization of equivalent dual norms solves an open problem on the extension of equivalent norms.

Corollary 2.4. *Let X be a real Banach space. Let $|\cdot|$ be an equivalent norm on X^*. Consider the equivalent norm on X given by $\lfloor\cdot\rfloor := |\cdot|^*|_X$. The following conditions are equivalent:*

(*1*) *The norm $|\cdot|$ is a dual norm.*
(*2*) $\lfloor\cdot\rfloor^{**} = |\cdot|^*$.

Proof. Assume that $\lfloor\cdot\rceil^{**} = |\cdot|^*$. We will show that the Goldstine Theorem is verified by $|\cdot|^*$, that is,

$$\mathrm{cl}_{w^*}\left(\mathsf{B}_{|\cdot|^*} \cap X\right) = \mathsf{B}_{|\cdot|^*}.$$

Indeed, by bearing in mind Remark 2.2 we have that

$$\mathrm{cl}_{w^*}\left(\mathsf{B}_{|\cdot|^*} \cap X\right) = \mathrm{cl}_{w^*}\left(\mathsf{B}_{\lfloor\cdot\rceil}\right) = \mathsf{B}_{\lfloor\cdot\rceil^{**}} = \mathsf{B}_{|\cdot|^*}$$

in virtue of the fact that the bidual norm $\lfloor\cdot\rceil^{**}$ verifies the Goldstine Theorem. Now it only suffices to apply the previous theorem. $\square$

An informally open question among Functional Analysts is to determine whether two equivalent dual norms on a dual space are equal provided that they coincide on a w^*-dense subspace. The next example shows that this is not true in virtue of Corollary 2.4.

Example 2.5. Let X be a non-reflexive real Banach space. Let $|\cdot|$ be an equivalent norm on X^* which is not a dual norm. Consider the equivalent norm on X given by $\lfloor\cdot\rceil := |\cdot|^* |_X$. In accordance to Corollary 2.4 we have that both $\lfloor\cdot\rceil^{**}$ and $|\cdot|^*$ are equivalent dual norms on X^{**} which coincide on the w^*-dense subspace X but which are not equal.

3. The Hahn-Banach Theorem

This section is devoted to prove an equivalent form of the Hahn-Banach Separation Theorem over the Zermelo-Fraenkel Axiomatic System (ZF) plus the Axiom of Dependent Choices (DC). It is well known that DC is a weaker form of the Axiom of Choice (AC) and is actually equivalent to the Baire Category Theorem (see [1]). It is also known that DC strictly implies the Axiom of Countable Choice (AC_ω). The literature involving AC and Hahn-Banach Theorem shows, among other things, that:

(1) The Ultrafilter Lemma, which is strictly weaker than AC, implies the Hahn-Banach Theorem, although the converse is not the case.
(2) The Hahn-Banach Theorem can in fact be proved using even weaker hypotheses than the Ultrafilter Lemma (see [5] and [4]).
(3) For separable Banach spaces, Brown and Simpson proved that the Hahn-Banach Theorem follows from WKL_0, a weak subsystem of second-order arithmetic (see [2]).

Over $\mathsf{ZF} + \mathrm{DC}$ the following three results hold (notice that the existence of converging sequences is guaranteed by AC_ω which is implied by DC as we mentioned at the beginning of this section).

Lemma 3.1. *Let X be a real Banach space. Let $c \in \mathsf{S}_X$ and $0 < r < 1$ such that $\mathsf{B}_X(c, r) \cap \mathsf{S}_X$ is convex. Then $[\mathsf{B}_X(c, r) \cap \mathsf{S}_X] - c$ is absolutely convex.*

Proof. Let $x \in [\mathsf{B}_X(c, r) \cap \mathsf{S}_X] - c$. All we need to show is that $-x \in [\mathsf{B}_X(c, r) \cap \mathsf{S}_X] - c$. Note that in order to show this it is sufficient to prove that $\|-x + c\| \leq 1$.

Indeed, since $x+c \in \mathsf{B}_X(c,r) \cap \mathsf{S}_X$, we have that $-x+c \in \mathsf{B}_X(c,r)$, and since

$$c = \frac{1}{2}(x+c) + \frac{1}{2}(-x+c),$$

we necessarily have that $\|-x+c\| = 1$. Let us show then that $-x+c$ must be in B_X. Assume not. Let $Y := \operatorname{span}\{x,c\}$. Let $y \in [c,-x+c]$ be the closest to $-x+c$ that still remains on S_Y. Notice that $\|y-c\| < r$. We can find $z \in \mathsf{S}_Y$ sufficiently close to y so that $\|z-c\| < r$ and z is not on the segment $[c,y]$. Since $\mathsf{B}_Y(c,r) \cap \mathsf{S}_Y$ is convex, z must lie on the segment $(y,-x+c)$, which contradicts the fact that y is the closest in $[c,-x+c]$ to $-x+c$ that still remains on S_Y. $\square$

Theorem 3.2. *Let X be a real Banach space. Let $C \subset \mathsf{S}_X$ be a convex set with non-empty interior relative to S_X. There exists a unique $f \in \mathsf{S}_{X^*}$ such that $C \subset f^{-1}(1)$.*

Proof. Let $c \in \operatorname{int}_{\mathsf{S}_X}(C)$ and $0 < r < 1$ such that $\mathsf{B}_X(c,r) \cap \mathsf{S}_X \subseteq C$. We will show the following:

(1) $\operatorname{span}(C-c) = \operatorname{span}([\mathsf{B}_X(c,r) \cap \mathsf{S}_X] - c)$. Indeed, let $x \in C$. There exists $t \in (0,1)$ sufficiently small such that $tx + (1-t)c \in \mathsf{B}_X(c,r) \cap \mathsf{S}_X$. Then

$$\begin{aligned} x - c &= \frac{1}{t}([tx+(1-t)c] - c) \\ &\in \operatorname{span}([\mathsf{B}_X(c,r) \cap \mathsf{S}_X] - c). \end{aligned}$$

(2) $X \neq \operatorname{span}(C-c)$. Indeed, assume that $X = \operatorname{span}(C-c)$, then $X = \operatorname{span}([\mathsf{B}_X(c,r) \cap \mathsf{S}_X] - c)$. By Lemma 3.1, we have that $[\mathsf{B}_X(c,r) \cap \mathsf{S}_X] - c$ is a barrel of X. Since X is complete, we are allowed to apply the Baire Category Theorem (which is equivalent to DC as we mentioned at the beginning of this section). Therefore, $[\mathsf{B}_X(c,r) \cap \mathsf{S}_X] - c$ has nonempty interior, which means that $\mathsf{B}_X(c,r) \cap \mathsf{S}_X$ has non-empty interior, and this is not possible.

(3) $X = \operatorname{span}(C-c) + \mathbb{R}c$. Indeed, let $x \in X$. There exists $\alpha \in \mathbb{R} \setminus \{0\}$ small enough such that

$$\frac{c - \alpha x}{\|c - \alpha x\|} \in C.$$

Then

$$\begin{aligned} x &= \frac{-\|c-\alpha x\|}{\alpha}\left(\frac{c-\alpha x}{\|c-\alpha x\|} - c\right) + \frac{1 - \|c-\alpha x\|}{\alpha}c \\ &\in \operatorname{span}(C-c) + \mathbb{R}c. \end{aligned}$$

(4) $\mathsf{B}_{\mathrm{span}(C-c)}(0,r/2) \subseteq C-c$. Indeed, let $y \in \mathsf{B}_{\mathrm{span}(C-c)}(0,r/2)$. Then

$$\begin{aligned}
\left\|\frac{c-y}{\|c-y\|}-c\right\| &= \left\|\frac{(1-\|c-y\|)\,c-y}{\|c-y\|}\right\| \\
&= \left\|\frac{(1-\|c-y\|)\,(c-y)-\|c-y\|\,y}{\|c-y\|}\right\| \\
&\leq |1-\|c-y\||+\|y\| \\
&= |\|c\|-\|c-y\||+\|y\| \\
&\leq \|c-(c-y)\|+\|y\| = 2\,\|y\| \leq r.
\end{aligned}$$

Now,

$$y = -\,\|c-y\|\left(\frac{c-y}{\|c-y\|}-c\right)+(1-\|c-y\|)\,c,$$

which shows the contradiction that $c \in \mathrm{span}\,(C-c)$ unless $1-\|c-y\|=0$. Therefore

$$-y = \frac{c-y}{\|c-y\|}-c \in C-c.$$

(5) $\mathrm{dist}\left(\frac{rc}{2}, C-c\right) \geq \frac{r}{2}$. Indeed, if $x \in C$, then

$$\begin{aligned}
\left\|\frac{r}{2}c-(x-c)\right\| &= \left\|\left(1+\frac{r}{2}\right)c-x\right\| \\
&\geq \left|\left\|\left(1+\frac{r}{2}\right)c\right\|-\|x\|\right| \\
&= \frac{r}{2}.
\end{aligned}$$

(6) $\mathrm{span}\,(C-c)$ is closed. If it is not closed, then it must be dense, since it is of codimension 1. Therefore, there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathrm{span}\,(C-c)$ that converges to c. Then

$$\left(\frac{ry_n}{\|2y_n\|}\right)_{n\in\mathbb{N}} \subseteq \mathsf{B}_{\mathrm{span}(C-c)}(0,r/2) \subseteq C-c$$

converges to $\frac{rc}{2}$, which is impossible because $\mathrm{dist}\left(\frac{rc}{2}, C-c\right) \geq \frac{r}{2}$.

(7) Since $\mathrm{span}\,(C-c)$ is closed and of codimension 1, it must be the kernel of a unique continuous linear functional $f \in \mathsf{S}_{X^*}$ such that $f(c) > 0$. All is left to prove is that $f(c) = 1$. Assume not. There exists $x \in \mathsf{S}_{X^*}$ such that $f(c) < f(x) \leq 1$. We can find $t \in (0,1)$ small enough so that

$$\frac{tx+(1-t)\,c}{\|tx+(1-t)\,c\|} \in C.$$

However, this is not possible because

$$tf(x)+(1-t)\,f(c) > f(c) \geq \|tx+(1-t)\,c\|\,f(c)\,. \qquad \square$$

Theorem 3.3. *The following statements are equivalent for a real Banach space X:*

(1) *There exists a continuous linear functional $f \in X^* \setminus \{0\}$.*

(2) *There exists a closed bounded absolutely convex subset B of X whose boundary contains a convex subset C with non-empty interior relative to the boundary of B.*

Proof. If there exists such B, then one can call on Theorem 3.2 to assure the existence of a continuous linear functional $f \in X^* \setminus \{0\}$. Conversely, assume the existence of a continuous linear functional $f \in X^* \setminus \{0\}$. It is sufficient to take $B := \mathsf{B}_X \cap f^{-1}\left[-\frac{1}{2}, \frac{1}{2}\right]$ and $C := \mathsf{B}_X \cap f^{-1}\left(\frac{1}{2}\right)$. □

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