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Some Geometric Properties of the Cesàro Function Spaces

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Some geometric properties of the Cesàro function spaces $C_{p,w}$, $1 \leqslant p < \infty$, induced by an arbitrary positive weight function w on an interval $(0, l)$ where $0 < l \leqslant \infty$ are studied in this paper. It is showed that all non-empty relatively weakly open sets in the unit ball of $C_{p,w}$ have diameter 2. Also $C_{p,w}$, $1 < p < \infty$ is strictly convex but no point of its unit ball is strongly extreme. Moreover, some connections between uniformly non-square points and various geometric properties in general Banach spaces are presented.

Keywords: Cesàro function space, diameter 2 property, weak neighborhoods, uniformly non-square points

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1. Introduction

The Cesàro function spaces were studied for the first time in 1970 [16] and then, for a long time they did not attract a lot of attention. Only recently, in a series of papers [4, 6, 5], Astashkin and Maligranda started to study thoroughly the structure of the Cesàro function spaces Ces_p, $1 < p < \infty$ with standard weight $w(x) = x^{-1}$. They showed, among others, that Ces_p is not reflexive, is not isomorphic to any L_q, $1 \leqslant q \leqslant \infty$, does not have Dunford-Pettis Property but has the weak Banach-Saks property. They also investigated the dual space of Ces_p [5]. They found a Banach space equipped with a norm equivalent to the dual norm, which is an isomorphic representation of the dual space of Ces_p. In [13] the dual norm of the Cesàro function space $C_{p,w}$ on $(0, l)$, $0 < l \leqslant \infty$, generated by $1 < p < \infty$ and an arbitrary positive weight function w has been found. In the same paper authors also show that the space $C_{p,w}$ has all slices of the unit ball of diameter 2 and hence $C_{p,w}$ does not have the Radon-Nikodym property. Moreover, $C_{p,w}$ is not locally uniformly convex and is not a dual space.

In this paper we continue the study of geometric properties of the Cesàro function space $C_{p,w}$.

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We start our presentation with results in general Banach spaces. In Section 2, we show, among others, that if there are no locally uniformly non-square points in the unit sphere then all slices of the unit ball have diameter 2.

Sections 3 to 5 are devoted to geometric properties of the Cesàro function space $C_{p,w}$. In Section 3, using results from Section 2, we show that all non-empty relatively weakly open sets of the unit ball of $C_{p,w}$ have diameter 2 and that $C_{p,w}$ is not uniformly rotund in every direction. It is also showed that there are no strongly extreme points in the unit ball while in Section 4 we show that $C_{p,w}$ is strictly convex.

In the final Section 5 we show that there is an asymptotically isometric copy of ℓ_1 in $C_{p,w}$. It follows that $C_{p,w}$ does not have the fixed point property.

We will use the standard notation. For a Banach space $(X, \|\cdot\|)$ by B_X and S_X we denote the unit ball and the unit sphere of X, and by X^* the dual space of X. By λ we denote the Lebesgue measure on the real line $\mathbb{R}$. For an interval $I \subset \mathbb{R}$ by $L_0(I)$ we denote the set of all (equivalence classes of) real valued Lebesgue measurable (finite almost everywhere) functions on I. Any Banach space $E = E(I) \subset L_0(I)$ with norm $\|\cdot\|$ satisfying the condition that $f \in E$ and $\|f\| \leqslant \|g\|$ whenever $0 \leqslant f \leqslant g$ a.e., $f \in L_0(I)$ and $g \in E$, is called a *Banach function lattice.* An element f in a Banach function lattice E is called *order continuous* if for every $0 \leqslant f_n \leqslant |f|$ a.e. such that $f_n \downarrow 0$ a.e. it holds $\|f_n\| \downarrow 0$. We say that E is *order continuous* if every element in E is order continuous. A Banach function lattice $(E, \|\cdot\|)$ has the *Fatou property* if for any sequence $(f_n) \subset E$ and any $f \in L_0(I)$ such that $0 \leqslant f_n \leqslant f$ a.e., $f_n \uparrow f$ a.e. and $\sup_n \|f_n\| < \infty$ it holds $f \in E$ and $\|f\| = \lim_n \|f_n\|$. Definition of the Cesàro function space $C_{p,w}$ and related notion are provided in Section 3.

2. Some results in general Banach spaces

Let $(X, \|\cdot\|)$ be a (real) Banach space. For any $x^* \in S_{X^*}$ and $\epsilon > 0$ the set $S(x^*; \epsilon) = \{x \in B_X : x^*x > 1 - \epsilon\}$ is called *a slice* determined by x^* and ϵ.

A finite convex combination of slices is a set of the form

$$S = \sum_{i=1}^{n} \lambda_i S(x_i^*, \epsilon_i), \qquad \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geqslant 0,$$

where $x_i^* \in S_{X^*}$ and $\epsilon_i > 0$, $i = 1, 2, \dots n$.

We say that a Banach space $(X, \|\cdot\|)$ has the

a) *local diameter* 2 *property* if every slice of B_X has diameter 2.
b) *diameter* 2 *property* if every non-empty relatively weakly open subset of B_X has diameter 2.
c) *strong diameter* 2 *property* if every finite convex combination of slices of B_X has diameter 2.

Examples of spaces with diameter 2 properties are c_0, $L_1[0,1]$ and $C[0,1]$. For other examples and history of these properties we refer to [2]. It is known that the following implications hold true between diameter 2 properties

$$c \implies b \implies a.$$

It has also been reported that strong diameter 2 property is not equivalent to diameter 2 property [1]. However, it is not known whether local diameter 2 property implies diameter 2 property.

We show that there are connections between uniformly non-square points and diameter 2 properties.

We say that a point $x \in X$ is *uniformly non-square* if there exists $\rho > 1$ such that

$$\rho \min\{\|x\|, \|y\|\} \leqslant \max\{\|x+y\|, \|x-y\|\} \text{ for all } y \in X.$$

If all points $x \in X$ are uniformly non-square then the space $(X, \|\cdot\|)$ is called *locally uniformly non-square*. Uniformly non-square points were introduced by Schäffer in 1976 [15, p. 131].

It should be noted that in the literature there is another geometric property, also called local uniform non-squareness, which is not equivalent to one stated above (cf. [11] and references therein and [7, p. 127]).

The following lemma is easy to show.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a Banach space.*

(*i*) *If $x \in X$ is a uniformly non-square point then so is λx for any $\lambda \in \mathbb{R}$.*
(*ii*) *The space X is locally uniformly non-square if and only if all points $x \in S_X$ are uniformly non-square.*

Now we give equivalent conditions under which a point $x \in S_X$ is not uniformly non-square. Some of them are known. We deliver a proof for the sake of completeness.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a Banach space, and let $x \in S_X$. The following are equivalent.*

(*i*) *The point x is not uniformly non-square.*
(*ii*) *There exists a sequence $(y_n) \subset X$ such that $\|y_n\| \to 1$ and $\|x \pm y_n\| \to 1$ as $n \to \infty$.*
(*iii*) *There exists a sequence $(y_n) \subset B_X$ such that $\|y_n\| \to 1$ and $\|x \pm y_n\| \to 1$ as $n \to \infty$.*
(*iv*) *There exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$ as $n \to \infty$.*
(*v*) *There exists a sequence $(y_n) \subset B_X$ such that $\|\lambda x \pm y_n\| \to 1$ as $n \to \infty$ for all $\lambda \in [0,1]$.*

Proof. *(i)* $\implies$ *(ii)* First observe that, for any $x, y \in X$,

$$\|x\| = \left\| \frac{x-y}{2} + \frac{x+y}{2} \right\| \leqslant \frac{1}{2}\|x+y\| + \frac{1}{2}\|x-y\|.$$

Hence

$$\|x+y\| + \|x-y\| \geqslant 2\|x\|. \tag{1}$$

Changing the roles of x and y we can infer that $\|x+y\|+\|x-y\| \geqslant 2\max(\|x\|, \|y\|)$. Hence

$$\max(\|x+y\|, \|x-y\|) \geqslant \max(\|x\|, \|y\|), \text{ for all } x, y \in X. \tag{2}$$

It follows that

$$\frac{\max(\|x+y\|, \|x-y\|)}{\min(\|x\|, \|y\|)} \geqslant 1, \text{ for all } x, y \in X. \tag{3}$$

Let $x \in S_X$ be a point which is not uniformly non-square. By inequality (3), we get that there exists sequence $(y_n) \subset X$ such that

$$\frac{\max(\|x+y_n\|, \|x-y_n\|)}{\min(1, \|y_n\|)} \to 1 \text{ as } n \to \infty. \tag{4}$$

It follows that $\|y_n\| \to 1$ as $n \to \infty$. Indeed, if there is a subsequence y_{n_k} such that $\|y_{n_k}\| \leqslant \eta < 1$ for all $k \in \mathbb{N}$ then $\max(\|x+y_{n_k}\|, \|x-y_{n_k}\|)/\min(1, \|y_{n_k}\|) \geqslant 1/\eta > 1$. Similarly, if $\|y_{n_k}\| \geqslant \eta > 1$ for all $k \in \mathbb{N}$ then by (2) $\max(\|x+y_{n_k}\|, \|x-y_{n_k}\|)/\min(1, \|y_{n_k}\|) \geqslant \eta/1 > 1$. In both cases we have a contradiction with (4). Now it follows from (4) that $\max(\|x+y_n\|, \|x-y_n\|) \to 1$ as $n \to \infty$. Since $\|x\| = 1$, by (1) we get that $\|x \pm y_n\| \to 1$ as $n \to \infty$.

(ii) $\implies$ *(iv)* By *(ii)* we can find a subsequence of (y_n), again called (y_n), such that $\|y_n\| \to 1$, $\|x \pm y_n\| \to 1$ as $n \to \infty$ and either $(y_n) \subset B_X$ or $(y_n) \subset X \setminus B_X$.

Suppose first that $(y_n) \subset B_X$. Let $\gamma, \gamma_n \geqslant 0$ be such that $\gamma_n \to \gamma$ as $n \to \infty$. We have

$$\begin{aligned}\|(1+\gamma_n)y_n \pm x\| &= \|(2+\gamma_n)y_n - (y_n \mp x)\| \\ &\geqslant |(2+\gamma_n)\|y_n\| - \|y_n \mp x\|| \to 2+\gamma-1 = 1+\gamma.\end{aligned}$$

Hence $\underline{\lim}_{n\to\infty}\|(1+\gamma_n)y_n \pm x\| \geqslant 1+\gamma$. But $\|(1+\gamma_n)y_n \pm x\| \leqslant \gamma_n\|y_n\| + \|y_n \pm x\| \to 1+\gamma$ whence $\overline{\lim}_{n\to\infty}\|(1+\gamma_n)y_n \pm x\| \leqslant 1+\gamma$ and so $\lim_{n\to\infty}\|(1+\gamma_n)y_n \pm x\| = 1+\gamma$. Equivalently

$$\lim_{n\to\infty} \frac{1+\gamma_n}{1+\gamma}\|(1+\gamma_n)^{-1}x \pm y_n\| = 1.$$

Since $(1+\gamma_n)/(1+\gamma) \to 1$ as $n \to \infty$ we get that $\lim_{n\to\infty}\|(1+\gamma_n)^{-1}x \pm y_n\| = 1$. Now taking $\gamma_n = 1/\|y_n\| - 1$ we obtain that $\gamma_n \to 0$ and

$$\left\| x \pm \frac{y_n}{\|y_n\|} \right\| = \frac{1}{\|y_n\|}\,\|\|y_n\|x \pm y_n\| = (1+\gamma_n)\|(1+\gamma_n)^{-1}x \pm y_n\| \to 1.$$

In case when $(y_n) \subset X \setminus B_X$ we proceed similarly. Let $0 \leqslant \gamma_n \leqslant 1$, $n \in \mathbb{N}$ be such that $\gamma_n \to 0$ as $n \to \infty$. We have

$$\begin{aligned}\|(1-\gamma_n)y_n \pm x\| &= \|(2-\gamma_n)y_n - (y_n \mp x)\| \\ &\geqslant |(2-\gamma_n)\|y_n\| - \|y_n \mp x\|| \to 2-1 = 1.\end{aligned}$$

Hence $\underline{\lim}_{n\to\infty}\|(1-\gamma_n)y_n \pm x\| \geqslant 1$. But $\|(1-\gamma_n)y_n \pm x\| \leqslant \|\gamma_n y_n\| + \|y_n \pm x\| \to 1$ whence $\overline{\lim}_{n\to\infty}\|(1-\gamma_n)y_n \pm x\| \leqslant 1$ and so $\lim_{n\to\infty}\|(1-\gamma_n)y_n \pm x\| = 1$. Equivalently

$$\lim_{n\to\infty}(1-\gamma_n)\|(1-\gamma_n)^{-1}x \pm y_n\| = 1.$$

Now taking $\gamma_n = 1 - 1/\|y_n\|$ we obtain that $\gamma_n \to 0$ and

$$\left\|x \pm \frac{y_n}{\|y_n\|}\right\| = \frac{1}{\|y_n\|}\|\|y_n\|x \pm y_n\| = (1-\gamma_n)\|(1-\gamma_n)^{-1}x \pm y_n\| \to 1.$$

Implications $(iv) \implies (iii)$ and $(iii) \implies (i)$ are obvious. Hence conditions (i)–(iv) are equivalent.

The implication $(v) \implies (iii)$ is clear. While $(iii) \implies (v)$ can be proved in the same way as the first part of $(ii) \implies (iv)$ by taking $\gamma_n = 1/\lambda - 1$ to be a constant sequence and $\gamma = 1/\lambda - 1$ in case when $\lambda \in (0, 1]$. For $\lambda = 0$ the claim is clear. □

Observe that $x \in S_X$ is uniformly non-square if and only if there exists $\delta > 0$ such that

$$\max\{\|x+y\|, \|x-y\|\} \geqslant 1 + \delta \quad \text{for all } y \in S_X. \tag{5}$$

Indeed, if $x \in S_X$ is uniformly non-square then (5) is satisfied. If $x \in S_X$ is not uniformly non-square then by Proposition 2.2 (iv) there exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$. Hence (5) is not satisfied.

A point $x \in S_X$ is called *strongly extreme* or *midpoint locally uniformly rotund*, if for every sequence $(x_n) \subset B_X$ the condition $\|x \pm x_n\| \to 1$ as $n \to \infty$ implies that $x_n \to 0$ as $n \to \infty$. If all points of the unit sphere S_X are strongly extreme points then the space $(X, \|\cdot\|)$ is called *midpoint locally uniformly rotund.* A Banach space $(X, \|\cdot\|)$ is called *uniformly rotund in every direction* if $y_n, x \in X$, $\|y_n\| \to 1$, $\|x + y_n\| \to 1$ and $\|x + 2y_n\| \to 2$ implies that $x = 0$.

The following fact is clear.

Proposition 2.3. *If $x \in S_X$ is a strongly extreme point then it is a uniformly non-square point.*

It follows that a midpoint locally uniformly rotund space is necessarily locally uniformly non-square. Similar claim holds true for uniformly rotund in every direction space.

Proposition 2.4. *If a Banach space $(X, \|\cdot\|)$ is uniformly rotund in every direction then it is locally uniformly non-square.*

Proof. Let $(X, \|\cdot\|)$ be an uniformly rotund in every direction space. Suppose that $x \in S_X$ is not an uniformly non-square point. By Proposition 2.2 (v) there exists a sequence $(y_n) \subset B_X$ such that $\|y_n\| \to 1$, $\|x+y_n\| \to 1$ and $\|x+2y_n\| \to 2$ as $n \to \infty$. From the assumption on X we get that $x = 0$ which is a contradiction with $x \in S_X$. $\square$

On the other hand, the following interesting fact holds true.

Proposition 2.5. *Let $(X, \|\cdot\|)$ be a Banach space. If there are no uniformly non-square points in the unit sphere S_X then X has the local diameter 2 property.*

Proof. Let $x^* \in S_{X^*}$ and $\epsilon > 0$ be arbitrary. Let $\delta < \epsilon/4$ be positive and choose $x \in S_X$ such that $x^*x > 1-\delta$. In particular $x \in S(x^*;\epsilon)$. By assumption x is not a uniformly non-square point, so by Proposition 2.2 (iii) there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$ as $n \to \infty$. Since

$$x^*(x \pm y_n) \leqslant |x^*(x \pm y_n)| \leqslant \|x^*\|_{X^*}\|x \pm y_n\| \to 1,$$

we get that $x^*(x \pm y_n) < 1+\delta$, whence $|x^*y_n| < 2\delta$ for all n sufficiently large. Now, since $\|x \pm y_n\| \to 1$ as $n \to \infty$, for all n large enough

$$\left\| \frac{x \pm y_n}{1+\delta} \right\| \leqslant 1.$$

Moreover

$$x^*\left(\frac{x \pm y_n}{1+\delta}\right) = \frac{1}{1+\delta}(x^*x \pm x^*y_n) > \frac{1}{1+\delta}(1-\delta-2\delta) > 1-4\delta > 1-\epsilon,$$

hence $(x \pm y_n)/(1+\delta) \in S(x^*;\epsilon)$ for all n large enough. We also get that

$$\left\| \frac{x+y_n}{1+\delta} - \frac{x-y_n}{1+\delta} \right\| = \frac{1}{1+\delta}\|2y_n\| \to \frac{2}{1+\delta} \quad \text{as } n \to \infty.$$

Since δ can be taken arbitrarily small we get that diameter of $S(x^*;\epsilon)$ is equal 2. $\square$

The following result is a part of the proof of Theorem 3.5 in [3]. We provide it here for reader's convenience.

Proposition 2.6. *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. If for every $x \in S_X$ there is a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly in X as $n \to \infty$ then X has the diameter 2 property.*

Proof. Let $W \neq \emptyset$ be a relatively weakly open subset of B_X. Since X is infinite dimensional there is $x \in W$ such that $\|x\| = 1$. By assumption there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly in X as $n \to \infty$. Let $x_n^{'} = x + y_n$ and $x_n^{''} = x - y_n$. Clearly $x_n^{'}, x_n^{''} \to x$ weakly in X, $\|x_n^{'}\|$, $\|x_n^{''}\| \to 1$ and $\|x_n^{'} - x_n^{''}\| = 2\|y_n\| \to 2$ as $n \to \infty$. Taking $z_n = x_n^{'}/\|x_n^{'}\|$ and $w_n = x_n^{''}/\|x_n^{''}\|$, we clearly have that $z_n, w_n \to x$ weakly in X, $z_n, w_n \in B_X$ and $\|z_n - w_n\| \to 2$ as $n \to \infty$. $\square$

Recall that a point $x \in S_X$ is called an *H-point* of B_X if for every sequence $(x_n) \subset X$ such that $\|x_n\| \to \|x\|$ and $x_n \to x$ weakly it follows that $\|x - x_n\| \to 0$ as $n \to \infty$. If all points of the unit sphere are H-points then $(X, \|\cdot\|)$ is said to have *Kadec-Klee property* or *H property* or *Radon-Riesz property.*

In connection with points which appear in the Proposition 2.6 we have the following result.

Proposition 2.7. *If for $x \in S_X$ there is a sequence $(y_n) \subset B_X$ such that $y_n \nrightarrow 0$, $y_n \to 0$ weakly in X and either $\|x + y_n\| \to 1$ or $\|x - y_n\| \to 1$ as $n \to \infty$ then x is not an H-point of B_X.*

Proof. It is enough to take $x_n = x + y_n$ or $x_n = x - y_n$, if $\|x + y_n\| \to 1$ or $\|x - y_n\| \to 1$ as $n \to \infty$, respectively. By assumption $\|x_n\| \to 1 = \|x\|$, $x_n \to x$ weakly and $\|x_n - x\| = \|y_n\| \nrightarrow 0$. $\square$

3. Diameter 2 properties

Let $I = (0, l)$ where $0 < l \leqslant \infty$ is fixed and $w \in L_0(I)$ be a positive a.e. function called weight in the sequel. The *weighted Cesàro function space* on I is defined as $(1 \leqslant p < \infty)$

$$\begin{aligned} C_{p,w} &= C_{p,w}(I) \\ &:= \left\{ f \in L_0(I) : \|f\|_{C_{p,w}} := \left(\int_I \left(w(x) \int_0^x |f(t)|\, dt \right)^p dx \right)^{1/p} < \infty \right\}. \end{aligned}$$

Note that for $f \in C_{p,w}(I)$,

$$\|f\|_{C_{p,w}} = \|\mathcal{H}_w f\|_p \quad \text{where } \mathcal{H}_w f(x) = w(x) \int_0^x |f(t)|\, dt, \;\; x \in I,$$

and $\|\cdot\|_p$ is the norm in the space $L_p(I)$.

It is known that the space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ is an order continuous Banach function lattice with the Fatou property. Its Köthe dual $(C_{p,w})'$ is isometrically isomorphic to its Banach dual $(C_{p,w})^*$. Moreover $C_{p,w} \neq \{0\}$ if and only if $\int_c^l w(x)^p\, dx < \infty$ for some $c \in I$. For $p = 1$, space $C_{p,w}$ is just a weighted Lebesgue space [13].

In the sequel we assume that $C_{p,w} \neq \{0\}$ and $1 \leqslant p < \infty$, unless stated otherwise.

In this section, applying results from Section 2, we show that the Cesàro function space has diameter 2 property.

Definition 3.1. Let $(r_n)_{n=1}^\infty$ be the sequence of Rademacher functions on $[0, 1]$, that is $r_n(t) = \operatorname{sign}(\sin(2^n \pi t))$, $t \in [0, 1]$. Let $\tilde{r}_n(t) = r_n(t - m)$ if $t \in [m, m+1)$, $m \in \mathbb{N} \cup \{0\}$ for all $n \in \mathbb{N}$. We define the sequence of Rademacher functions $(p_n)_{n=1}^\infty$ on an interval I by restricting $\tilde{r}_n$ to the interval I, that is $p_n = \tilde{r}_n|_I$, $n \in \mathbb{N}$.

Lemma 3.2. *For any function $f \in L_1(I)$, $\int_I f p_n \to 0$ as $n \to \infty$.*

Proof. Suppose that $f \geqslant 0$. Let $\epsilon > 0$ be arbitrary. Since $f \in L_1(I)$ and $L_1(I)$ is order continuous, there exists a set $A \subset I$ of finite positive measure such that $f|_A \in L_\infty(I)$ and $\int_{I\setminus A} f(t)\,dt < \epsilon$. Let sets $B_i \subset A$, $i = 1, 2, \ldots, N$, be disjoint and such that $A = \cup_{i=1}^N B_i$ and $\operatorname{ess\,sup}_{t,s\in B_i} |f(t) - f(s)| < \epsilon/\lambda(A)$, $i = 1, 2, \ldots, N$. For each $i = 1, 2, \ldots, N$ and each $n \in \mathbb{N}$ we can find pairwise disjoint sets $C_i^{(n)}$, $D_i^{(n)}$, and $E_i^{(n)}$ such that $B_i = C_i^{(n)} \cup D_i^{(n)} \cup E_i^{(n)}$, $\lambda(C_i^{(n)}) = \lambda(D_i^{(n)})$ and for all n large enough $p_n(C_i^{(n)}) = \{1\}$, $p_n(D_i^{(n)}) = \{-1\}$, and $\lambda(E_i^{(n)}) \to 0$ as $n \to \infty$. Observe that

$$\int_{\cup_{i=1}^N E_i^{(n)}} f(t)\,dt \leqslant \| f|_A\|_\infty \, \lambda(\cup_{i=1}^N E_i^{(n)}) \to 0 \quad \text{as } n \to \infty,$$

and for all $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned} \left| \int_{B_i \setminus E_i^{(n)}} f(t) p_n(t)\,dt \right| &= \left| \int_{C_i^{(n)}} f(t)\,dt - \int_{D_i^{(n)}} f(t)\,dt \right| \\ \leqslant \lambda(C_i^{(n)}) \operatorname*{ess\,sup}_{t\in B_i} f(t) &- \lambda(D_i^{(n)}) \operatorname*{ess\,inf}_{t\in B_i} f(t) < \epsilon\lambda(B_i)/\lambda(A). \end{aligned}$$

Now, for all n large enough,

$$\begin{aligned} \left| \int_I f(t) p_n(t)\,dt \right| &\leqslant \left| \int_A f(t) p_n(t)\,dt \right| + \int_{I\setminus A} f(t)\,dt \\ &= \sum_{i=1}^N \left| \int_{B_i} f(t) p_n(t)\,dt \right| + \epsilon \\ &\leqslant \sum_{i=1}^N \left| \int_{B_i \setminus E_i^{(n)}} f(t) p_n(t)\,dt \right| + \int_{\cup_{i=1}^N E_i^{(n)}} f(t)\,dt + \epsilon \leqslant 3\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get that $\int_I f(t) p_n(t)\,dt \to 0$ as $n \to \infty$. If $f \in L_1(I)$ is arbitrary then $f = f^+ - f^-$, where $f^+, f^- \geqslant 0$ and applying the above result to both f^+, f^- we obtain the conclusion. □

Lemma 3.3. *For every function $f \in S_{Cp,w}$ there exists a sequence $(f_n) \subset B_{C_{p,w}}$ such that $\|f \pm f_n\|_{C_{p,w}} \to 1$, $\|f_n\|_{C_{p,w}} \to 1$ and $f_n \to 0$ weakly in $C_{p,w}$ as $n \to \infty$.*

Proof. Let $f_n = f p_n$. Clearly $\|f_n\| = \|f\| = 1$, $n \in \mathbb{N}$. Since $f\chi_{[0,x]} \in L_1(I)$, $x \in I$ by Lemma 3.2 we get that for all $x \in I$,

$$\begin{aligned} \int_0^x |f(t) \pm f_n(t)|\,dt &= \int_0^x |f(t) \pm f(t)p_n(t)|\,dt = \int_0^x |f(t)|(1 \pm p_n(t))\,dt \\ &= \int_0^x |f(t)|\,dt \pm \int_0^x |f(t)| p_n(t)\,dt \to \int_0^x |f(t)|\,dt \quad \text{as } n \to \infty. \end{aligned}$$

By the Lebesgue Dominated Convergence Theorem we get that $\|f \pm f_n\|_{C_{p,w}} \to 1$ as $n \to \infty$.

Observe that for any $f \in C_{p,w}$, $f_n = fp_n \to 0$ weakly in $C_{p,w}$. Indeed, for every function $g \in (C_{p,w})'$, $fg \in L_1(I)$ and hence by Lemma 3.2, $\int_I g(t)f_n(t)\,dt = \int_I g(t)f(t)p_n(t)\,dt \to 0$ as $n \to \infty$. $\square$

Lemma 3.3 and Propositions 2.6, 2.7 imply the following.

Corollary 3.4. *The Cesàro function space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ has the diameter 2 property. The unit sphere $S_{C_{p,w}}$ does not have uniformly non-square points nor strongly extreme points nor H-points. In particular the space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ is neither locally uniformly non-square nor midpoint locally uniformly rotund nor uniformly rotund in every direction and does not have the Kadec-Klee property.*

It is worth to mention that in opposition to function spaces, the Cesàro sequence spaces are locally uniformly non-square [8].

In connection with the strong diameter 2 property we have the following result. It can be proved (under assumption that $\Psi(x) := \int_x^l w(t)^p\,dt < \infty$ for all $x \in I$ and $\int_0^l w(t)^p\,dt = \infty$, $1 < p < \infty$) in a similar way as Theorem 4.1 in [13]. Given a function $f \in L_0(I)$ we define $\hat{f}$ to be the essential Ψ-concave majorant of $|f|$. That is, $\hat{f}$ is the smallest function greater than or equal to $|f|$ such that $\hat{f}$ is Ψ-concave on I (i.e. $\hat{f} \circ \Psi^{-1}$ is concave on $\Psi(I)$). For a precise definition of essential Ψ-concave majorant and its properties we refer to [13]. For a detailed proof of the below result we refer to [14].

Theorem 3.5. *Let $f_j \in S_{(C_{p,w})'}$, $j = 1, 2, \ldots, r$, $r \in \mathbb{N}$ be such that $\widehat{f_j} = \widehat{f_i}$, $i, j = 1, 2, \ldots, r$. The diameter of a finite convex combination of slices defined by bounded linear functionals determined by f_j, $j = 1, 2, \ldots, r$, is 2.*

4. Strict convexity

A point $x \in S_X$ is called *extreme* if for every $y \in X$ the condition $\|x \pm y\| = 1$ implies that $y = 0$. Equivalently, if $x = (y + z)/2$, where $y, z \in B_X$ then $y = z = x$. If all points of the unit sphere S_X are extreme then the space $(X, \|\cdot\|)$ is called *strictly convex.* By Corollary 3.4 we have that no point of $S_{C_{p,w}}$ is strongly extreme. However, all points of $S_{C_{p,w}}$ are extreme.

Theorem 4.1. *The space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$, $1 < p < \infty$ is strictly convex.*

Proof. Let $f \in S_{C_{p,w}}$ and suppose that $\|f \pm g\|_{C_{p,w}} = \|f\|_{C_{p,w}}$, $g \in C_{p,w}$. Since

$$|f| = \frac{|f + g + f - g|}{2} \leqslant \frac{1}{2}|f + g| + \frac{1}{2}|f - g|, \tag{6}$$

we get that

$$\|f\|_{C_{p,w}} \leqslant \left\| \frac{1}{2}|f+g| + \frac{1}{2}|f-g| \right\|_{C_{p,w}} \leqslant \frac{1}{2}\|f+g\|_{C_{p,w}} + \frac{1}{2}\|f-g\|_{C_{p,w}} = \|f\|_{C_{p,w}}.$$

It follows that

$$\int_I \left(w(x) \int_0^x \frac{|f(t)+g(t)| + |f(t)-g(t)|}{2}\, dt \right)^p dx \\ - \int_I \left(w(x) \int_0^x |f(t)|\, dt \right)^p dx = 0.$$

The latter and inequality (6) implies that

$$w(x) \int_0^x \frac{|f(t)+g(t)| + |f(t)-g(t)|}{2}\, dt = w(x) \int_0^x |f(t)|\, dt \quad \text{for almost all } x \in I,$$

that is

$$\frac{\mathcal{H}_w(f+g) + \mathcal{H}_w(f-g)}{2} = \mathcal{H}_w f.$$

By assumption $\mathcal{H}_w f$, $\mathcal{H}_w(f+g)$, $\mathcal{H}_w(f-g) \in S_{L_p}$. Since $L_p(I)$ space is strictly convex for $1 < p < \infty$ we get that $\mathcal{H}_w f = \mathcal{H}_w(f+g) = \mathcal{H}_w(f-g)$. Since the weight function w is strictly positive a.e.,

$$\int_0^x |f(t) \pm g(t)| - |f(t)|\, dt = 0 \quad \text{for almost all } x \in I.$$

This implies that $|f(t) \pm g(t)| = |f(t)|$ a.e. on I. Whence $g = 0$. $\square$

The above result can be also obtained by applying Corollary 1 from [12]. A proof of strict convexity of the Cesàro function space Ces_p appeared in [5].

5. Copy of ℓ_1

Recall [10] that a Banach space $(X, \|\cdot\|)$ contains an *asymptotically isometric copy of* ℓ_1 if there exists a sequence $(\epsilon_n) \subset (0,1)$, $\epsilon_n \to 0$ as $n \to \infty$ and a sequence $(x_n) \subset X$ such that for arbitrary $(\alpha_n) \in \ell_1$ we have

$$\sum_{n=1}^{\infty} (1-\epsilon_n)|\alpha_n| \leqslant \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| \leqslant \sum_{n=1}^{\infty} |\alpha_n|.$$

In 2008 Astashkin and Maligranda proved that the Cesàro function space $\mathrm{Ces}_p = C_{p,w}$ with weight function $w(x) = 1/x$ contains an asymptotically isometric copy of ℓ_1 [4]. In fact their proof can be extended to work well also for arbitrary weights.

Theorem 5.1. *The space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ contains an asymptotically isometric copy of ℓ_1.*

Proof. Since $C_{p,w} \neq \{0\}$ there exits $r \in I = (0,l)$ such that $\int_r^l w(t)^p\,dt < \infty$ [13]. Denote $\Psi(x) = \int_x^l w(t)^p\,dt$, $x \in [r,l)$. Since $w > 0$ a.e. on I, the function Ψ is strictly decreasing on $[r,l)$.

Let $(b_n)_{n=1}^\infty$ be any strictly increasing sequence in $[r,l)$ such that $\lim_{n\to\infty} b_n = b < l$. Denote $a = b_1$. Let $g_n = \chi_{(b_n,b_{n+1})}$ and $f_n = g_n/\|g_n\|_{C_{p,w}}$, $n \in \mathbb{N}$. It is easy to check for any function $f \in L_1(I)$ with $\operatorname{supp} f \subset [c,d]$ for some bounded interval $[c,d] \subset [r,l)$,

$$\Psi(d)^{1/p}\|f\|_{L_1} \leqslant \|f\|_{C_{p,w}} \leqslant \Psi(c)^{1/p}\|f\|_{L_1}. \tag{7}$$

Let $(\alpha_n)_{n=1}^\infty \in \ell_1$. Since $\operatorname{supp}\sum_{n=1}^\infty \alpha_n f_n \subset [a,b]$ and $\operatorname{supp} g_n \subset [b_n, b_{n+1}]$, by (7) we get that

$$\begin{aligned}\left\|\sum_{n=1}^\infty \alpha_n f_n\right\|_{C_{p,w}} &\geqslant \Psi(b)^{1/p}\|\sum_{n=1}^\infty \alpha_n f_n\|_{L_1}\\ &= \Psi(b)^{1/p}\sum_{n=1}^\infty |\alpha_n|\|f_n\|_{L_1} = \Psi(b)^{1/p}\sum_{n=1}^\infty |\alpha_n|\|g_n\|_{L_1}/\|g_n\|_{C_{p,w}}\\ &\geqslant \sum_{n=1}^\infty |\alpha_n|\Psi(b)^{1/p}/\Psi(b_n)^{1/p} = \sum_{n=1}^\infty (1-\epsilon_n)|\alpha_n|,\end{aligned}$$

where $\epsilon_n = 1 - \Psi(b)^{1/p}/\Psi(b_n)^{1/p}$. Clearly $\epsilon_n \to 0$ as $n \to \infty$. The other inequality is obvious. □

Let X be a Banach space. Recall that a mapping $T : K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K \subset X$. A Banach space X has the *(weak) fixed point property* if every nonexpansive mapping of every (nonempty weakly compact convex) closed bounded convex subset K into itself has a fixed point.

Similarly as in [4] by results in [9], we conclude this section with the following corollary.

Corollary 5.2. *The Cesàro function space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ and its dual $((C_{p,w})^*, \|\cdot\|_{(C_{p,w})^*})$ fail the fixed point property. Moreover $((C_{p,w})^*, \|\cdot\|_{(C_{p,w})^*})$ contains an isometric copy of $L_1[0,1]$, hence it also fails the weak fixed point property.*

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